Webs, pockets, and buildings

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Based on joint work with subsets of *Christian Gaetz, Oliver Pechenik, Stephan Pfannerer, Jessica Striker, and Haihan Wu*

Slides: https://www.jpswanson.org/talks/2026_BIRS_pockets.pdf

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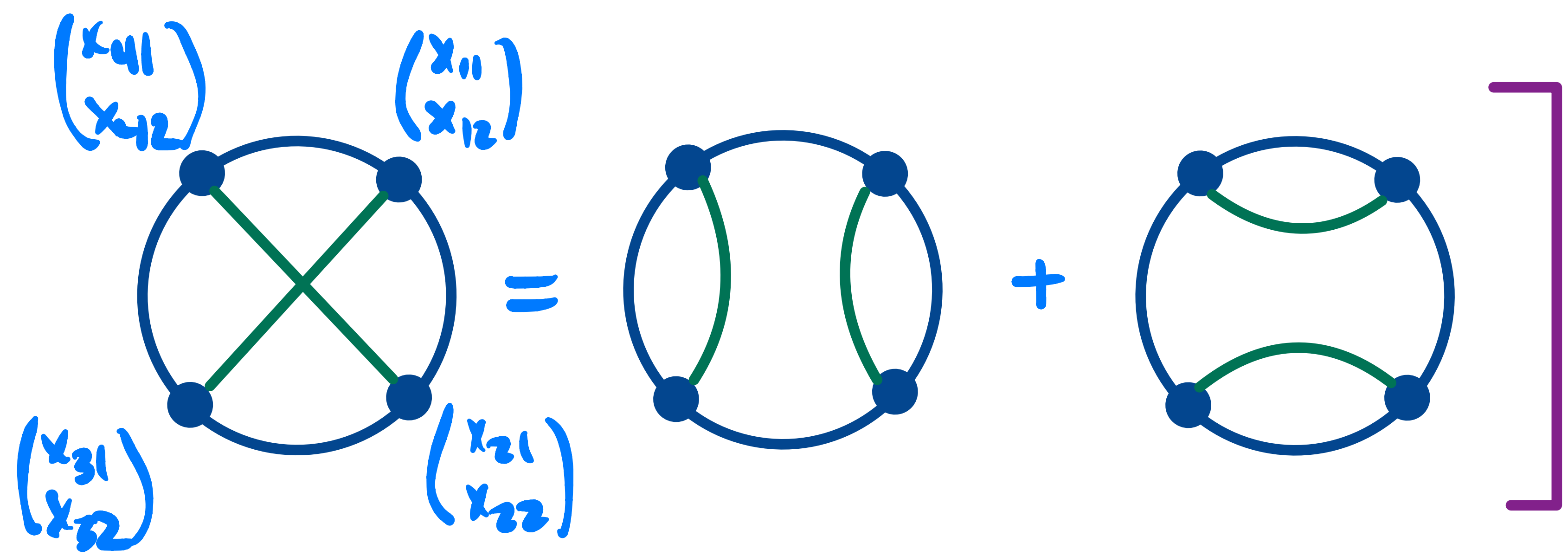
Temperley-Lieb web basis

Thm Non-crossing perfect matchings form a basis for

$$\mathrm{Hom}_{SL_2}(V^{\otimes n}, \mathbb{C}) \qquad (V = \mathbb{C}^2)$$

Ex

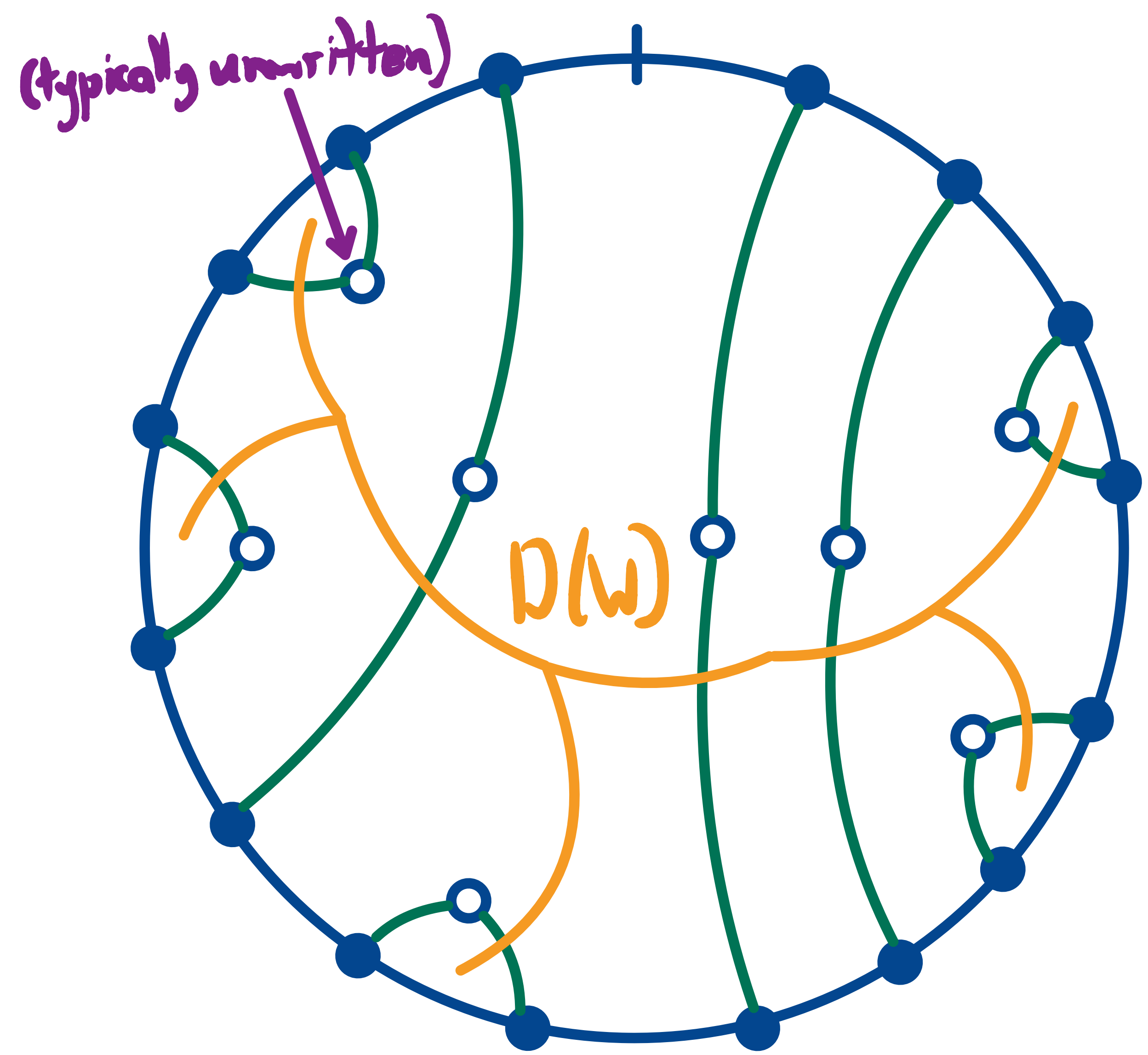


SL(2) webs

$$(x_{11}x_{23} - x_{21}x_{13})(x_{12}x_{24} - x_{22}x_{14}) = (x_{11}x_{22} - x_{21}x_{12})(x_{13}x_{24} - x_{23}x_{14})$$
$$+ (x_{11}x_{24} - x_{21}x_{14})(x_{12}x_{23} - x_{22}x_{13})$$

