

Webs, pockets, and buildings

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Based on joint work with subsets of *Christian Gaetz, Oliver Pechenik, Stephan Pfannerer, Jessica Striker, and Haihan Wu*

Slides: https://www.jpswanson.org/talks/2026_Harvard-MIT_pockets.pdf

Presented at the
Richard P. Stanley Seminar in Combinatorics
at Harvard/MIT
January 28th, 2026

Outline

- Effective graphical calculi
 - Web bases for $SL(r)$ -mod, $r=2,3$
 - Hourglass plabic graphs for $r=4$
- Pockets
- Weight-valued geodesics
- Affine buildings, geometric embeddings

Web Calculus

Q How do you make a representation category effectively computable?

Ex $V = \mathbb{C}^n$ $\det: V^{\otimes n} \rightarrow \mathbb{C}$ (in $SL(V)\text{-mod}$)

$$\bigotimes_{i=1}^n (\sum a_{ij} e_j) \mapsto \sum_{w \in S_n} \text{sgn}(w) \prod_{i=1}^n a_{i, w_i}$$

— Expanding into monomials is horribly inefficient!

Idea Use a graphical calculus w/ generators + relations

SL_2 -Webs

Def | An SL_2 -web is

- a bipartite graph embedded in

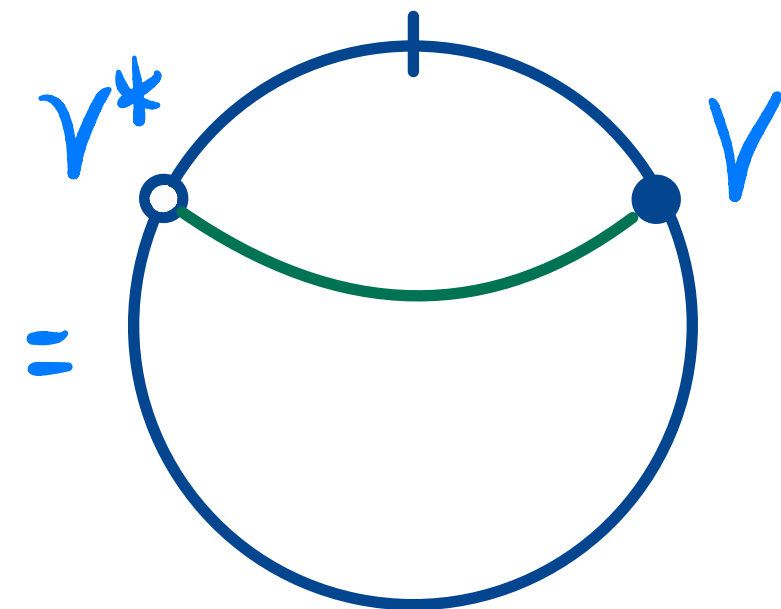
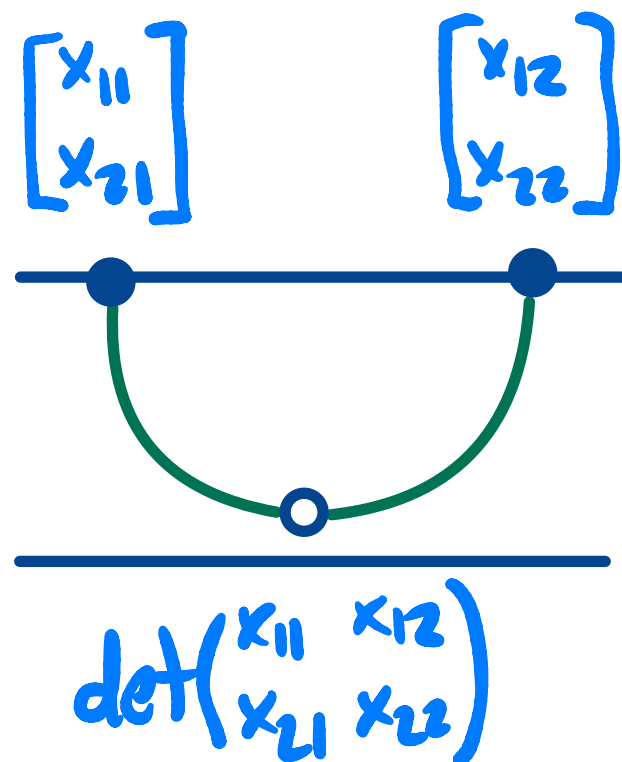
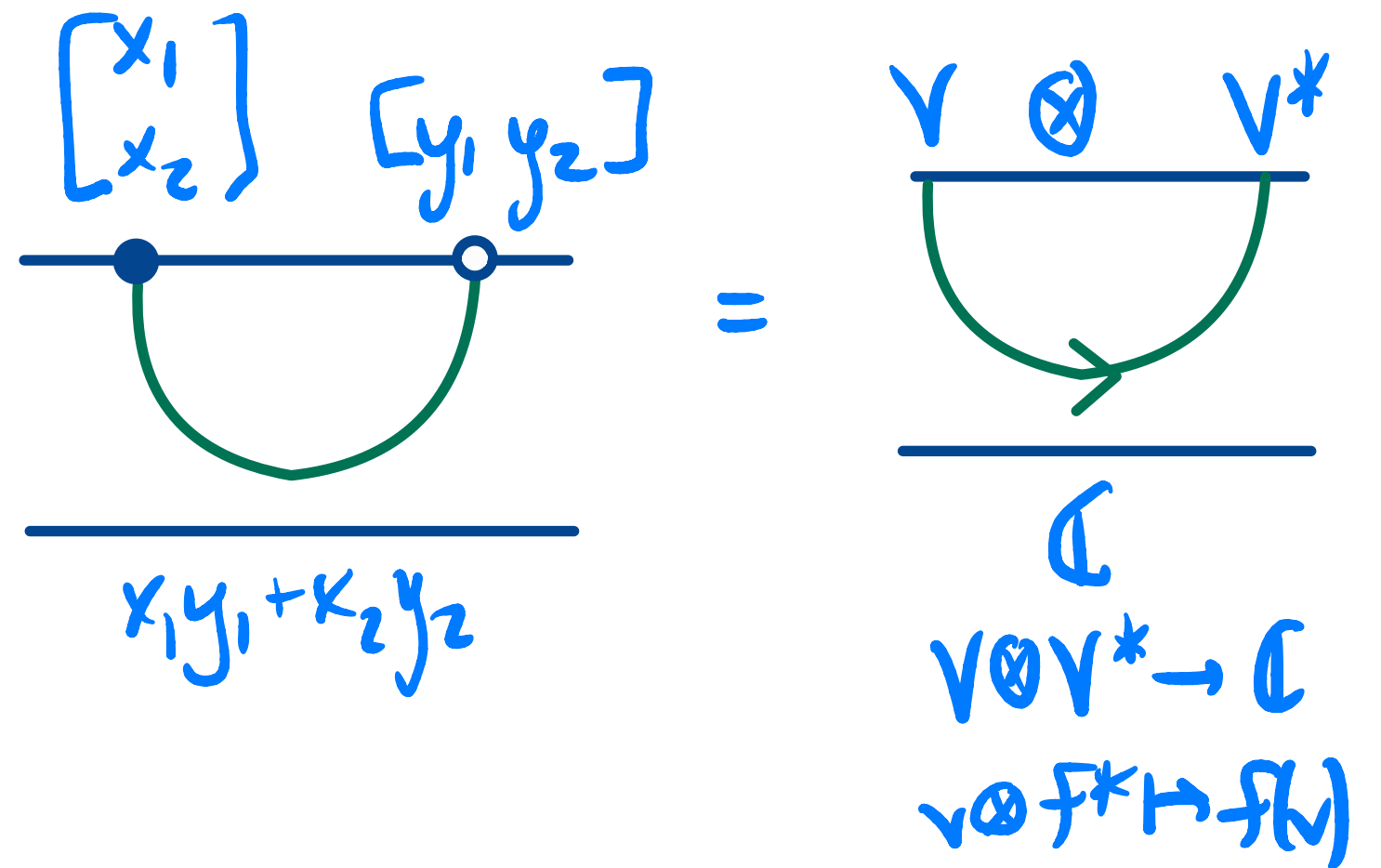
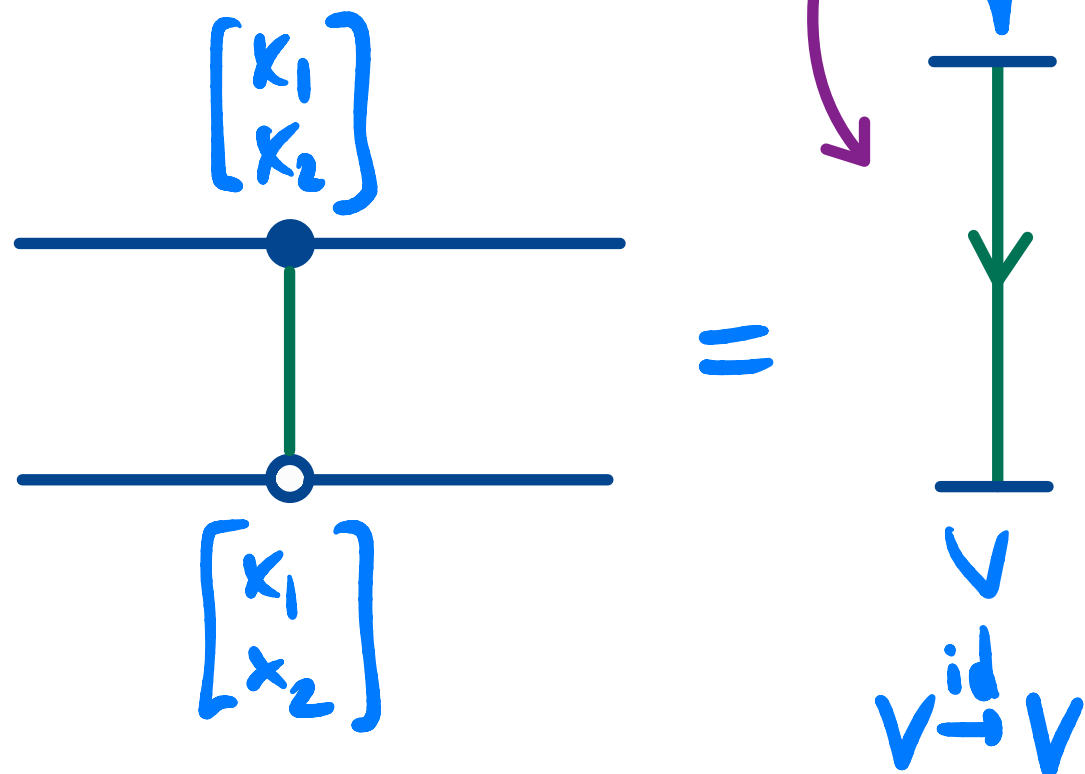


- with degree 2 internal vertices,
- and degree 1 boundary vertices.

Fact | Fully encodes category of $SL(2)$ -modules

SL_2 -Webs

Ex Let $V = \mathbb{C}^2$. (alternate notation)



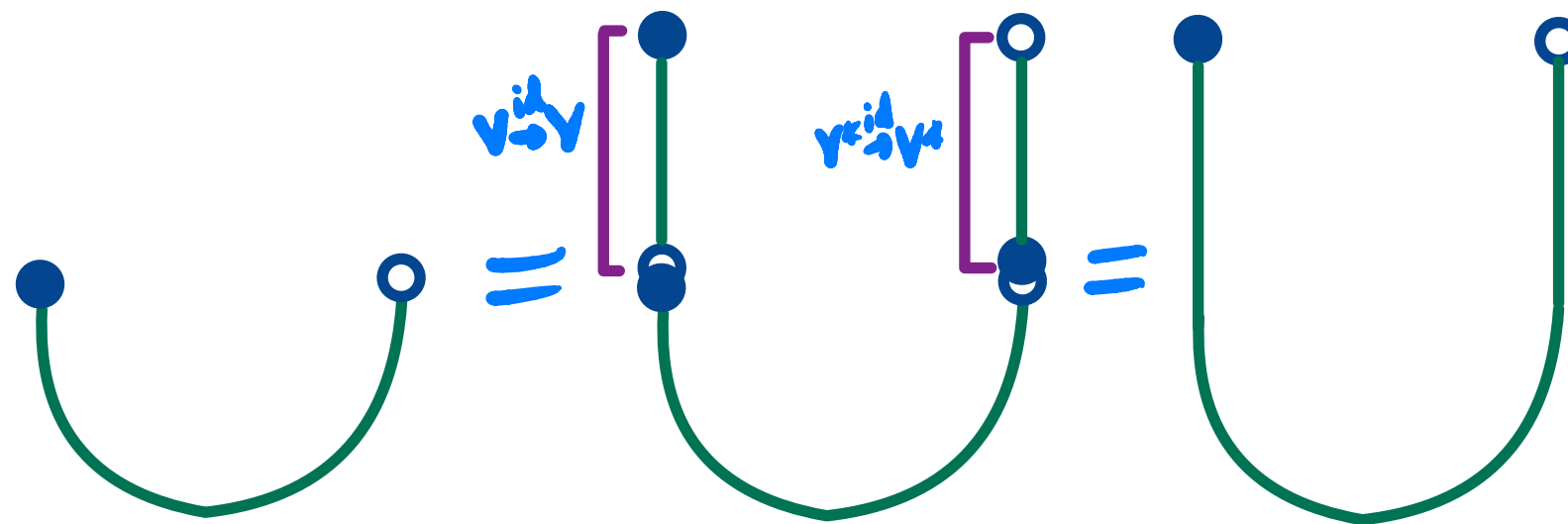
SL_2 -Webs

• Bipartite composition conventions

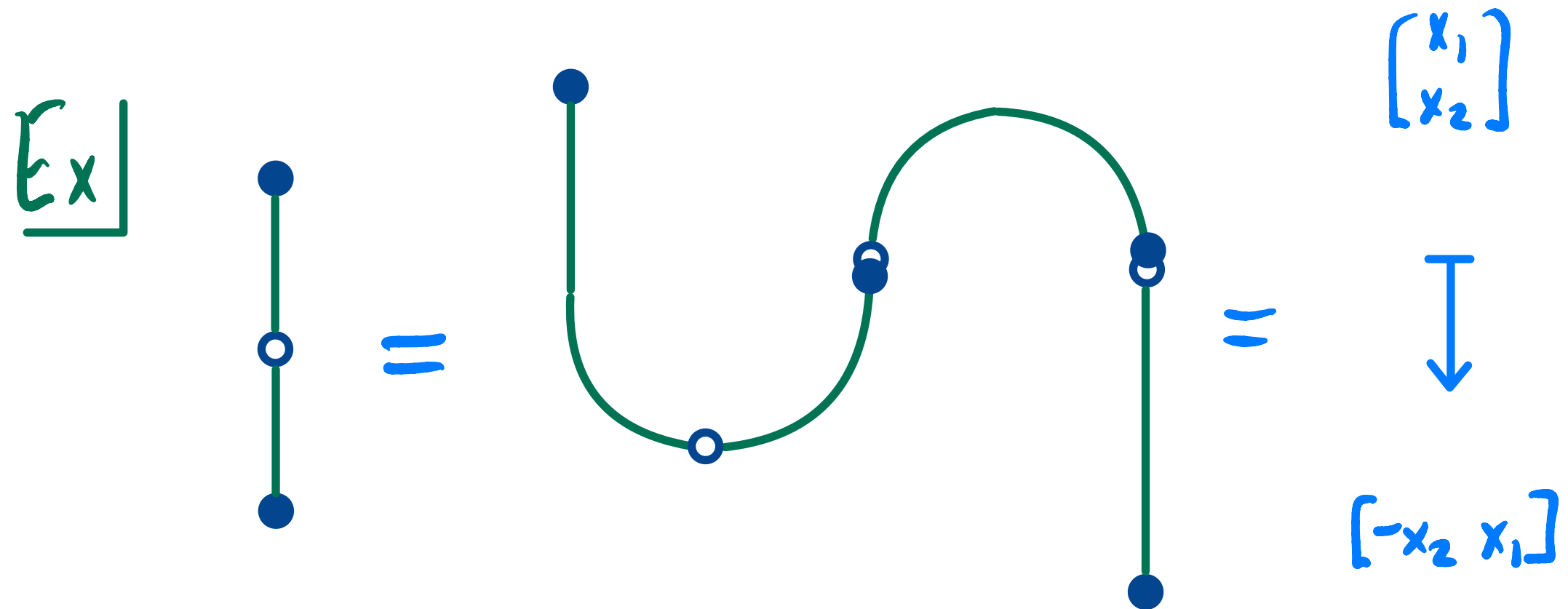
$$\begin{array}{ll} \bullet = V & \circ = V^* \quad \text{in domain} \\ \circ = V & \bullet = V^* \quad \text{in codomain} \end{array}$$

