Math 126 Challenge Problems/Solutions

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1. Given $|\mathbf{v}+\mathbf{w}|$ and $|\mathbf{v}-\mathbf{w}|$, compute $\mathbf{v}\cdot\mathbf{w}$.

Using the relationship between the magnitude and the dot product and the linearity of the dot product (see §12.3, Theorem 2), we have

$$\begin{aligned}|\mathbf{v}+\mathbf{w}|^2 &= (\mathbf{v}+\mathbf{w})\cdot(\mathbf{v}+\mathbf{w})\\ &= \mathbf{v}\cdot\mathbf{v}+\mathbf{v}\cdot\mathbf{w}+\mathbf{w}\cdot\mathbf{v}+\mathbf{w}\cdot\mathbf{w}\\ &= |\mathbf{v}|^2+2(\mathbf{v}\cdot\mathbf{w})+|\mathbf{w}|^2.\end{aligned}$$

In the same way we find

$$|\mathbf{v}-\mathbf{w}|^2 = |\mathbf{v}|^2-2(\mathbf{v}\cdot\mathbf{w})+|\mathbf{w}|^2.$$

Thus subtracting the second from the first gives $4(\mathbf{v}\cdot\mathbf{w})$, so that

$$\mathbf{v}\cdot\mathbf{w} = \frac{1}{4}(|\mathbf{v}+\mathbf{w}|^2-|\mathbf{v}-\mathbf{w}|^2).$$

This is the so-called "polarization identity" (for Euclidean spaces; there are more general versions). It's perhaps a bit surprising that we can compute the dot product function given only a way to compute the lengths of arbitrary vectors. In more general settings (Banach spaces), this relationship breaks down and provides for a potentially richer variety of objects to study. ∎

2. Find a non-zero vector perpendicular to both $\langle 0,1,2\rangle$ and $\langle 3,4,5\rangle$. Devise a general strategy for solving this type of problem.

Suppose such a vector is $\langle x,y,z\rangle$. The dot product of this with the given vectors must be zero, so we may solve the following system of equations:

$$\begin{aligned}\langle x,y,z\rangle\cdot\langle 0,1,2\rangle &= y+2z=0\\ \langle x,y,z\rangle\cdot\langle 3,4,5\rangle &= 3x+4y+5z=0\end{aligned}$$

An obvious solution to the first equation is $y=1, z=-1$. This forces $x=2$ in the second equation, so $\langle 2,1,-1\rangle$ is a solution. In general there are only two equations and three unknowns, so, except for degenerate cases where one of the input vectors is $\mathbf{0}$, there are always (infinitely many) solutions.

Another technique: say the input vectors are $\mathbf{u},\mathbf{v}$. Pick any vector $\mathbf{w}$ not in the plane they span; there are several ways to do this, though I'll leave that to you. Now compute

$$\mathbf{w}' = \mathbf{w}-\text{proj}_{\mathbf{u}}\mathbf{w}-\text{proj}_{\mathbf{v}}\mathbf{w}.$$

Since we subtract off the portions of $\mathbf{w}$ in the direction of the input vectors, the result is perpendicular to each. Since $\mathbf{w}$ was not in the plane they span, $\mathbf{w}'$ is non-zero, as needed. This operation is the key idea in the "Gram-Schmidt Orthogonalization" process.

Also, cross products solve this problem immediately, though they don't work in general dimensions whereas the above methods do. ∎