Duals and trees

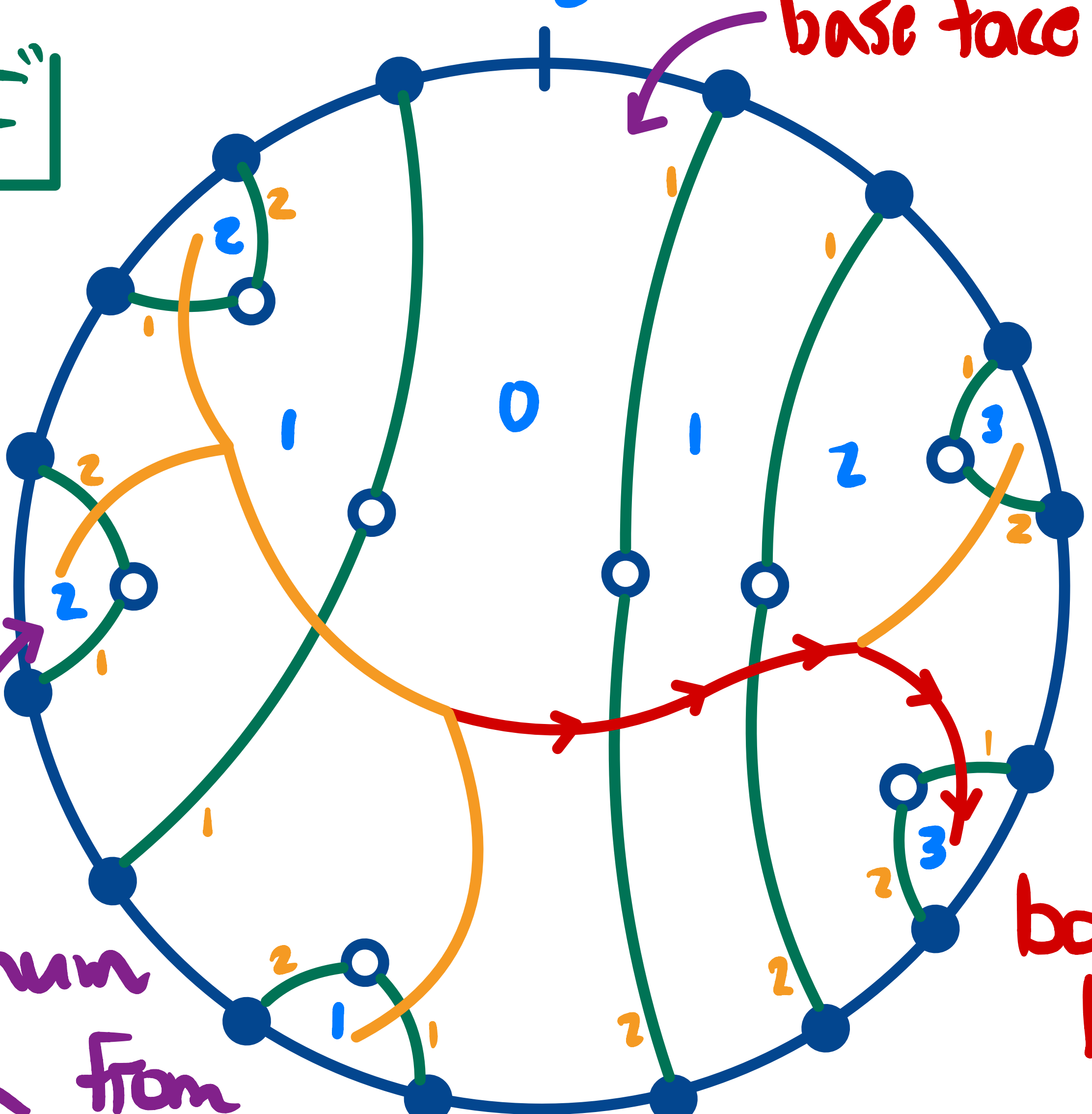
Obs The dual graph $D(W)$ of an $SL(2)$ basis web W is a tree, so 1-dimensional



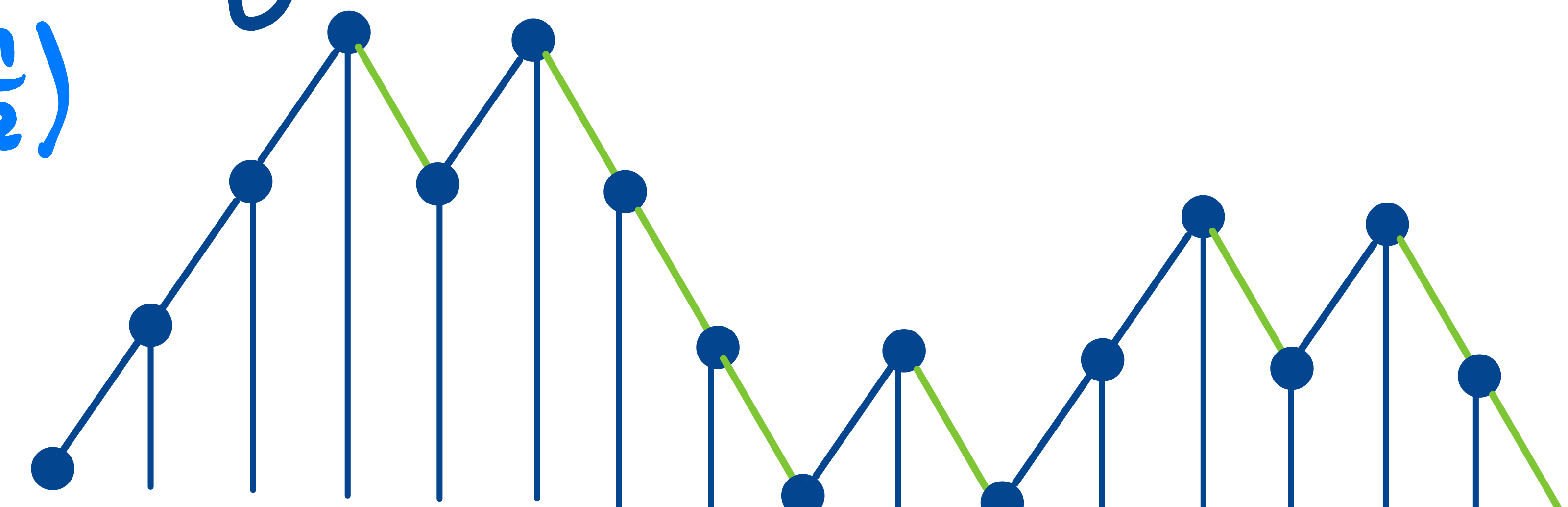
A Catalan bijection

Fact $\dim \mathrm{Inv}_{SL_2}(V^{\otimes n}) = \#SYT(2 \times \tfrac{n}{2})$

"Pf"