— When composing, must match and cancel $\bullet$ pairs:

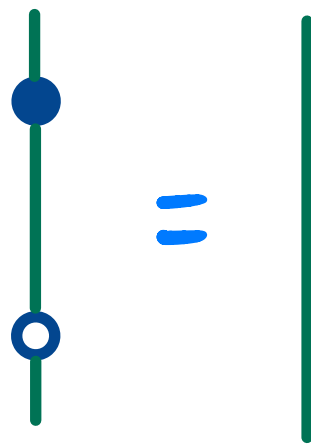
Ex



SL_2 -Webs



Ex Have contraction relation:



SL_2 -Webs

Facts | 1] Have well-defined map

$$\{SL(2) \text{ webs}\} / \text{isotopy} \longrightarrow \left\{ \begin{array}{c} SL(2) \text{ morphisms} \\ v^* \otimes v^* \otimes \dots \rightarrow v^* \otimes \dots \end{array} \right\}$$

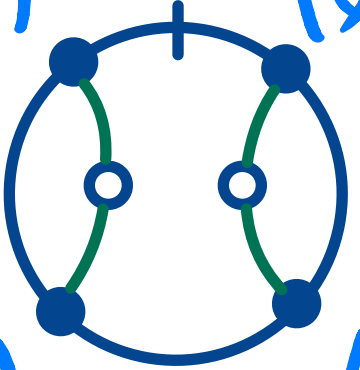
2] All of $\text{Rep}(SL_2)$ is encoded in disk case

(... surjective up to linear combinations, take Karoubi envelope...)

Q | Generators and relations? Bases??

SL_2 -Webs

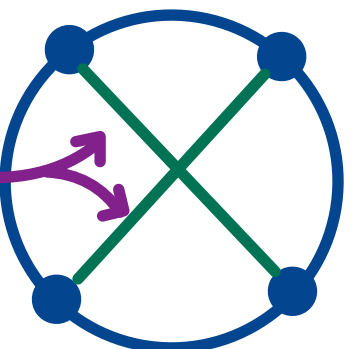
Ex

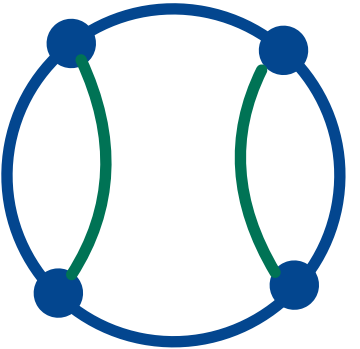


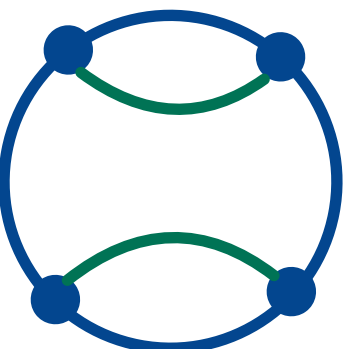
$$= \det \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \cdot \det \begin{pmatrix} x_{13} & x_{14} \\ x_{23} & x_{24} \end{pmatrix} \in \text{Inv}(V^{\otimes 4})$$

$$= \text{Hom}_{SL_2}(V^{\otimes 4}, \mathbb{C})$$

Ex Plücker relations:



$$=$$


$$+$$


$$(x_{11}x_{23} - x_{21}x_{13})(x_{12}x_{24} - x_{12}x_{14})$$

$$=$$

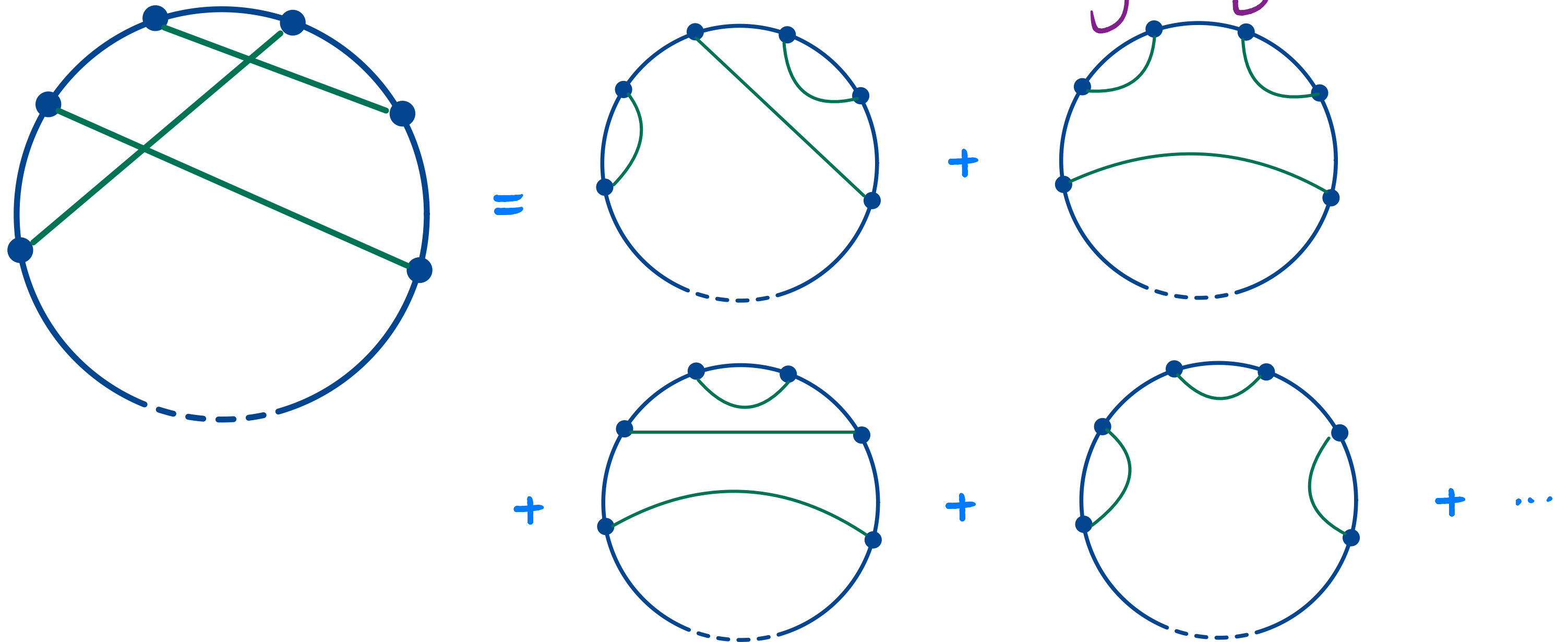
$$(x_{11}x_{22} - x_{21}x_{12})(x_{13}x_{24} - x_{23}x_{14})$$

$$+$$

$$(x_{11}x_{24} - x_{21}x_{14})(x_{12}x_{23} - x_{22}x_{13})$$

Temperley-Lieb basis

- Using $\text{diagram with crossing} = \text{diagram with two arcs} + \text{diagram with two arcs}$, can reduce any matching diagram to a linear combination of matching diagrams:



Temperley-Lieb basis

Thm The noncrossing 2-row webs are a basis for
 $\text{Inv}_{SL_2}(V_1 \otimes \dots \otimes V_n)$ $(V_i \in \{V, V^*\})$

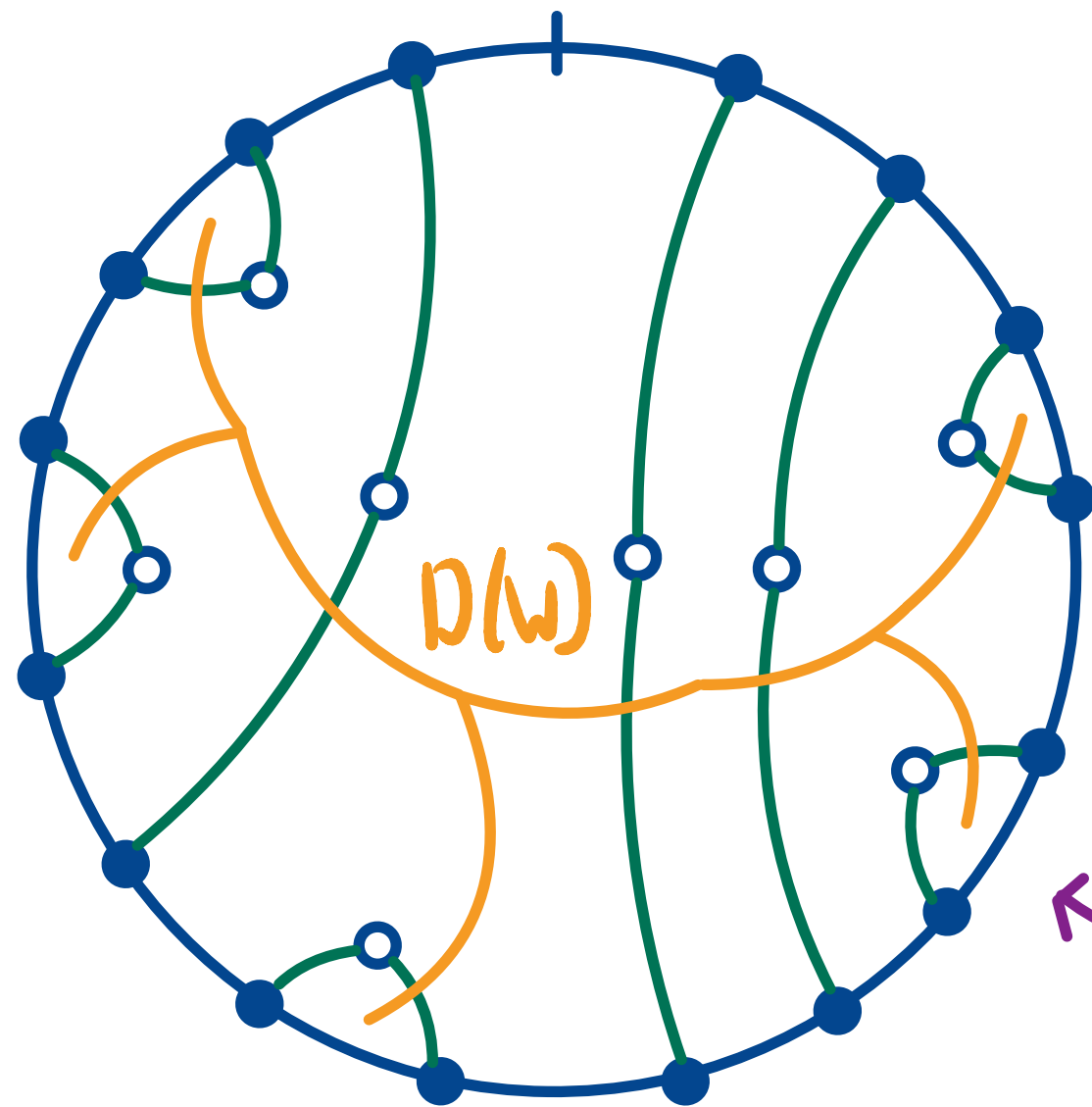
called the Temperley-Lieb basis.

Cor Effective equality testing in $SL(2)$ -mod!

- 1 Express $\alpha, \beta: X \rightarrow Y$ as $SL(2)$ -webs.
- 2 Expand in TL-basis by Plücker relations.
- 3 Compare coefficients.

Duals and trees

Obs | The dual graph $D(W)$ of an $SL(2)$ basis web W is a tree:



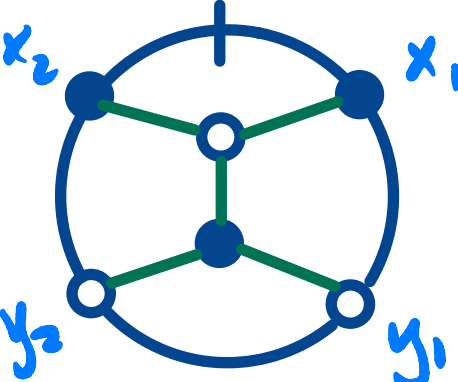
e.g. #faces of W
= #vertices of $D(W)$

$SL(2)$ basis
web

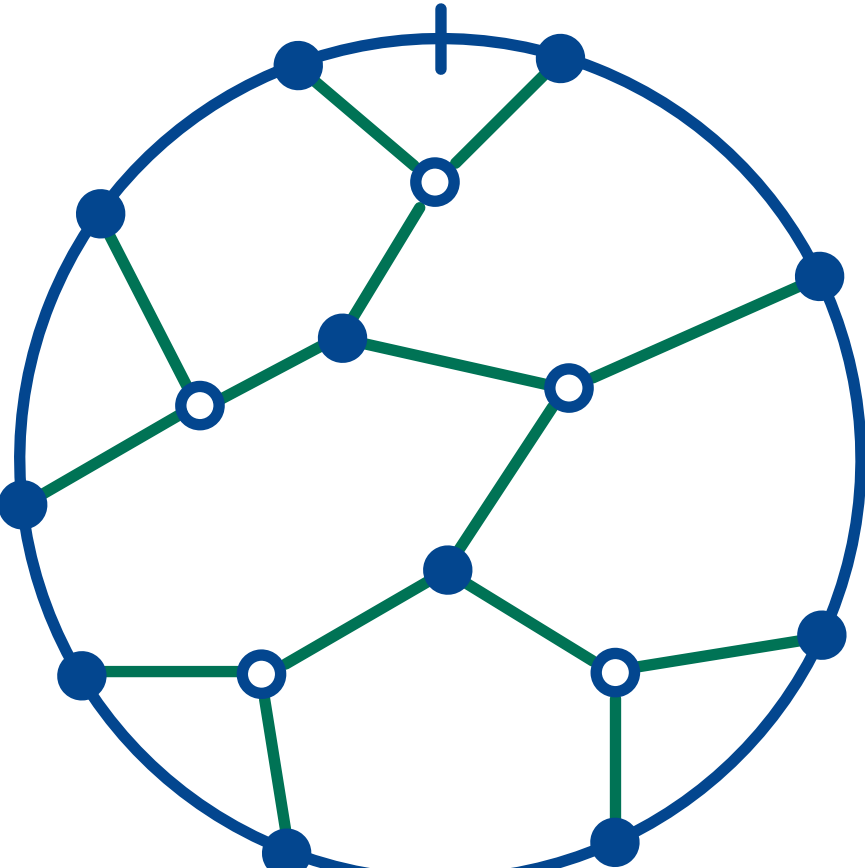
SL_3 -webs

Q General $SL(n)$ calculus? SL_3 ? (See [CKM '14])

Ex


$$= \det \begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & y_1 \wedge y_2 \end{pmatrix}$$

Ex


$$= \dots \text{a polynomial obtained by} \\ \text{summing over proper edge} \\ \text{3-colorings...}$$

SL_3 -webs

Thm (Kuperberg '94) The generating SL_3 -web relations are

$$\bigcirc = 3$$

$$\text{---} \bigcirc \text{---} = 2 \cdot \text{---}$$

$$\begin{array}{c} \diagup \quad \diagdown \\ \bigcirc \quad \bigcirc \\ \diagdown \quad \diagup \end{array} = \begin{array}{c} \diagup \quad \diagdown \\ \quad \quad \quad \\ \diagdown \quad \diagup \end{array} + \begin{array}{c} \diagup \quad \diagdown \\ \quad \quad \quad \\ \diagdown \quad \diagup \end{array}$$

Non-elliptic web basis

Thm (Kuperberg '94)

Call an SL_3 -web nonelliptic if it has
no 2-faces or 4-faces.

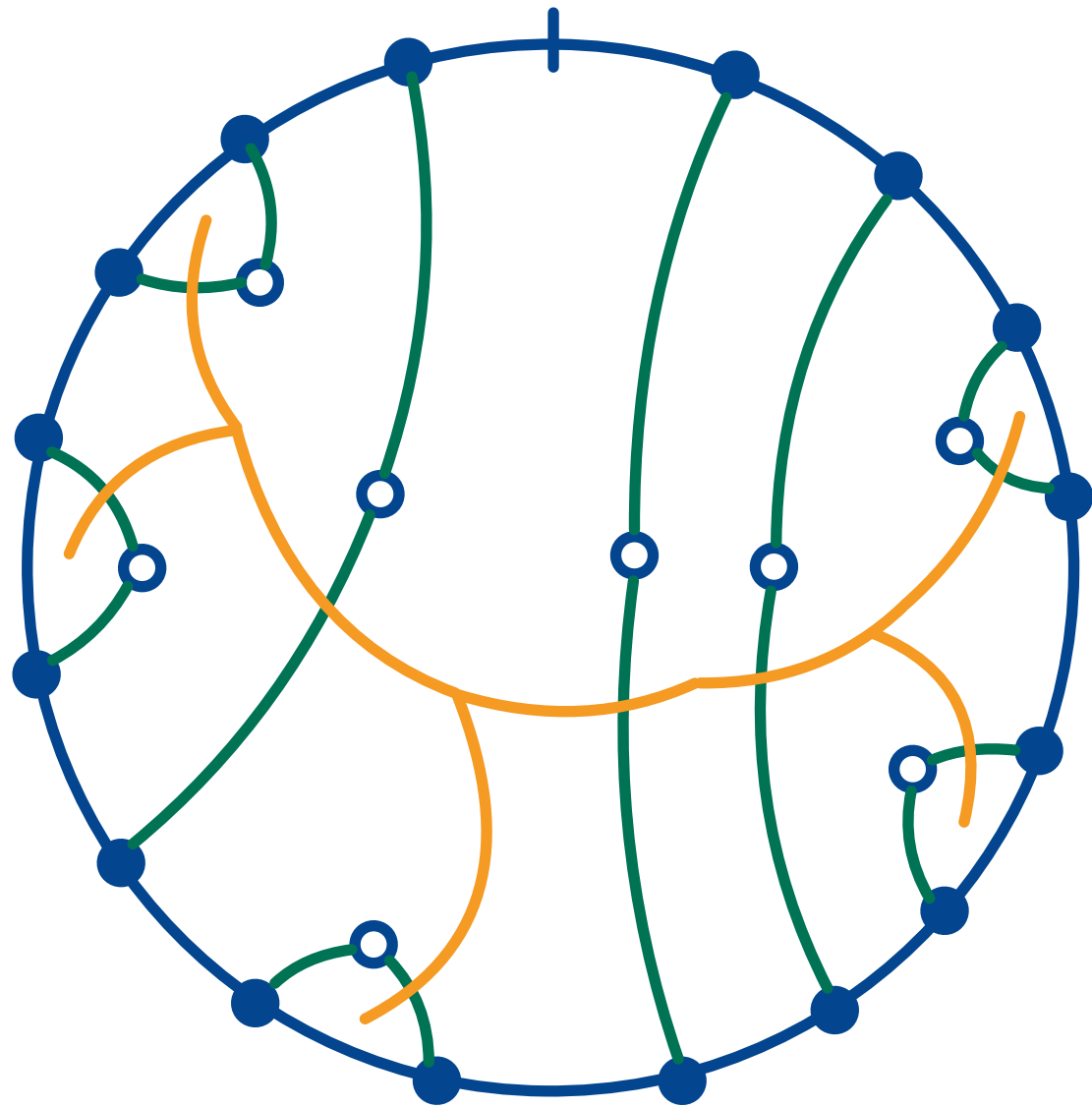
The nonelliptic webs form a basis of

$$\text{Inv}_{SL_3}(V_1 \otimes \dots \otimes V_n). \quad (v_i \in \{V, V^*\})$$

Cor Effective equality testing in $SL(3)$ -mod!

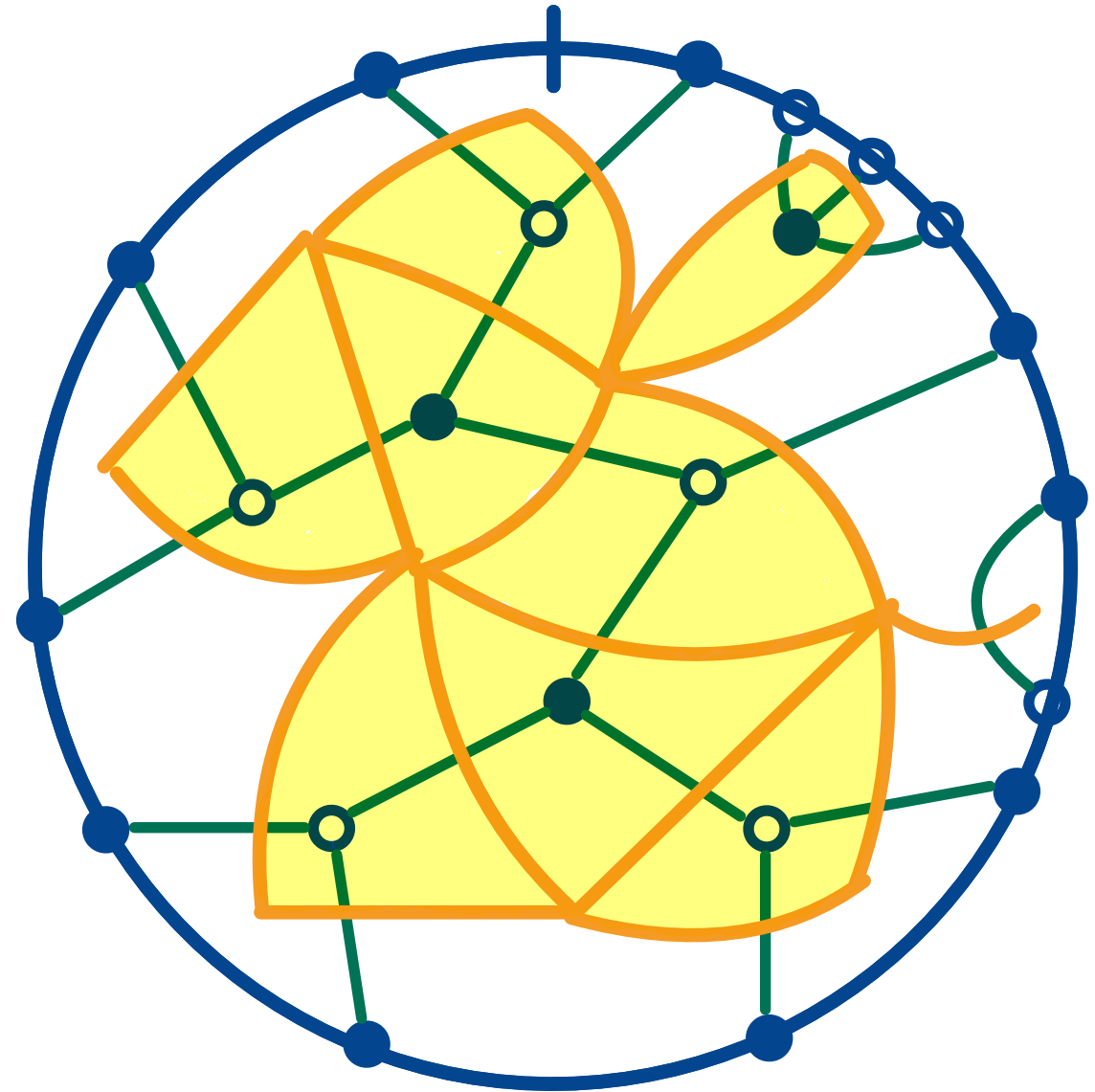
SL_3 -web duals

Obs | $D(W)$ for $SL(3)$ is a triangulation:



$D(W)$ is 10 for $SL(2)$

vs.



$D(W)$ is "20" for $SL(3)$

Hourglass plabic graphs

Thm | ([Gaetz-Pechenik-Pfannerer-Striker-S. '25])

There is a (rotation-invariant) $SL(4)$ web basis consisting of top fully reduced, 4-row hourglass plabic graphs.

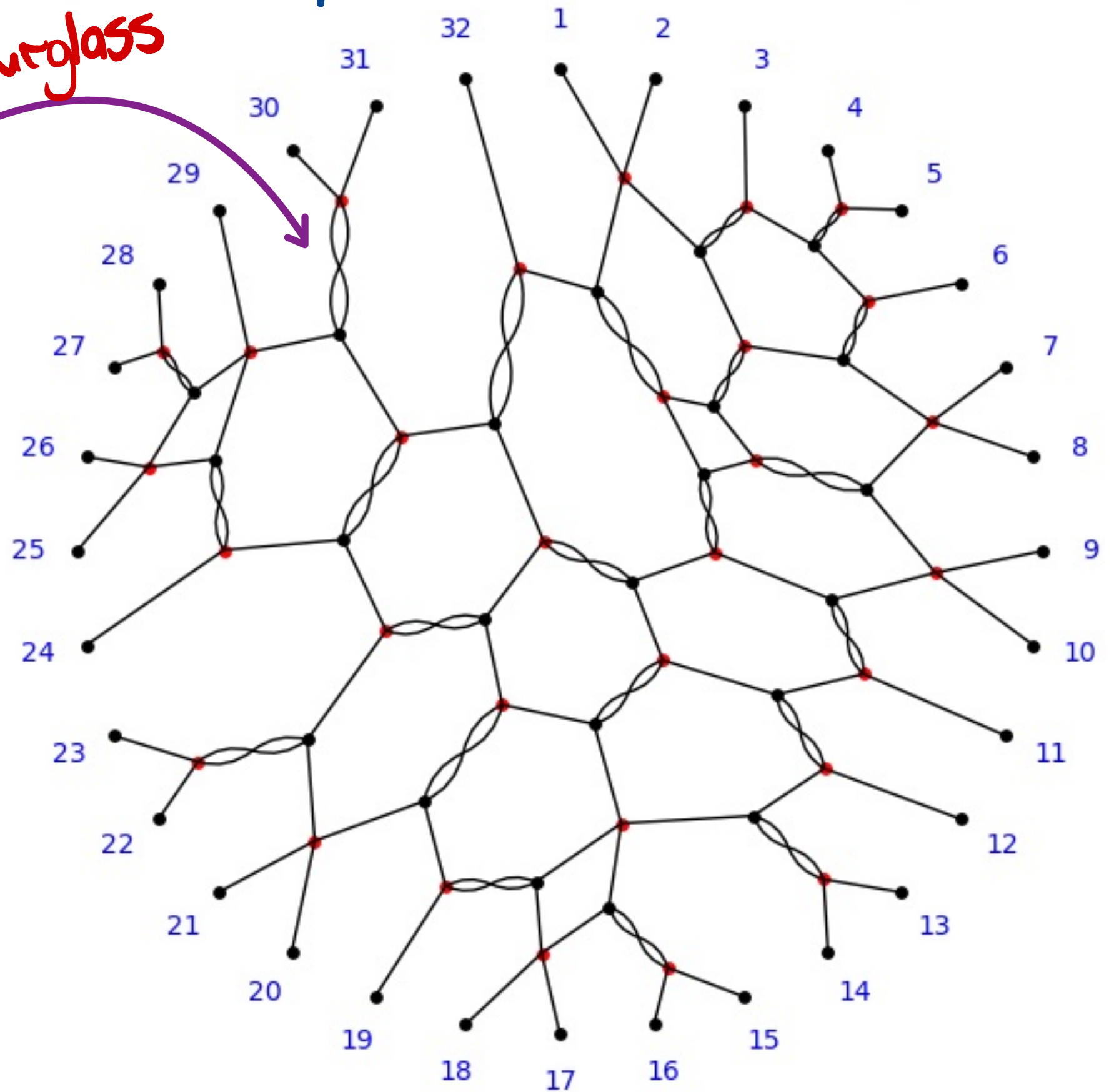
Cor | Effective equality testing in $SL(4)$ -mod!

HPG basis example

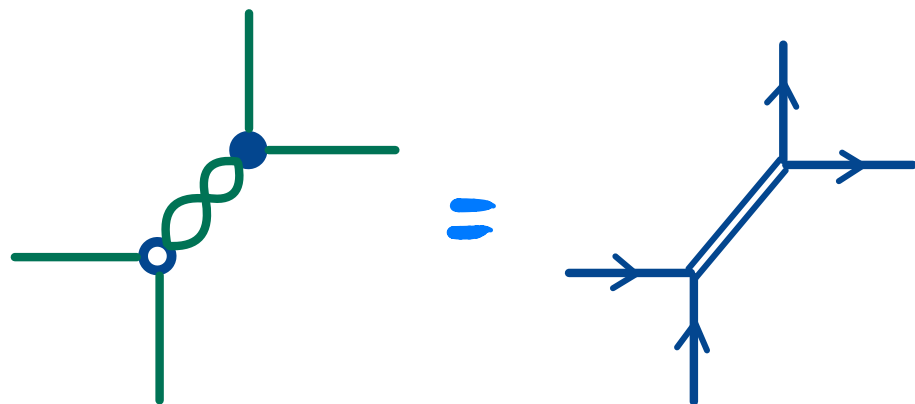
Ex

1	3	4	7	8	17	19	23
2	5	6	9	14	18	21	24
10	12	13	15	16	25	26	28
11	20	22	27	29	30	31	32

a Z-hourglass



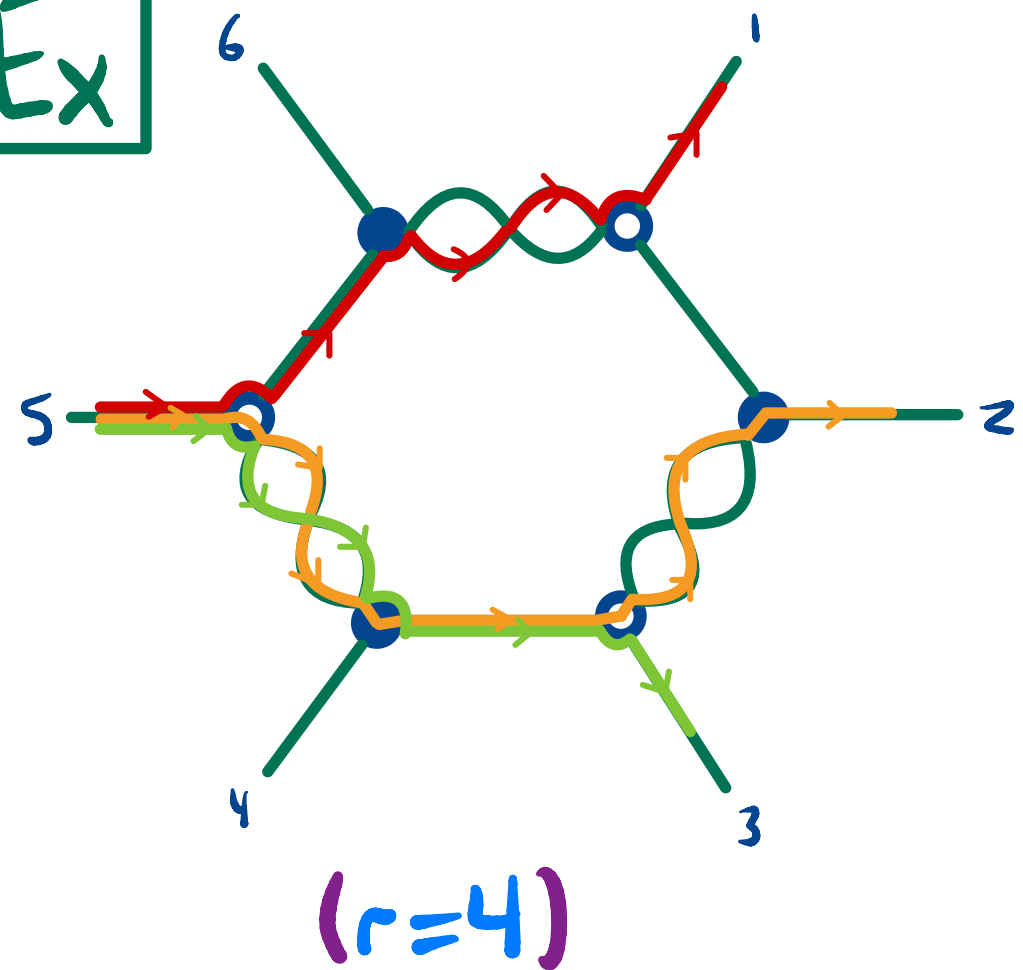
Note



Trip permutations

Def (GPPSS '25) An r -hourglass planar graph has trip permutations $\text{trip}_1, \dots, \text{trip}_{r-1}$ where trip_i takes the i th left at white and i th right at black:

Ex



$\rightarrow$ = $\text{trip}_1 = (135)(642)$
 $\rightarrow$ = $\text{trip}_2 = (14)(25)(36)$
 $\rightarrow$ = $\text{trip}_3 = (531)(246)$

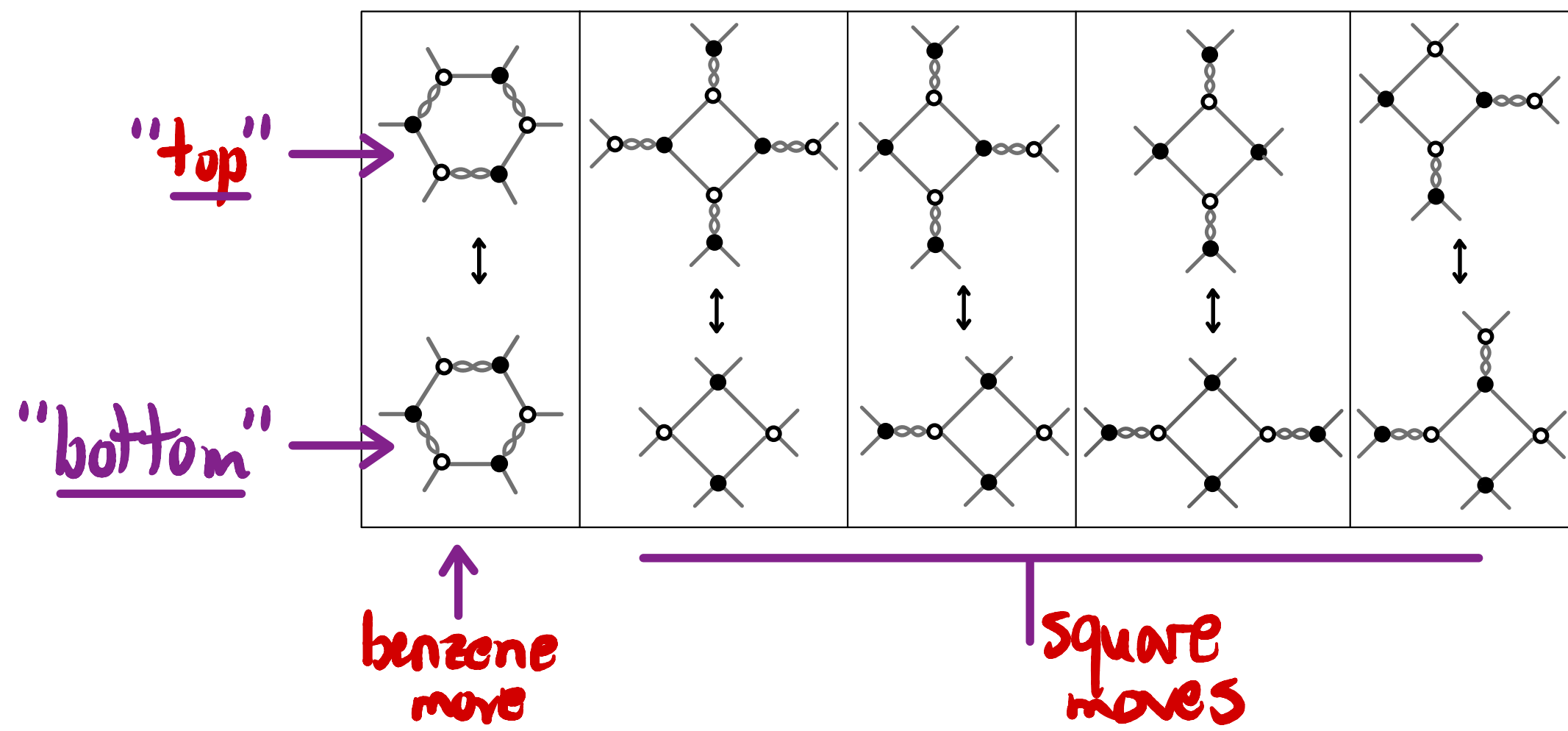
Note

$\text{trip}_i = \text{trip}_{r-i}^{-1}$!

$r=4$ moves

Thm (GPPSS '75) Two contracted, fully reduced
4-HPG's have the same trip permutations

$\Leftrightarrow$ they are related by moves:

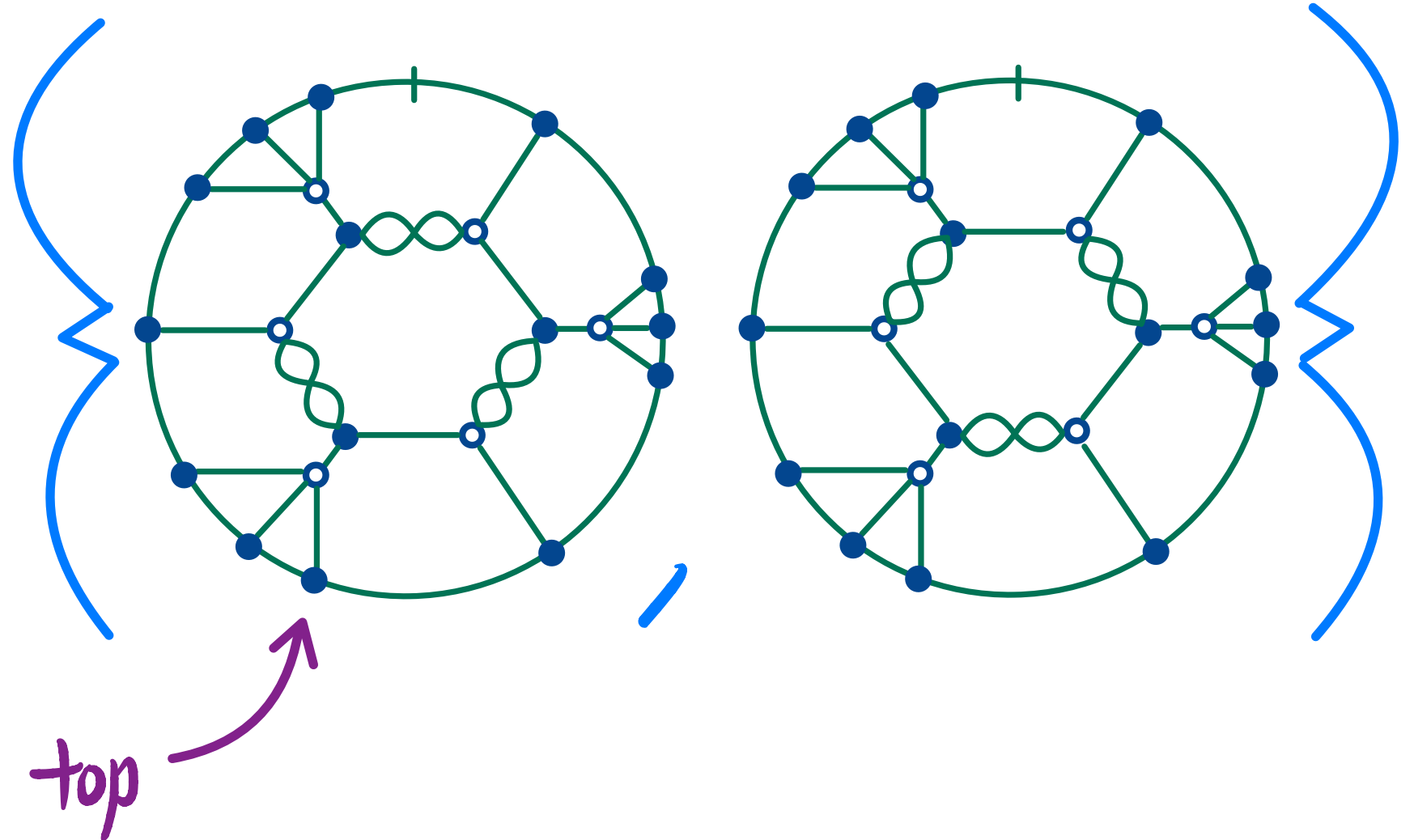


$r=4$ moves

Ex

$T =$

1	2	6
3	5	10
4	7	11
8	9	12



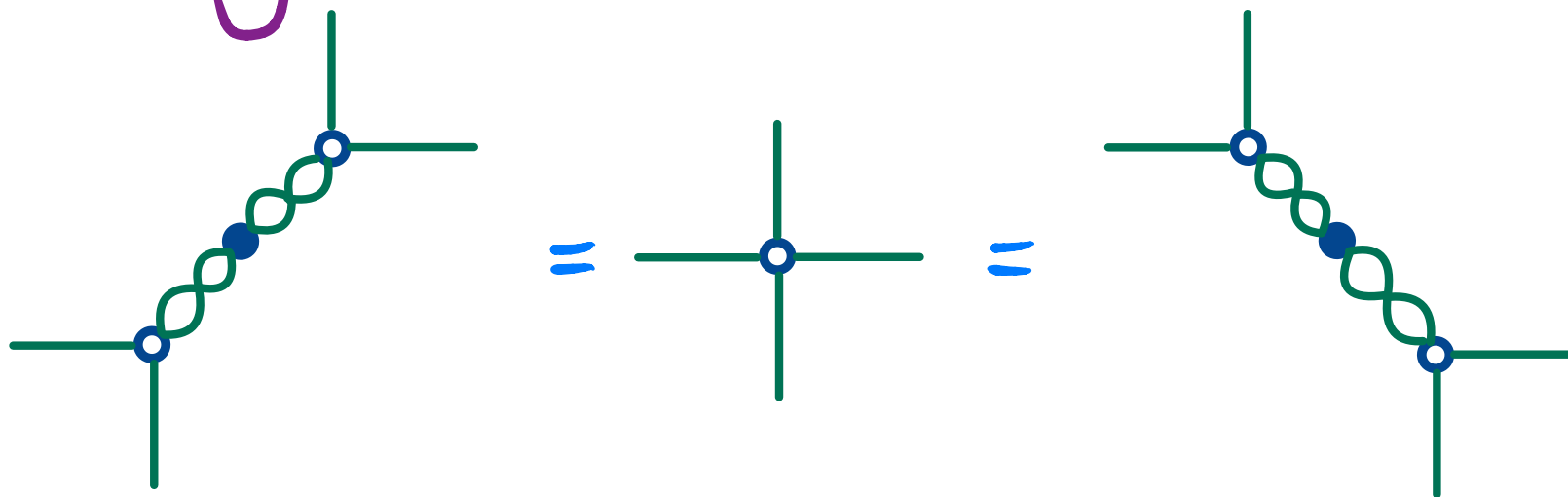
- Q
- Why moves for $r=4$ but not $r \leq 3$?
 - How to understand "top" step?

Pockets

Def (Gaetz-Striker-S.-Wu)

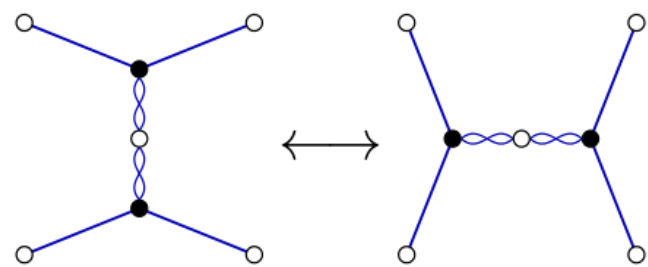
Let $\mathcal{C}$ be the move-class of some fully reduced 4-HPG W . Then there is a 3D flag simplicial complex $P(\mathcal{C})$ called the pocket of $\mathcal{C}$, which $D(W)$ embeds into.

- Requires expanding sources/sinks via $I=H$ moves:

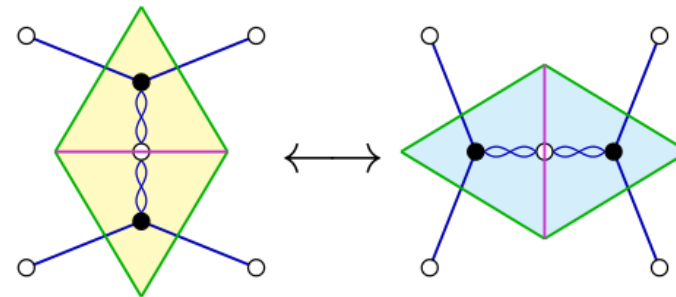


Pockets

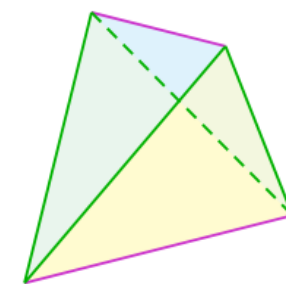
- Build a pocket from a move-class by gluing $D(w)$'s with tetrahedra between moves:



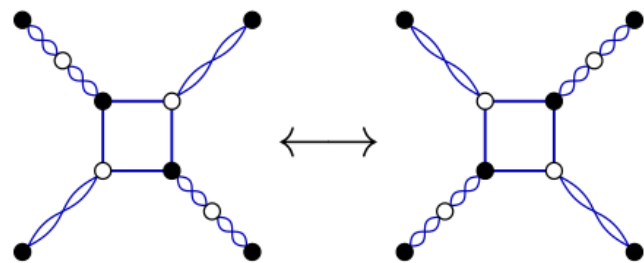
IH move between webs



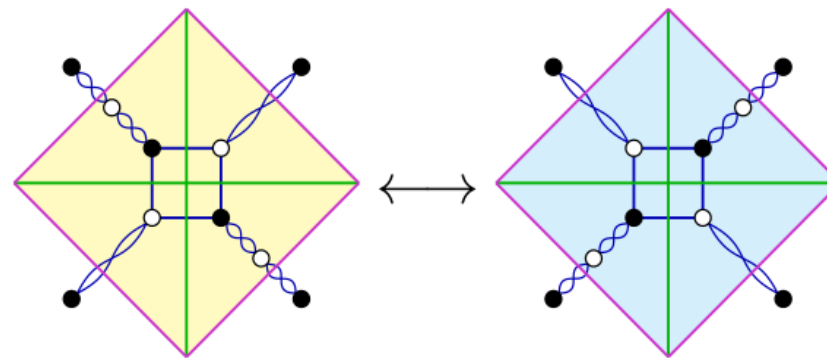
Dual diskoids



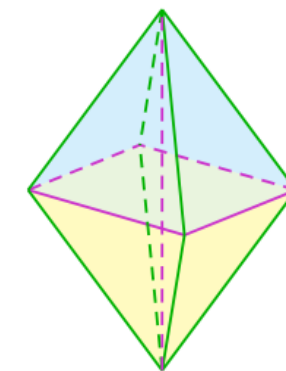
tetrahedron



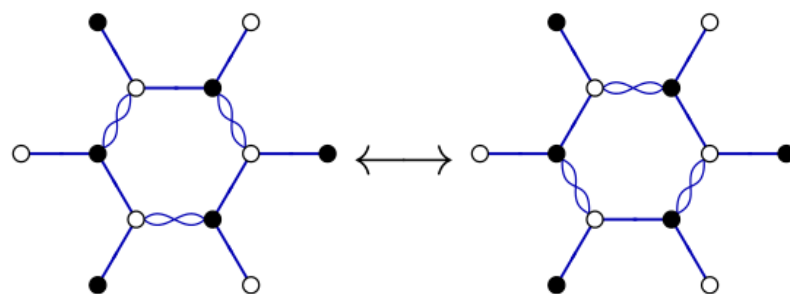
Square move between webs



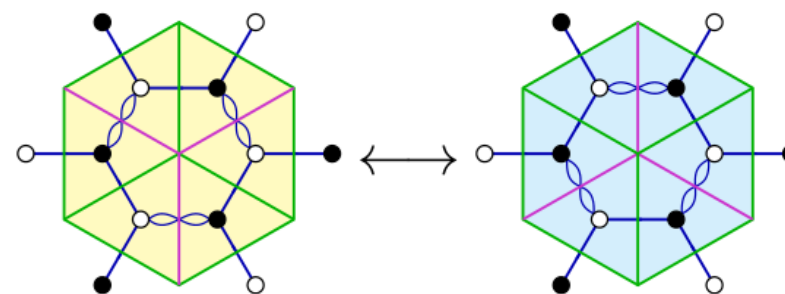
Dual diskoids



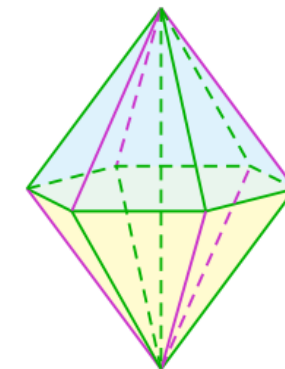
octahedron



Benzene move between webs



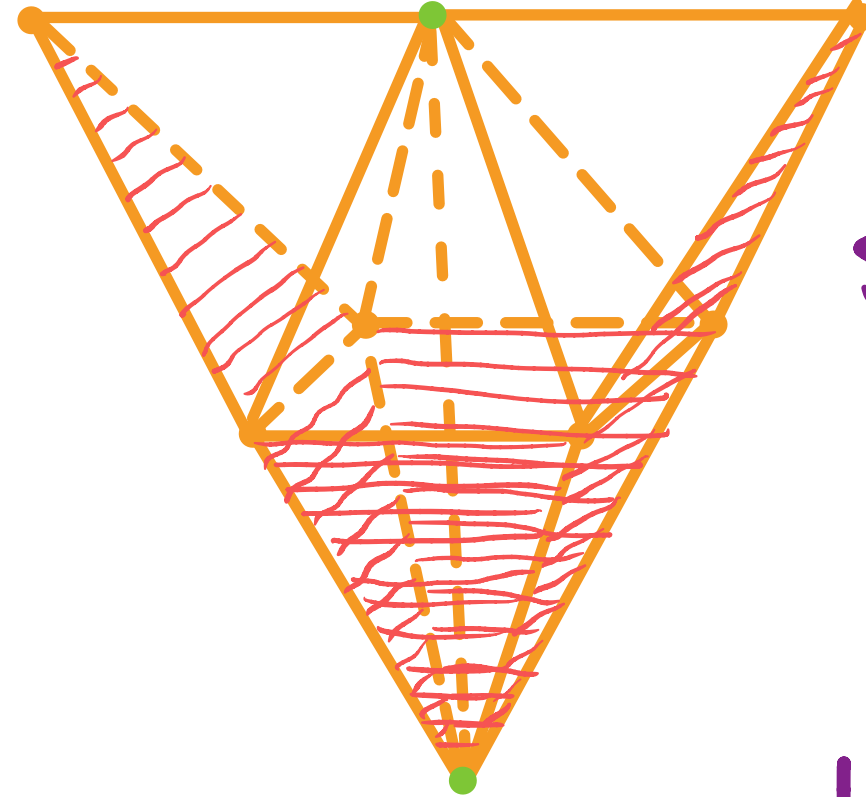
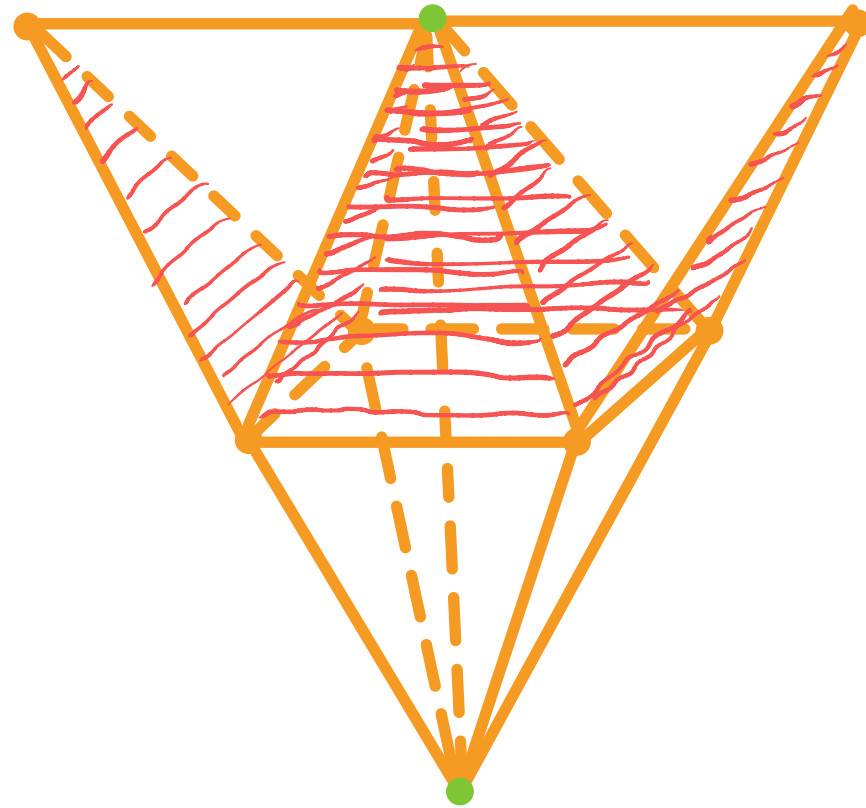
Dual diskoids



dodecahedron

Pockets

Ex



Pocket of
square + benzene + IH
class of

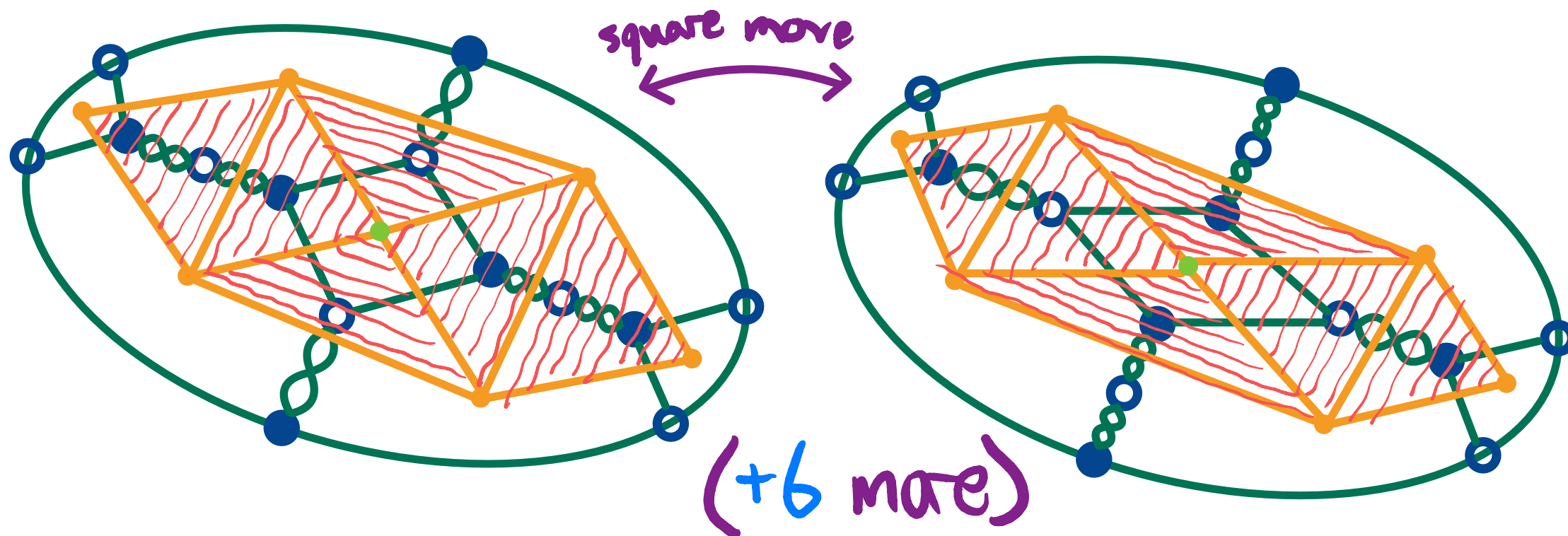
$\overline{4}\{1,2\}\overline{3}\overline{2}\{3,4\}\overline{1}$

has 6 tetrahedra

14 triangles

19 edges

8 vertices



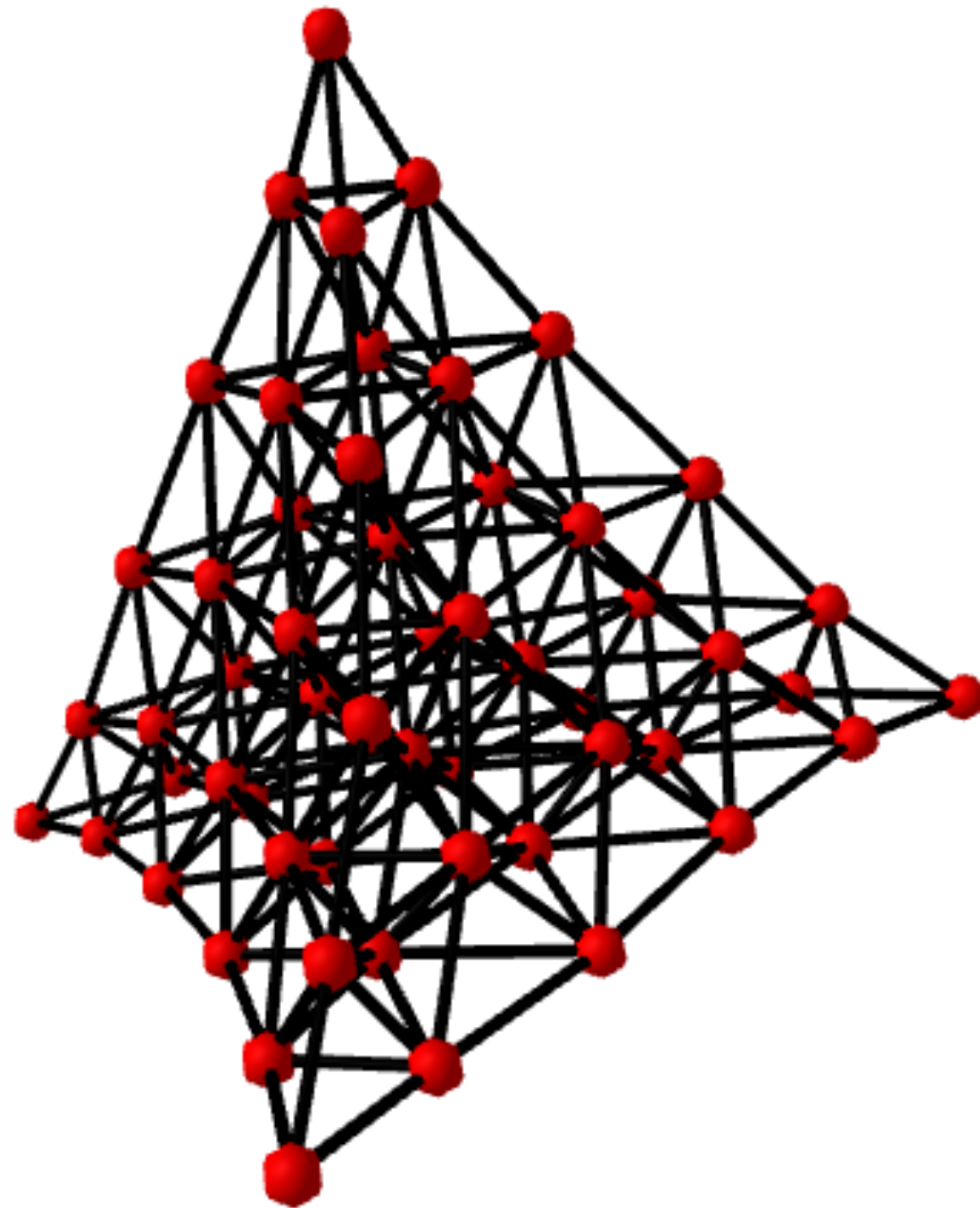
*

Podnets

Ex Product of 5×5 ASM:

$$L = 1^5 2^5 3^5 4^5$$

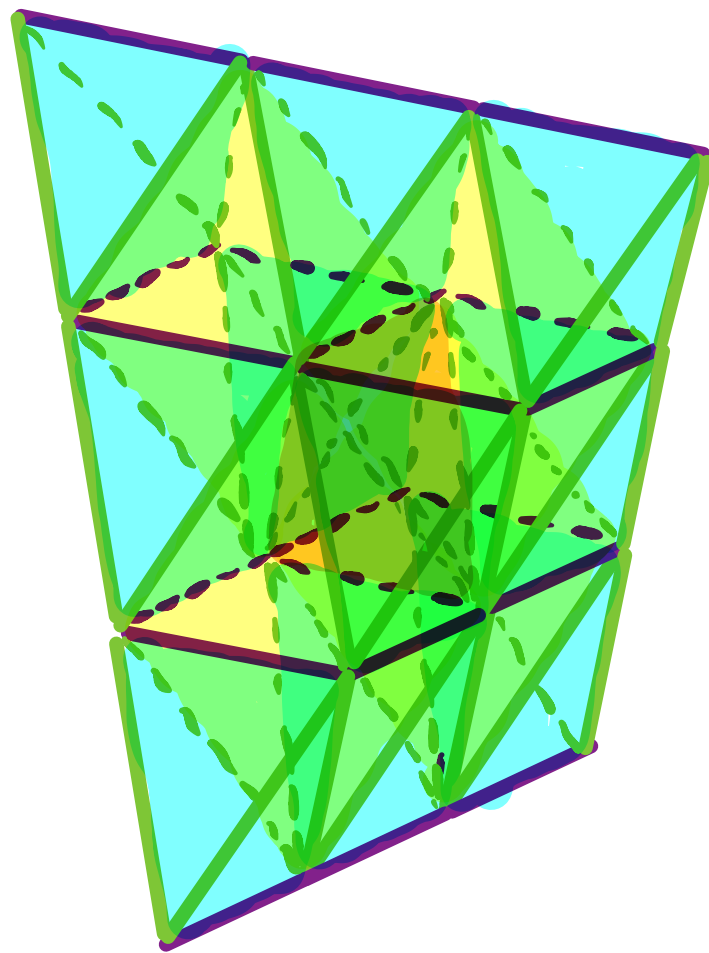
$$\left(T = \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 & 20 \end{array} \right)$$