$\longmapsto$



$\downarrow$

((( ) ( ))) ( ) (( ) ( ))

$\rotatebox{90}{$\equiv$}$

balanced lattice word [1 1 1 2 1 2 2 2 1 2 1 1 2 1 2 2

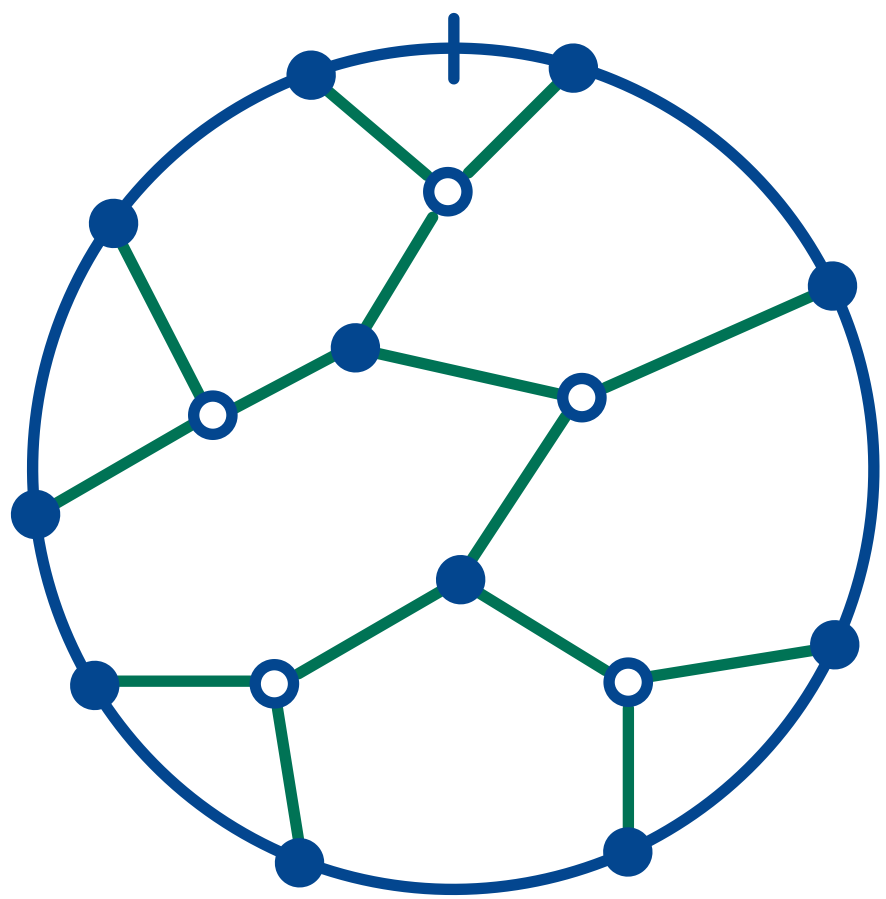
$\downarrow$

1	2	3	5	9	11	12	14
4	6	7	8	10	13	15	16

Non-elliptic web basis

Def (Kuperberg) An $SL(3)$ web is non-elliptic if it has no 2-gons or 4-gons.

Ex



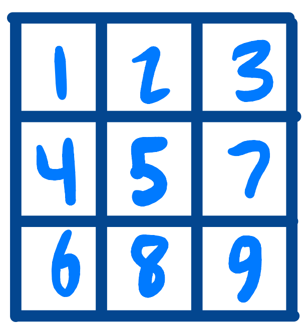
= …a polynomial obtained by summing over proper edge 3-colorings…

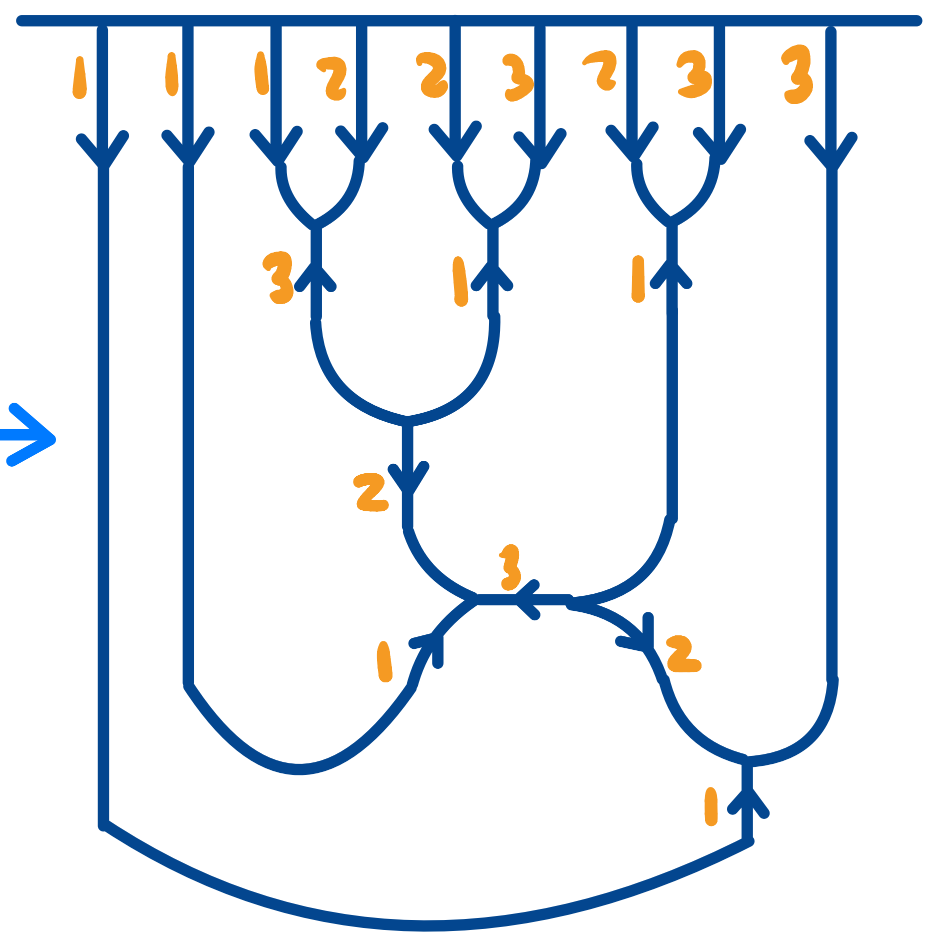
Thm Non-elliptic webs form a basis for $\mathrm{Inv}_{SL_3}(V^{\otimes n})$

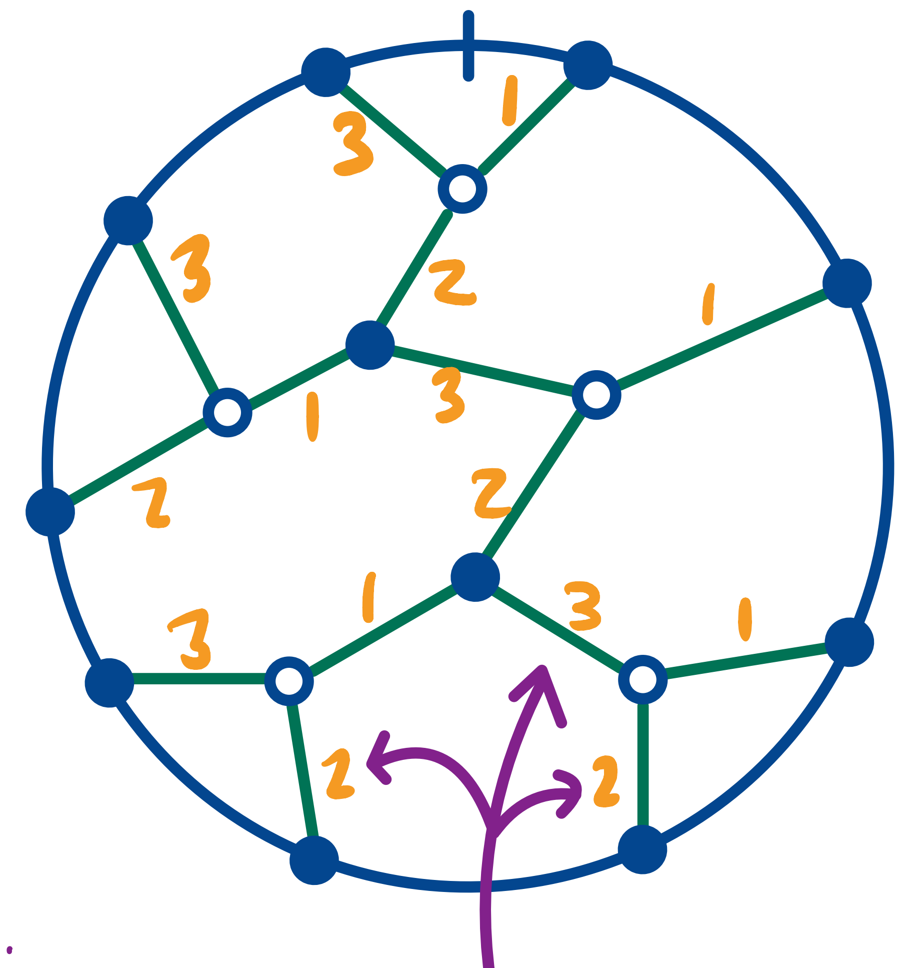
Growth labeling

Fact $\dim \mathrm{Inv}_{SL_3}(V^{\otimes n}) = \#SYT(3 \times \frac{n}{3})$

"Pf" Khovanov-Kuperberg growth rules give





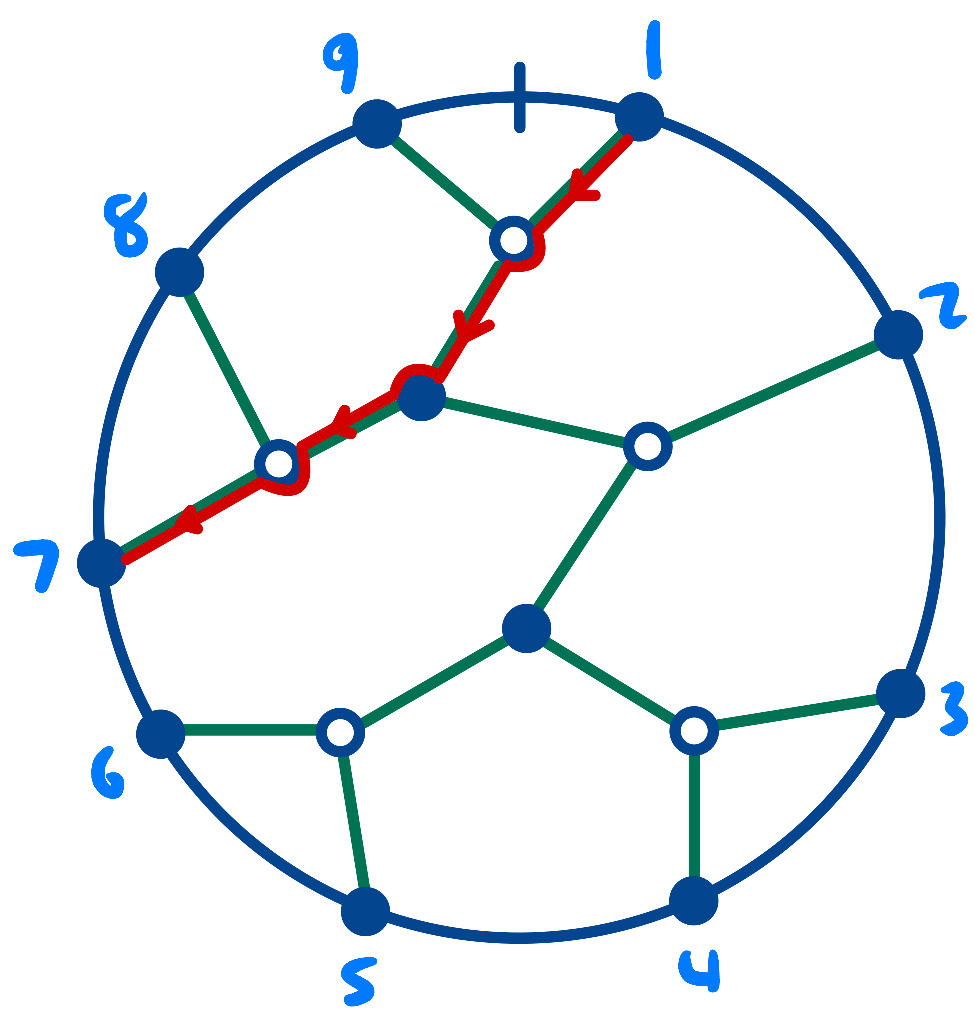


growth labeling (always proper)

- Boundary word of growth labeling is original lattice word

Plabic graphs and trips

$\boxed{\text{Fact}}$ Non-elliptic webs are reduced plabic graphs in the sense of Postnikov, hence determined by their trip permutation.



Rules of the road:

- left at ∘
- right at •

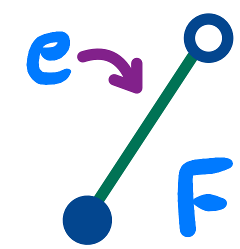
Here $1 \mapsto 7$ and

trip = 754963821

Separation labeling

Def (GPPSS) Let W be non-elliptic.

The separation labeling of W labels edge e by...