Note Related to
height functions,
octahedral
recurrence,
tilings of the
Aztec diamond,
distributive lattices

Podnets

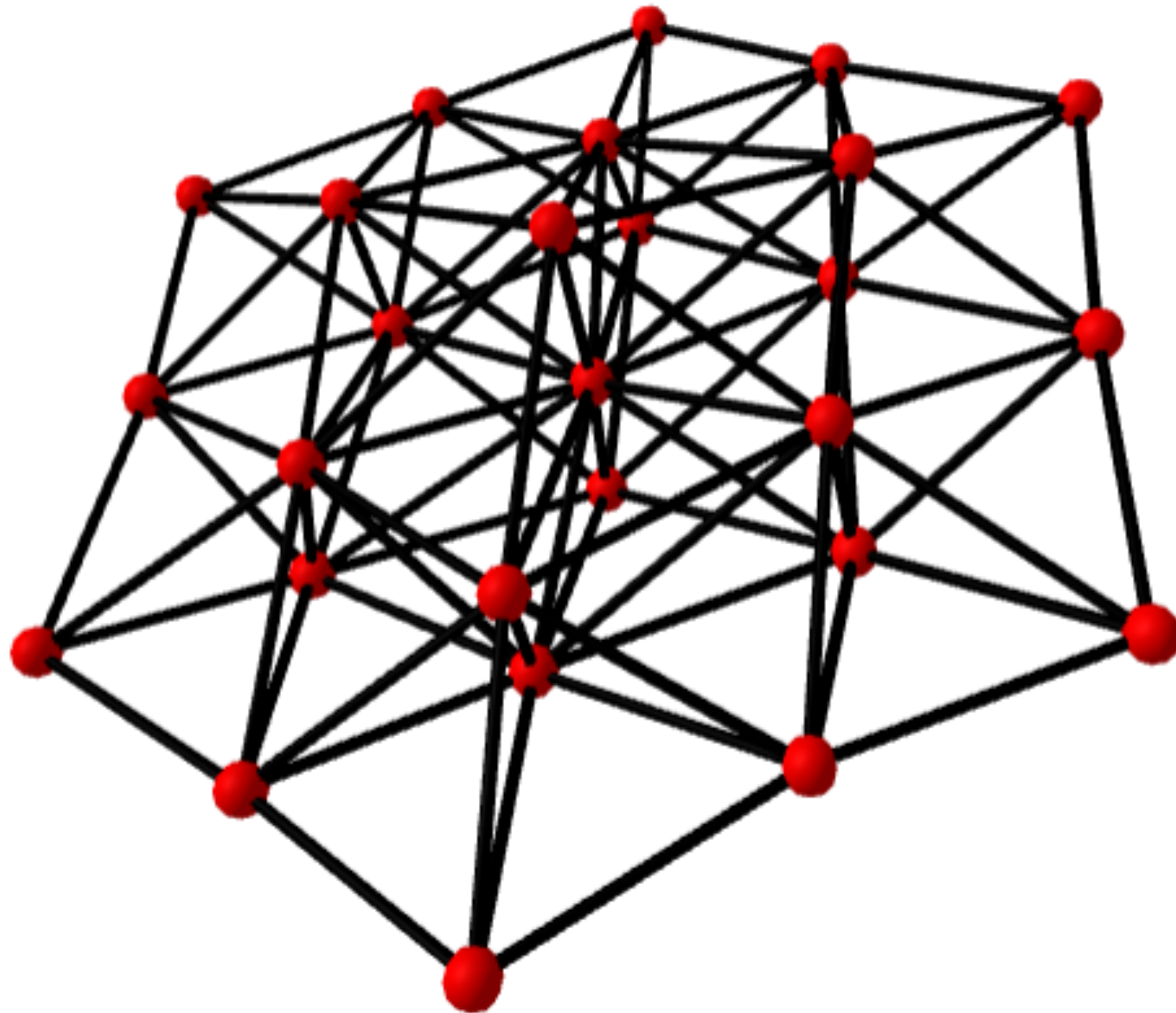
Ex Product of 3×3 ASM: tetrahedral-octahedral
honeycomb



(vertical edges)
omitted

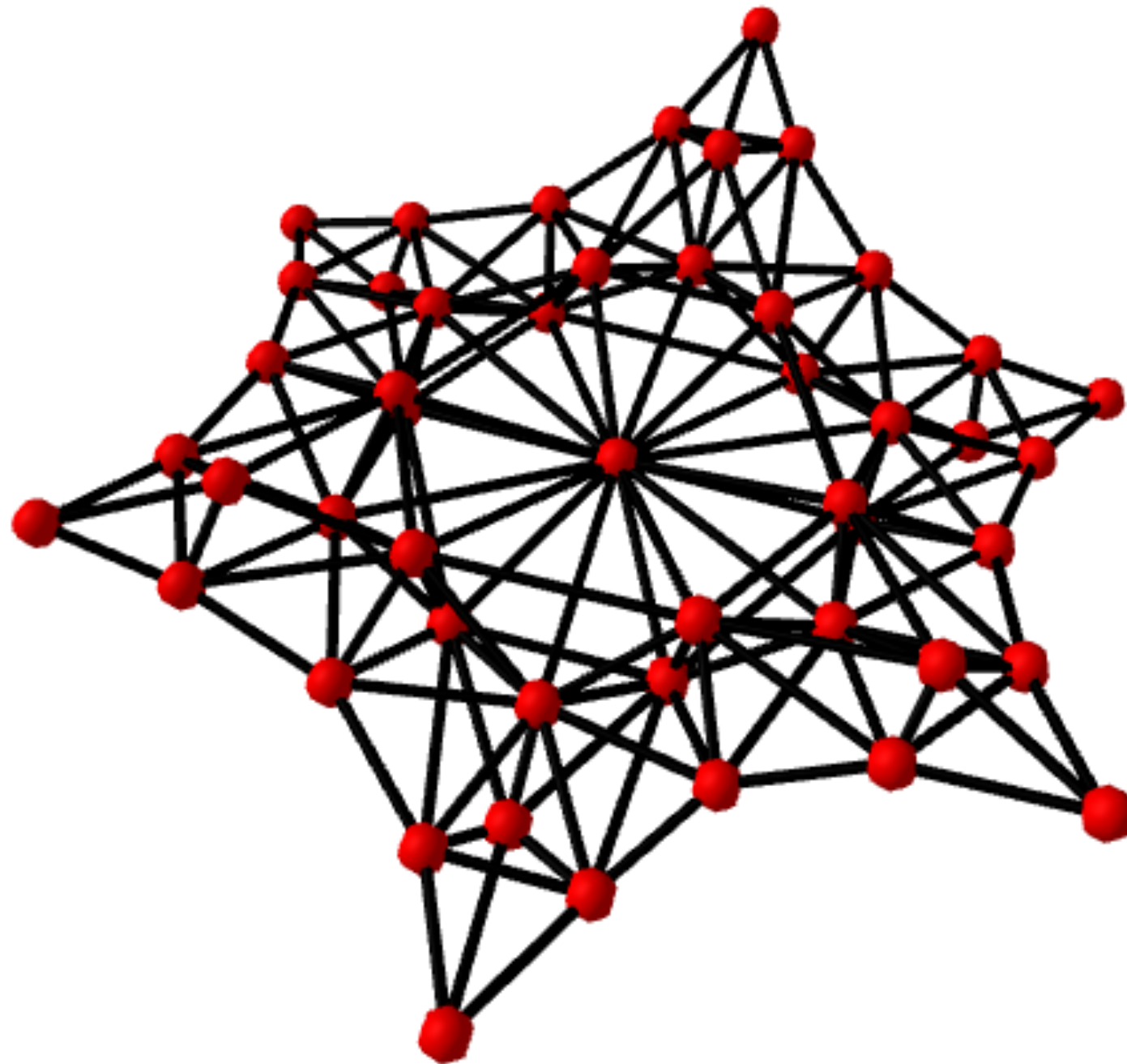
Products

Ex Product of $2 \times 2 \times 2$ PP:



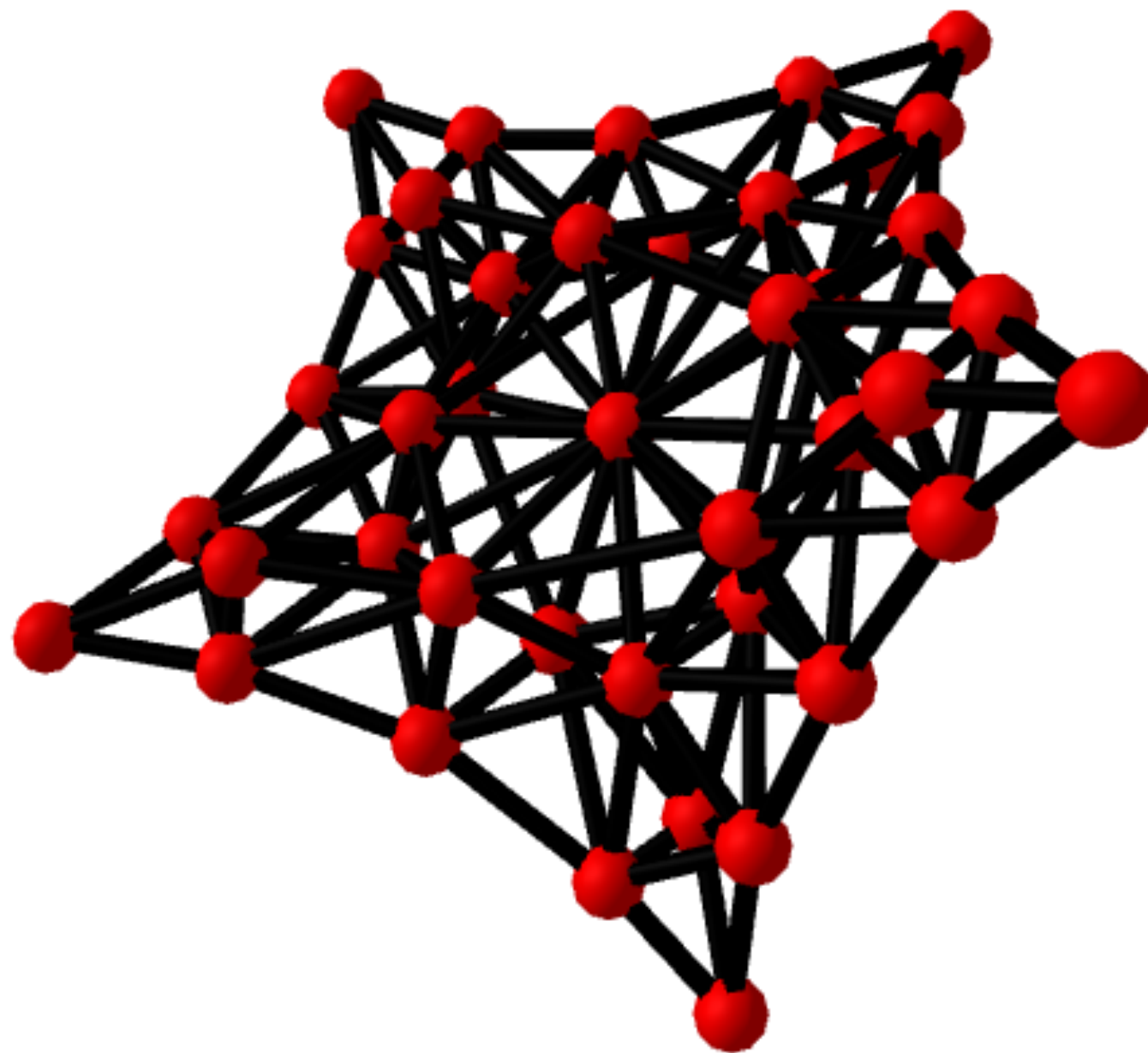
Podsets

Ex Product of a "chained hexagon":



Podsets

Ex | Product of a "chained pentagon":



Note | Not embeddable
in $\mathbb{R}^3$

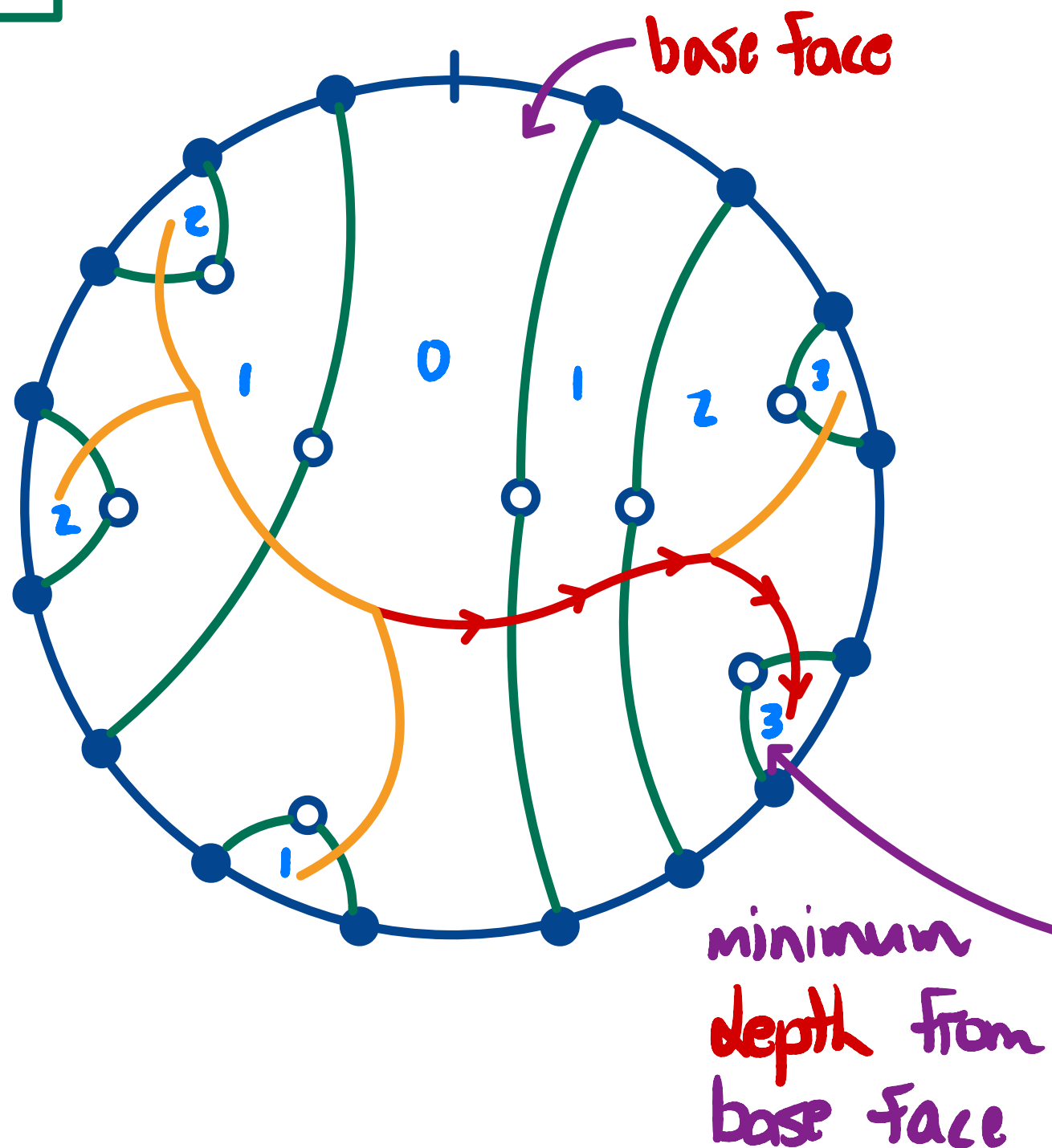
Pockets

Next | Pockets are highly structured!