1. Stand on face F as in 

("white is on the right")

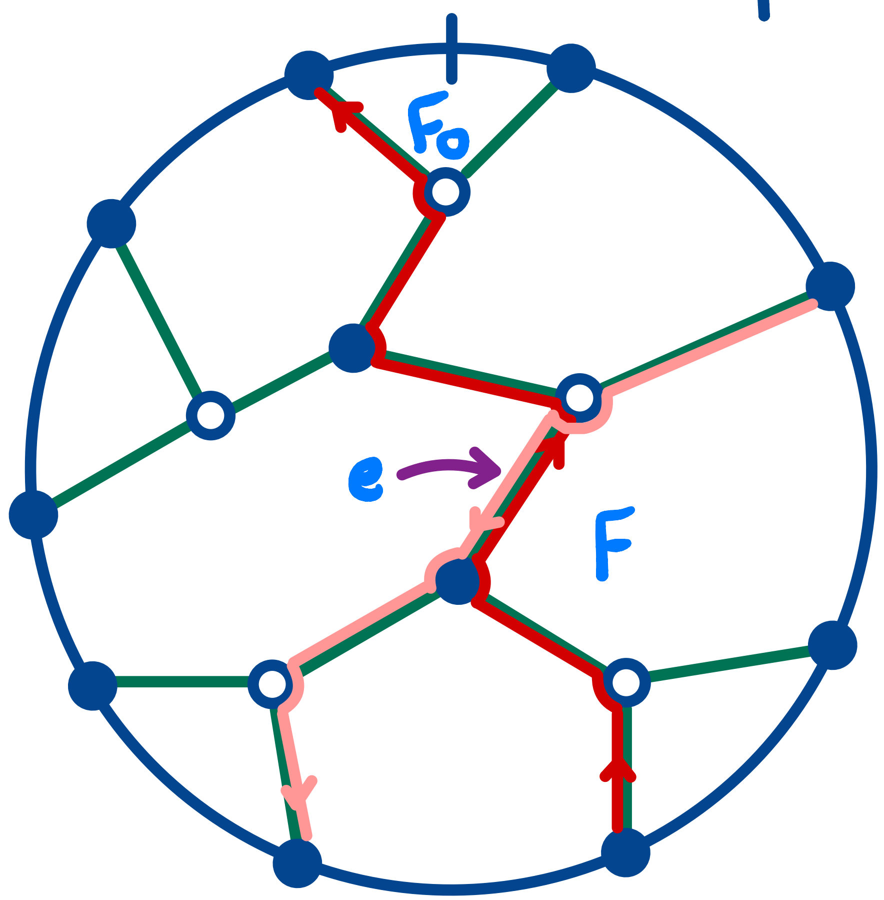
2. Let s = # trips thru e separating F from the base face F_0

3. Label e with $s+1$

Separation labeling

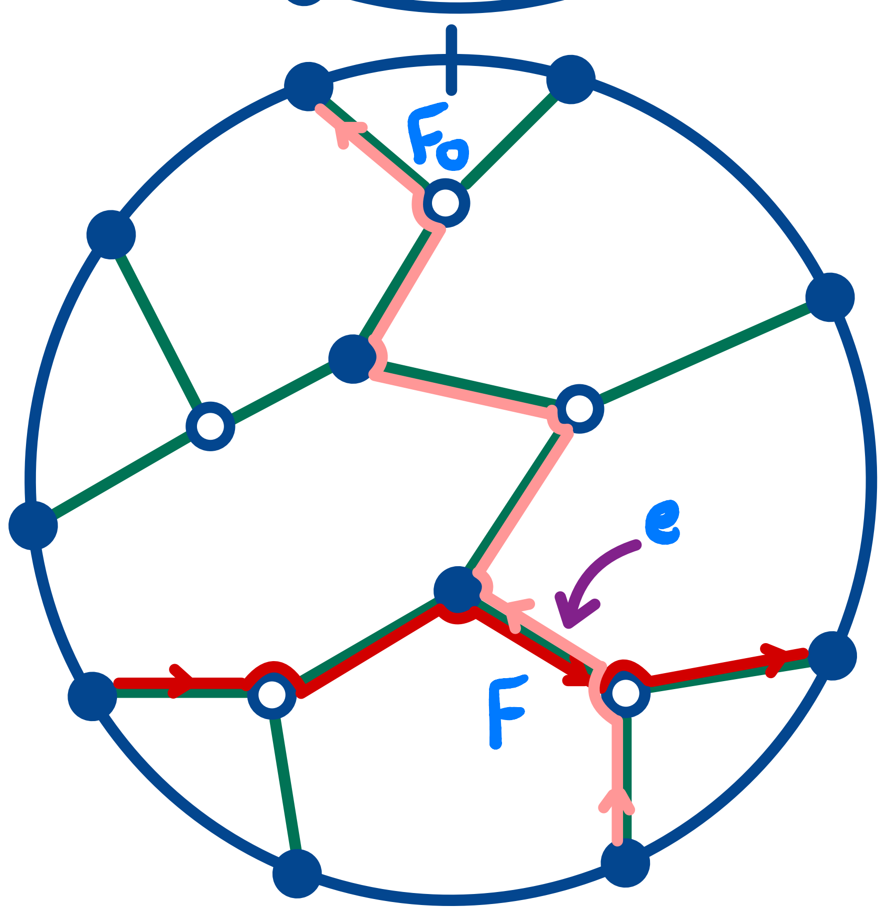
1. Stand on face F as in ("white is on the right")
2. Let $s=\#$trips thru e separating F from the base face F_0
3. Label e with $s+1$