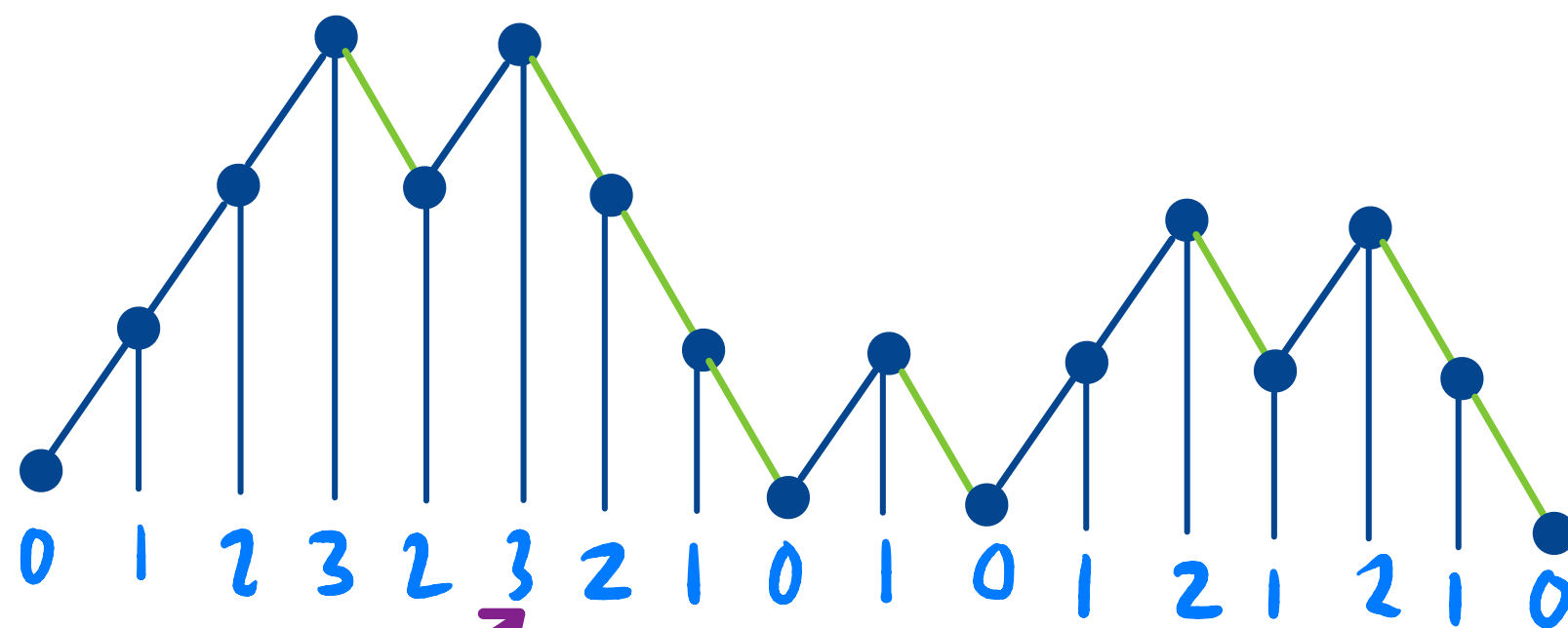
- Thm | Simply-connected, contractible, $CAT(0)$
- Distributive lattice on $\mathcal{C}$ (when embeddable in $\mathbb{R}^3$)
- • Coherent $SL(4)$ geodesics!
- Embeds in affine building Δ_4 ! ("Finite models")
- Same dihedral orbit structure as $SVT(4 \times c)$
under promotion + evacuation

Distances

E_x



1	2	3	5	9	11	12	14
4	6	7	8	10	13	15	16



- Hence depths around the boundary encode the tableau

Distances

Recall | The $SL(r)$ dominant weights are Λ_+ :

$$\lambda = (\lambda_1 \geq \dots \geq \lambda_r) \in \mathbb{Z}^r / \langle (1, \dots, 1) \rangle = \Lambda_r.$$

• The fundamental weights of $SL(r)$ are

$$\omega_i = (1^i, 0^{r-i})$$

$$\omega_i^* = (0^{r-i}, -1^i) \equiv \omega_{r-i}.$$

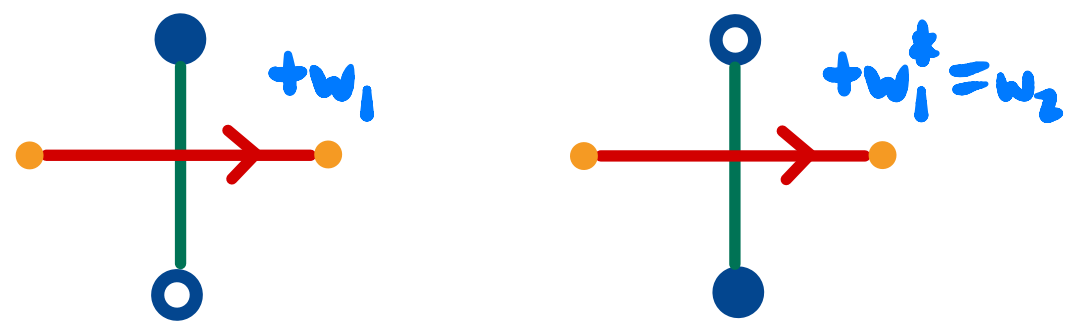
• If $\lambda, \mu \in \mathbb{Z}_{\text{dom}}^r$ where $|\lambda| = |\mu| + rk$ for some $k \in \mathbb{Z}$,

say

$$\lambda \geq \mu \Leftrightarrow \forall m, \sum_{j=1}^m \lambda_j \geq \sum_{j=1}^m (\mu_j + k).$$

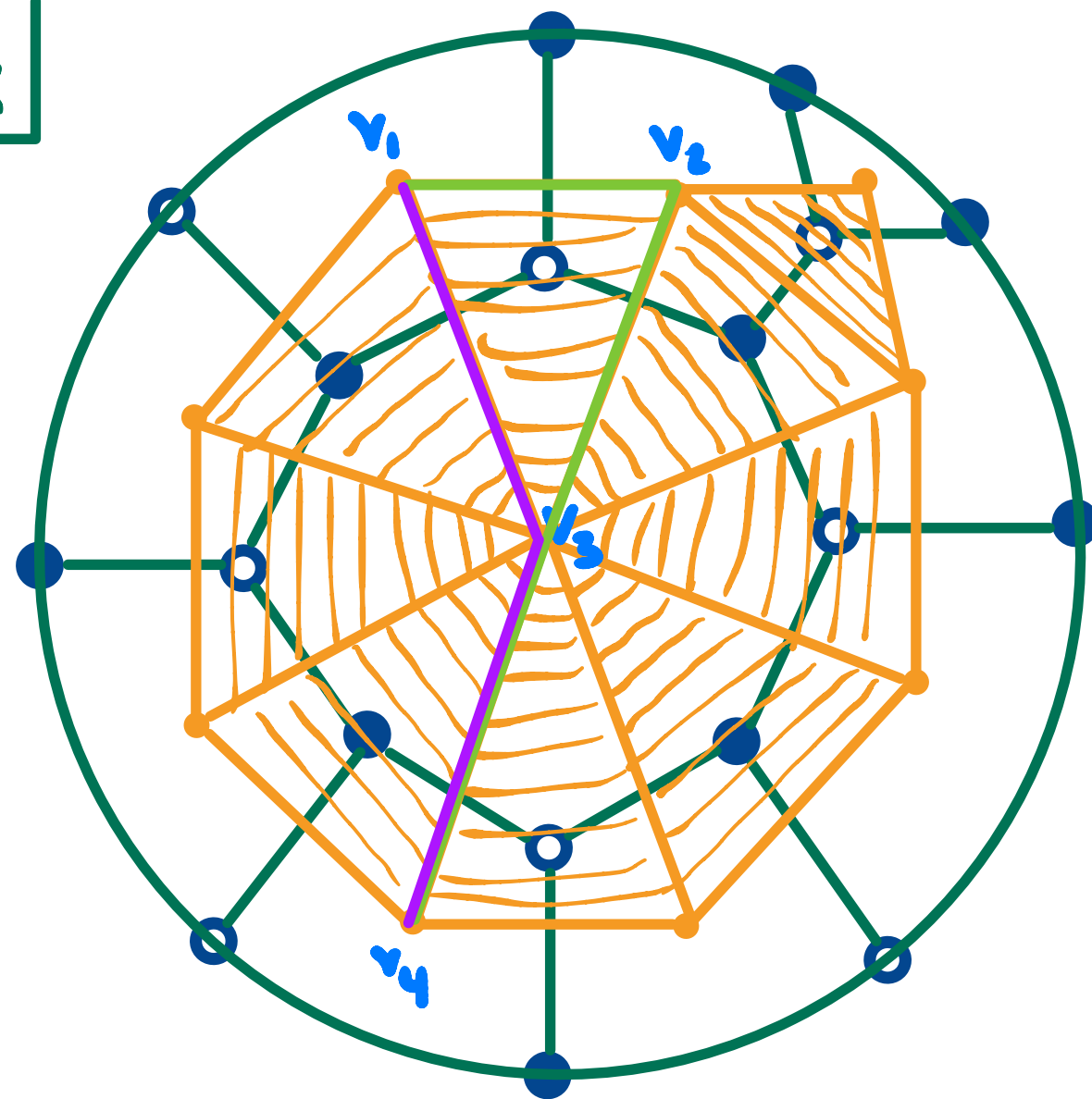
Distances

Idea for an $SL(3)$ web, walk along the dual graph, picking up $SL(3)$ weights according to:



Call this the **distance** of the walk.

Ex



$$d(v_1 \xrightarrow{w_1} v_2 \xrightarrow{w_1} v_3 \xrightarrow{w_2} v_4) = 2w_1 + w_2 = (3, 1, 0)$$

vs.

$$d(v_1 \xrightarrow{w_2} v_3 \xrightarrow{w_2} v_4) = 2w_2 = (2, 2, 0)$$

smaller!

Distances

Def (See [Kopovich-Leeb-Millson '08])

A KLM distance on a set X is $d: X \times X \rightarrow \mathbb{A}_r$ s.t.

1] $d(x,y) \in \mathbb{A}_+$ is dominant

2] $d(x,y) = 0 \Leftrightarrow x=y$

3] $d(y,x) = -\text{rev } d(x,y)$

• Contrast w/metric space axioms when $r=2$



Distances

• Let

$$\delta(a,b) = \left\{ \sum_{i=1}^k d(\gamma_{i-1}, \gamma_i) \mid \overbrace{a=\gamma_0, b=\gamma_k, d(\gamma_{i-1}, \gamma_i) \text{ miniscule}}^{\text{miniscule walk from } a \text{ to } b} \right\} \subset \mathbb{A}_+.$$

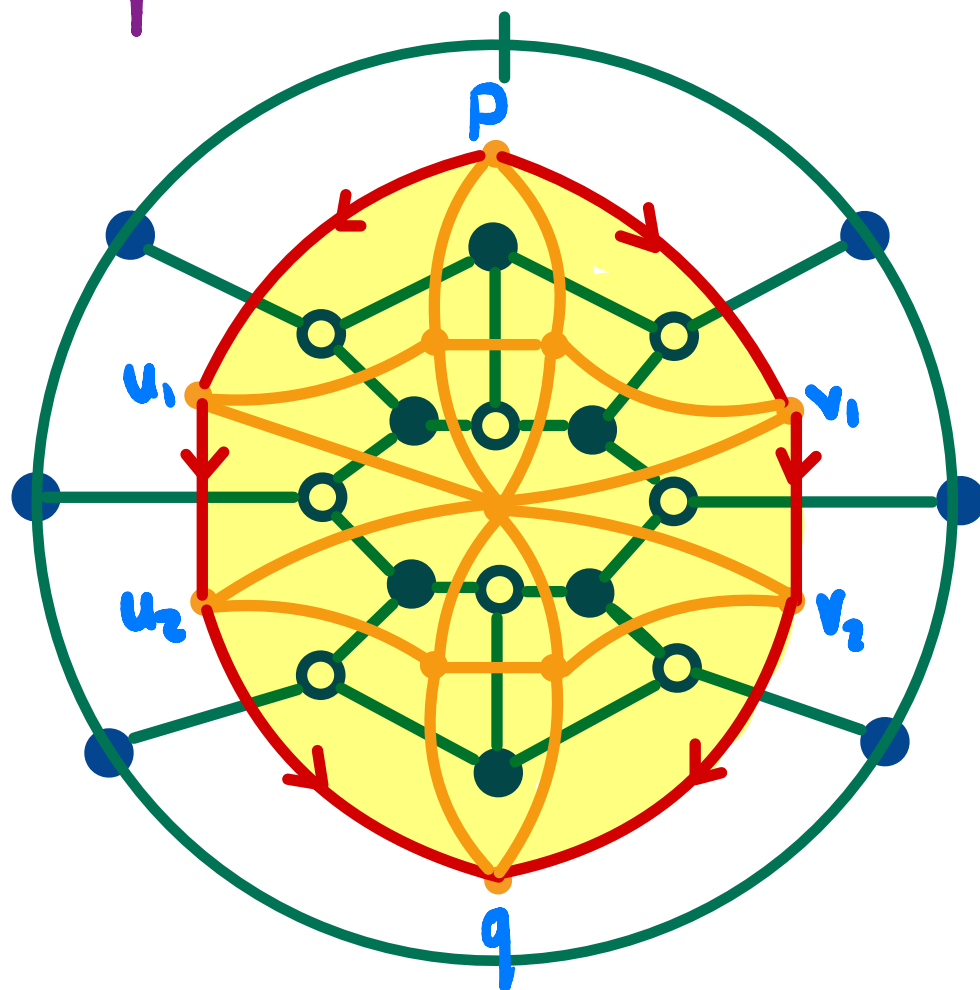
- A KLM geodesic is a path achieving a minima of $\delta(a,b)$
- X has coherent geodesics if $d(a,b)$ is always the unique minimum of $\delta(a,b)$.

Geodesics

Thm ([Kuperberg '96], [Fortaine-Kamnitzer-Kuperberg '13])

If W is a non-elliptic $SL(3)$ web, it has coherent geodesics.

Ex Non-example:



$$d(p \rightarrow u_1 \rightarrow u_2 \rightarrow q) = (3, 3, 0)$$

$$d(p \rightarrow v_1 \rightarrow v_2 \rightarrow q) = (3, 0, 0) \equiv (4, 1, 1)$$

incomparable!

(has squares)

Geodesics

Thm | (Gaetz - Striker - S. - Wu)

- Pockets have an $SL(4)$ KLM distance.
 - It has coherent geodesics.
 - The corresponding $T \in \mathcal{PT}(4 \times 1)$ is encoded in boundary distances to base face.
 - Geodesics avoid double-crossing trips.

Note | Sometimes geodesics must move through the pocket rather than a single dual diskoid!

Buildings

Q Why do products have such distance structure?