Ex



- trip_1 does not separate F and F_0
- trip_2 does separate F and F_0

$\Rightarrow s=1 \Rightarrow$ label is 2



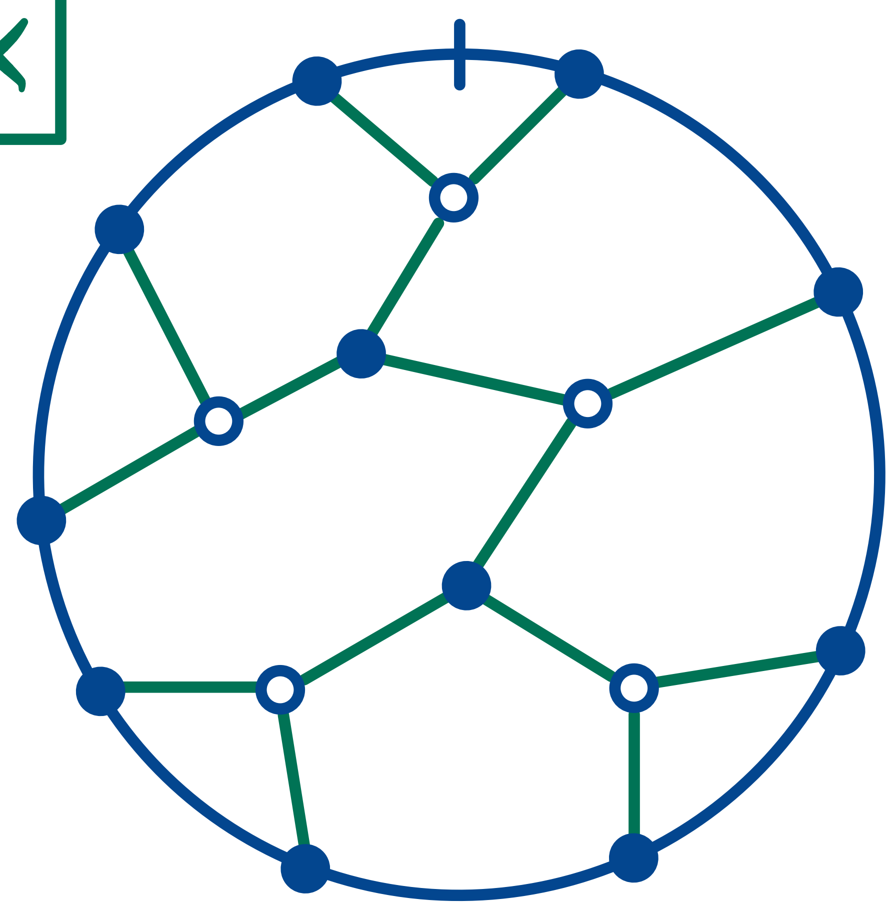
- trip_1 does separate F and F_0
- trip_2 does separate F and F_0

$\Rightarrow s=2 \Rightarrow$ label is 3

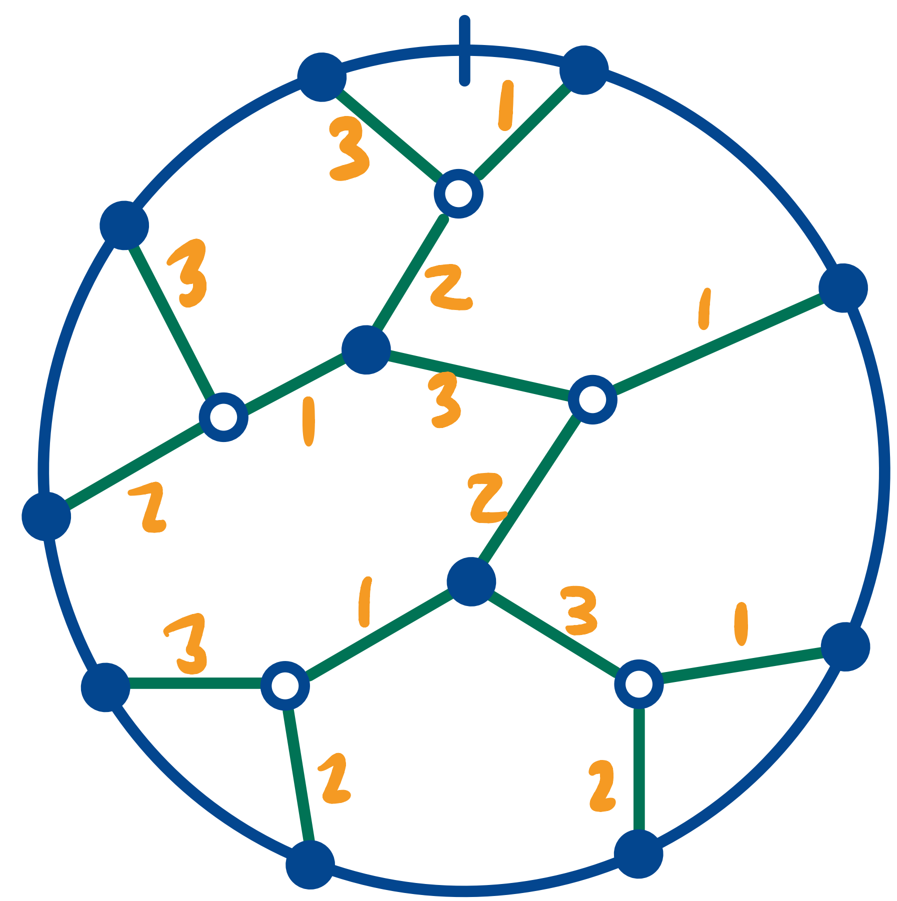
Separation labeling

Thm (GPPSS) The separation labeling is the growth labeling.

Ex



$\mapsto$



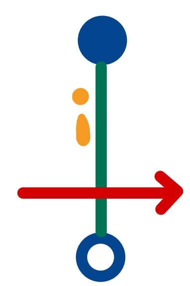
$\mapsto$ 111223233

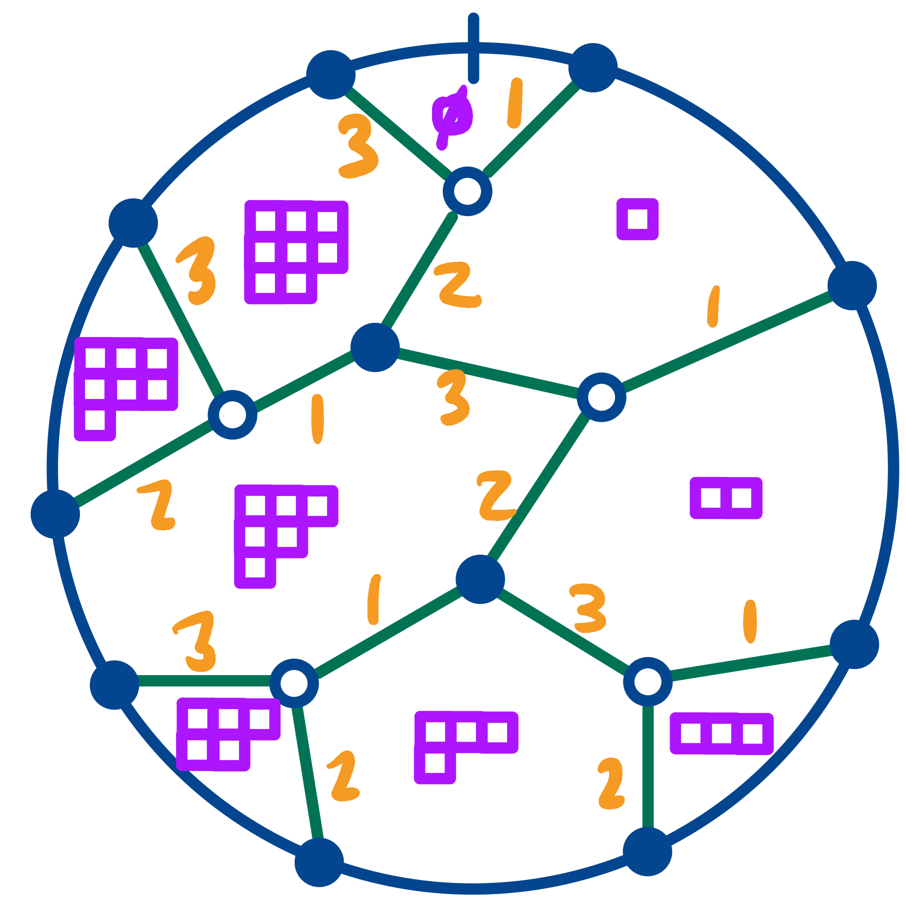
$\mapsto$

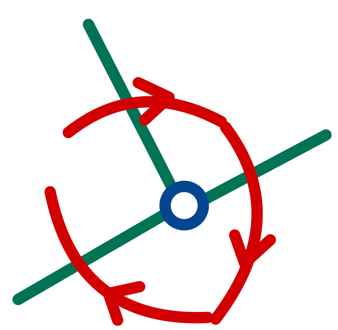
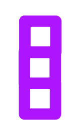
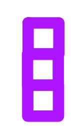
1	2	3
4	5	7
6	8	9