A They embed in the affine building $\Delta_4 = \Delta(\mathrm{SL}_4^\vee)$.

big simplicial complex
built from the
affine Grassmannian

- [FKK] proved this for $D(W)$ w/w non-elliptic

Affine Grassmannians

Def The affine Grassmannian of $SL(r)^\vee$ is

$$Gr = Gr_r = Gr(PGL(r)) = GL_r(\mathbb{C}(t)) / t^{-N} GL_r(\mathbb{C}[[t]]).$$

• Have points $[t^\lambda] = [\text{diag}(t^{\lambda_1}, \dots, t^{\lambda_r})]$

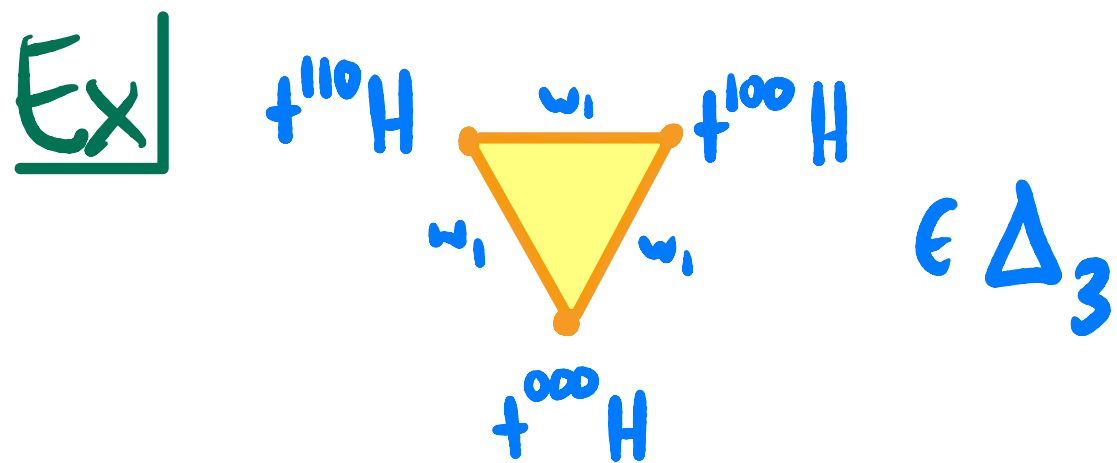
• Fact Double coset reps $H \backslash G / H \xrightarrow{\sim} \{t^\lambda \mid \lambda \in \Lambda_+\}$

• Gives KLM distance $d: Gr \times Gr \rightarrow \Lambda_+$ w/coherent geodesics

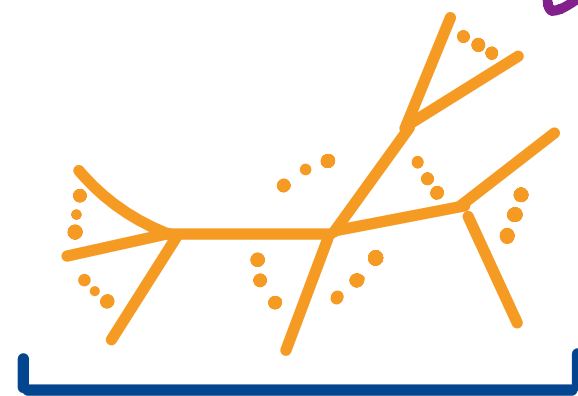
Ex $H(t^\lambda)^{-1} t^\mu H = H t^{\text{sort}(\mu - \lambda)} H \Rightarrow d([t^\lambda], [t^\mu]) = \text{sort}(\mu - \lambda)$

Affine Buildings

Def The affine building on G_r is the simplicial complex Δ_r whose vertices are the points of G_r and whose simplices are collections of points all of whose distances are fundamental weights.



Ex $\Delta_2 =$

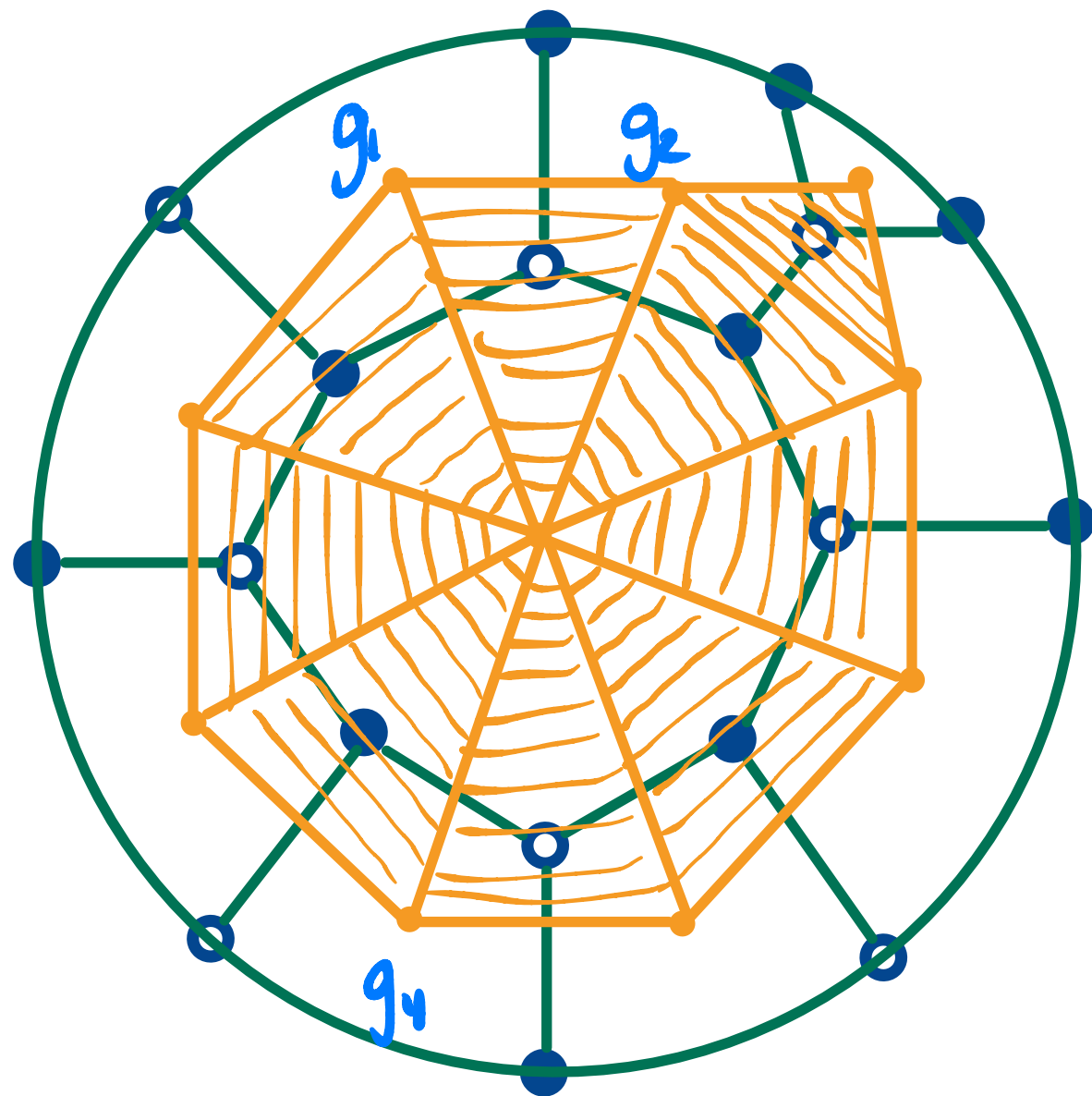


tree where every vertex has
uncountable degree
(Bruhat-Tits tree)

Affine Buildings

Thm ([FKK]) For W non-elliptic, $\exists$ isometric $D(W) \hookrightarrow \Delta_3$.

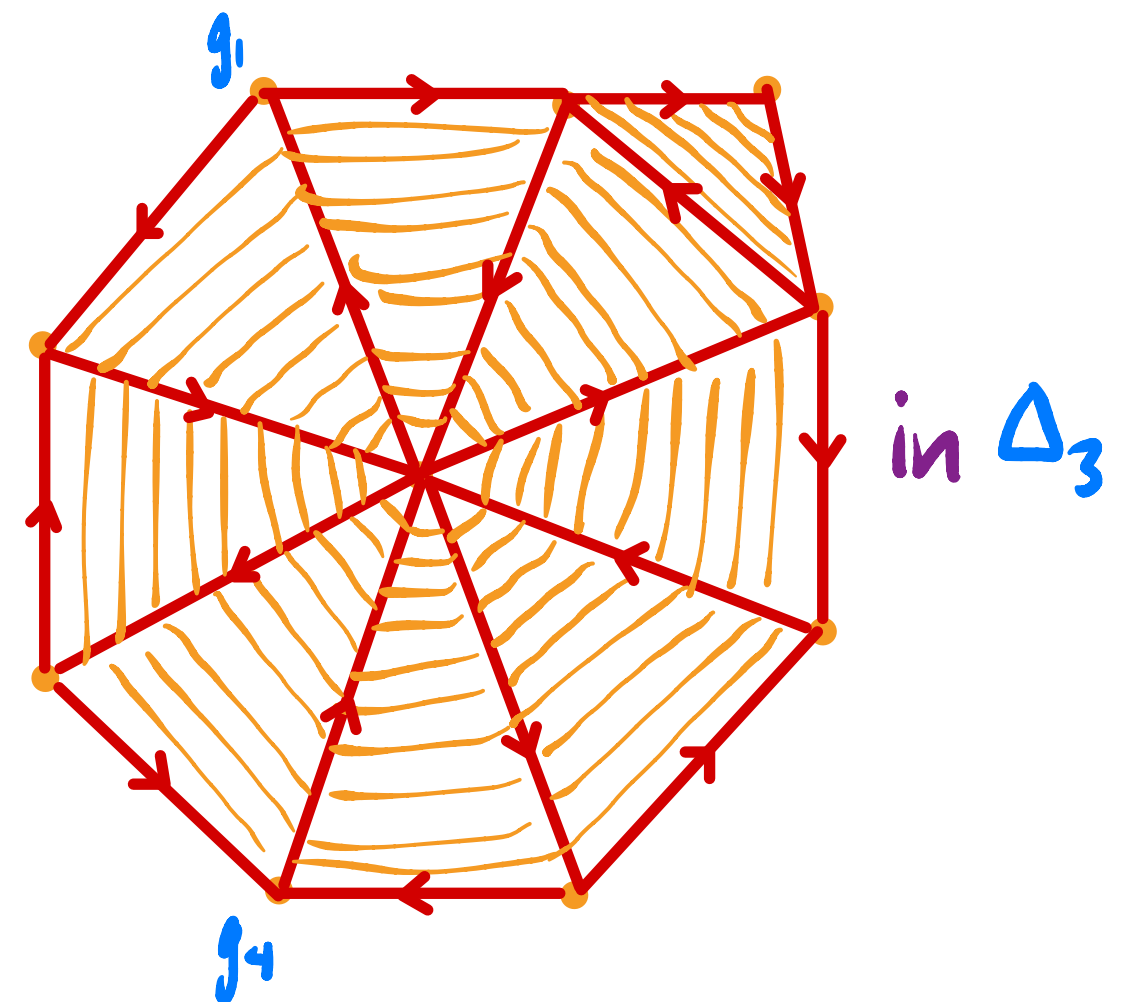
Ex There exist $g_1, g_2, g_3, g_4, \dots \in Gr_3$ s.t.



$$\begin{aligned} d(g_1, g_2) &= w_1 \\ d(g_2, g_3) &= w_1 \\ &\vdots \\ d(g_4, g_1) &= \underline{2w_1} \\ &\vdots \end{aligned}$$

geodesic distance

$$\begin{aligned} \overrightarrow{\hspace{1cm}} &= w_1 \\ \overleftarrow{\hspace{1cm}} &= w_1^* = w_2 \end{aligned}$$



Affine Buildings

Thm | (Gaetz - Striker - S. - Wu)

For a podet P , $\exists$ isometric $P \hookrightarrow \Delta_4$

Rem | Moves are a "feature" rather than a bug!

- Suggests higher rank web bases will have even more moves
- Seeking "top" reps has geometric meaning
- Pockets are generally non-affine (not in a single apartment)

Satake Components

- The geometric Satake correspondence relates G_F and SL_r -mod

(...equivalence of the category of equivariant perverse sheaves on G_F and $\text{rep}^*(SL_r)$ as symmetric and pivotal tensor categories...)

Def Let $F_n = \{ \text{loops in 1-skeleton of } \Delta_r \text{ of length } n \text{ starting at } [1^0] \text{ w/ steps of distance } w_i \}$

- This is a Satake fiber.
It is a projective variety.

Satake Components

Fact | $\text{Inv}_{\text{SL}_r}(V^{\otimes n}) \cong H_{\text{top}}(F_n, \mathbb{C})$ ($V = \mathbb{C}^r$)

• Hence top-dim'd irreducible components of F_n correspond to a basis of $\text{Inv}_{\text{SL}_r}(V^{\otimes n})$, the Satake basis

Q | What are those components?

Satake Components

Def For $\text{TEST}(r \times c)$, let $n=rc$ and set

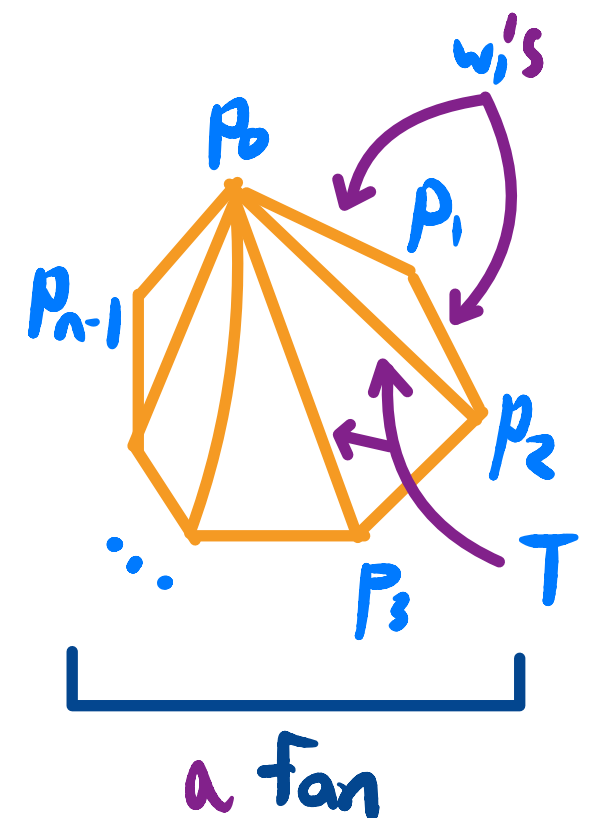
$$F_T = \{(p_0, p_1, \dots, p_{n-1}) \in F_n \mid p_0 = [t^0], d(p_0, p_i) = T_{\leq i} \forall i\}$$

partition w/ boxes $\leq i$

— These are fan configurations

Thm ([Haines '06]; see [FKK, §4])

The irreducible components of F_n are $\{F_T \mid \text{TEST}(r \times c)\}$.



Satake Components

Def Geometric rotation on F_n is given by

$$\text{rot}(p_0, p_1, \dots, p_{n-1}) = (\bar{p}_1^{-1} p_1, \bar{p}_1^{-1} p_2, \dots, \bar{p}_1^{-1} p_{n-1}, \bar{p}_1^{-1} p_0)$$

Thm For every $T \in \text{SYT}(4 \times c)$, we have $\text{rot}(\bar{F}_T) = \overline{F_{\text{Prom}(T)}}$.

Moreover, the subset

$$U_T = \bigcap_i \text{rot}^i F_{\text{Prom}(T)} = \{(p_0, p_1, \dots, p_{n-1}) \in F_n \mid d(p_i, p_{i+j}) = \text{Prom}(T)_{i,j}\}$$

is dense in F_T .

Satake Components

Thm (Gaetz - Striker - S.-Wu)

For every $T \in \text{SYT}(4 \times c)$, every point of U_T extends uniquely to an isometric embedding of the packet $P(T)$ into Δ_4 .

- In this sense, packets are generic finite models of the geometry of Δ_4 !
- Dihedral action on packets $\cong$ prom. equiv on $\text{SYT}(4 \times c)$

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THANKS!