Partition labels

Obs Can label a non-elliptic web's faces w/partitions using separation labels:

- Start w/ $\emptyset$ in base face
- If  add box to ith row

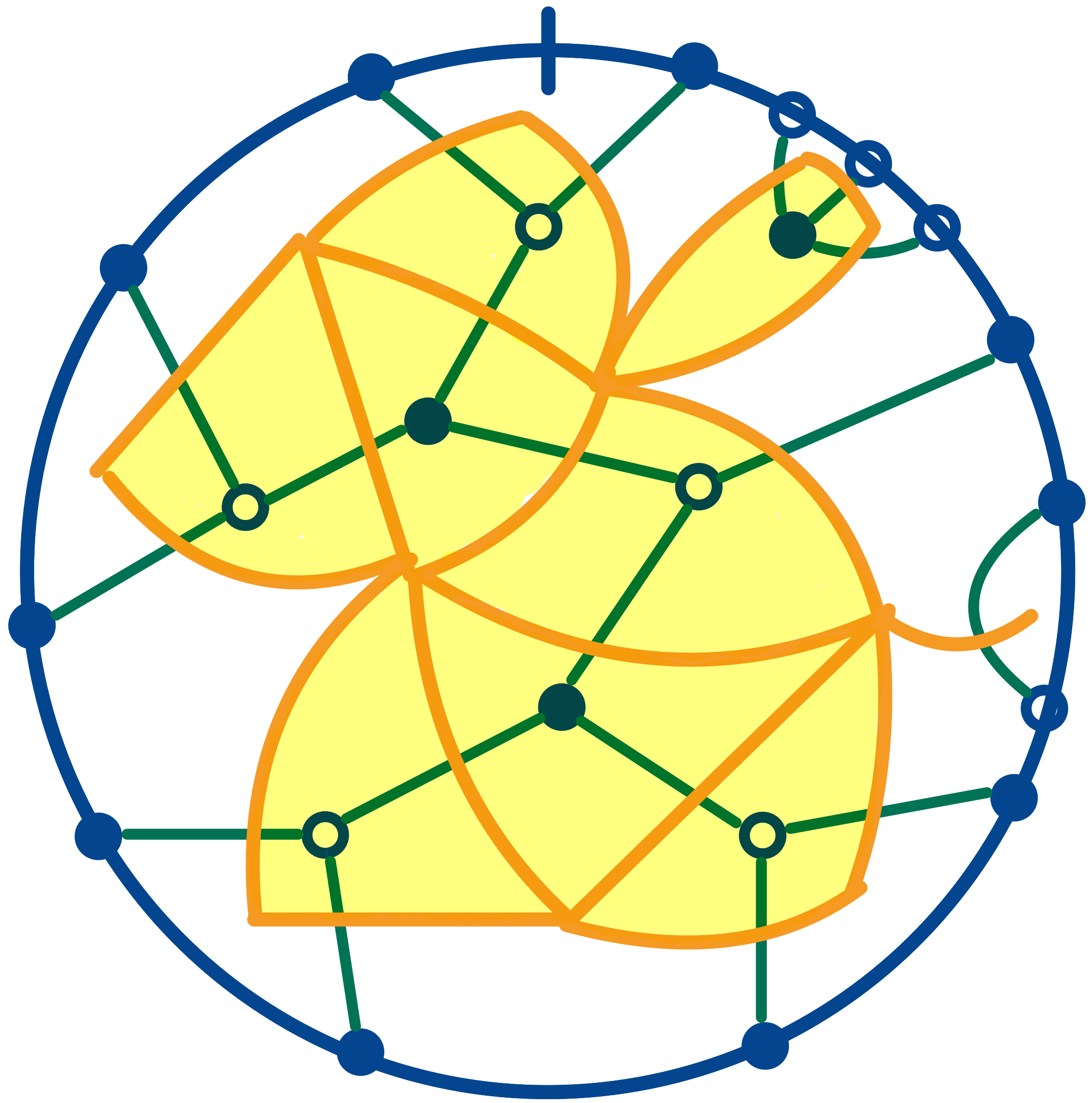
Ex 

Note ·  adds ; need  $\equiv \emptyset$

· Really need SL_3-weight labels

SL_3-web duals

Obs $D(W)$ for $SL(3)$ is a triangulation:



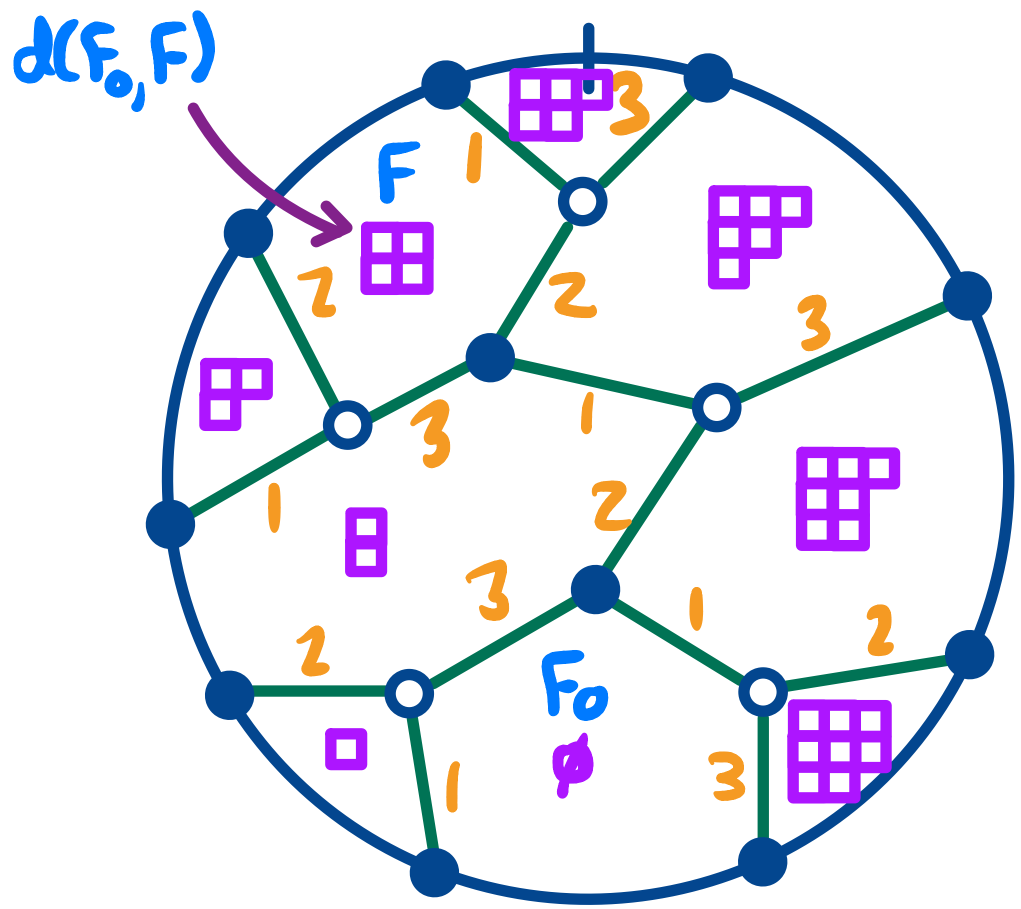
$D(W)$ is "2D" for $SL(3)$

Note Depth from base face to face labeled λ is $\lambda_1 - \lambda_3$. Not enough info to recover separation labels!

Idea Need a space with an <u>$SL(3)$-weight-valued distance</u>

SL_3-web distances

Fact Letting F_0 be any face of non-elliptic W, separation labels wrt F_0 give some dominant weight $d(F_0, F)$:



Fact F is adjacent to F_0

$\Leftrightarrow$ $d(F_0,F) = \square$ or $\boxed{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$

Fact For $T \in SYT(3 \times \frac{n}{3})$,

$\mathrm{rot}\, W(T) \mapsto \mathrm{pro}\, T$

$SL(r)$-weights

Recall

- The $SL(r)$-weights are $\Lambda = \mathbb{Z}^r / \{(c, \ldots, c) \mid c \in \mathbb{Z}\}$
- The dominant weights are $\Lambda_+ = \{\lambda_1 \geq \cdots \geq \lambda_r\} \subset \Lambda$
- The fundamental weights are

$$\omega_i = (1^i, 0^{r-i})$$

$$\omega_i^* = (0^{r-i}, -1^i) \equiv \omega_{r-i}.$$

- If $\lambda, \mu \in \Lambda_+$ w/ representatives with same sum, then

$$\lambda \geq \mu \Leftrightarrow \forall m, \sum_{i=1}^m \lambda_i \geq \sum_{i=1}^m \mu_i \quad \text{(dominance order)}$$

KLM Distances

Def (See [Kapovich-Leeb-Millson '08])

A **KLM distance** on a set X is $d: X \times X \to \Lambda$ s.t.

1. $d(x,y) \in \Lambda_+$ is dominant
2. $d(x,y) = 0 \Leftrightarrow x = y$
3. $d(y,x) = -\mathrm{rev}\, d(x,y)$

- Contrast w/ metric space axioms when $r=2$

KLM Distances

Obs ([Fontaine-Kamnitzer-Kuperberg '13])

1. If W is non-elliptic, then $D(W)$ has a natural KLM distance: $d(F_0, F)$ as before

2. The affine building $\Delta(SL(3)^\vee)$ does too!
(next topic)

3. **Thm** $D(W)$ embeds isometrically in $\Delta(SL(3)^\vee)$!

Affine Grassmannians

$\boxed{\text{Def}}$ The affine Grassmannian of $SL(r)^\vee$ is

$$Gr = Gr_r = Gr(PGL(r)) = \underbrace{GL_r(\mathbb{C}((t)))}_{\text{"G"}} / \underbrace{t^{\mathbb{Z}} GL_r(\mathbb{C}[[t]])}_{\text{"H"}}.$$

- For $\lambda \in \mathbb{Z}^r$, let $t^\lambda = \mathrm{diag}(t^{\lambda_1}, \ldots, t^{\lambda_r}) \in G$

  e.g. $[t^0] \in Gr$ is the "origin"

- Have $\Lambda \hookrightarrow Gr$ as $\{[t^\lambda] \mid \lambda \in \Lambda\}$

Affine Grassmannians

Key Fact Have double coset representatives

$$H \backslash G / H \xleftrightarrow{\sim} \{ t^\lambda \mid \lambda \in \Lambda_+ \}$$

Key Cor Have a KLM distance $d : Gr \times Gr \to \Lambda_+$

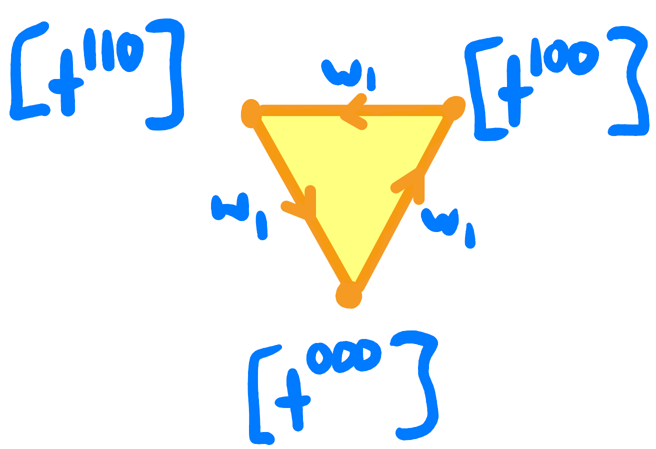
where $d([g_1],[g_2]) = \lambda \in \Lambda_+$ s.t. $H g_1^{-1} g_2 H = H t^\lambda H$

Ex If $\lambda, \mu \in \mathbb{Z}^r$, then $H (t^\lambda)^{-1} t^\mu H = H t^{\mathrm{sort}(\mu - \lambda)} H$

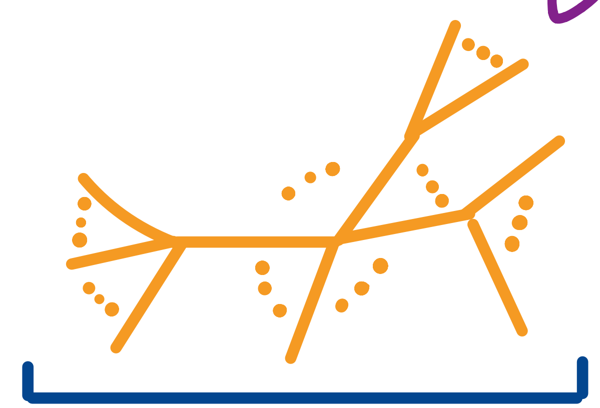
so that $d([t^\lambda],[t^\mu]) = \mathrm{sort}(\mu - \lambda)$

Affine Buildings

Def The **affine building** on Gr_r is the simplicial complex Δ_r whose <u>vertices</u> are the points of Gr_r and whose <u>simplices</u> are collections of points all of whose distances are fundamental weights.

Ex 

$\in \Delta_3$

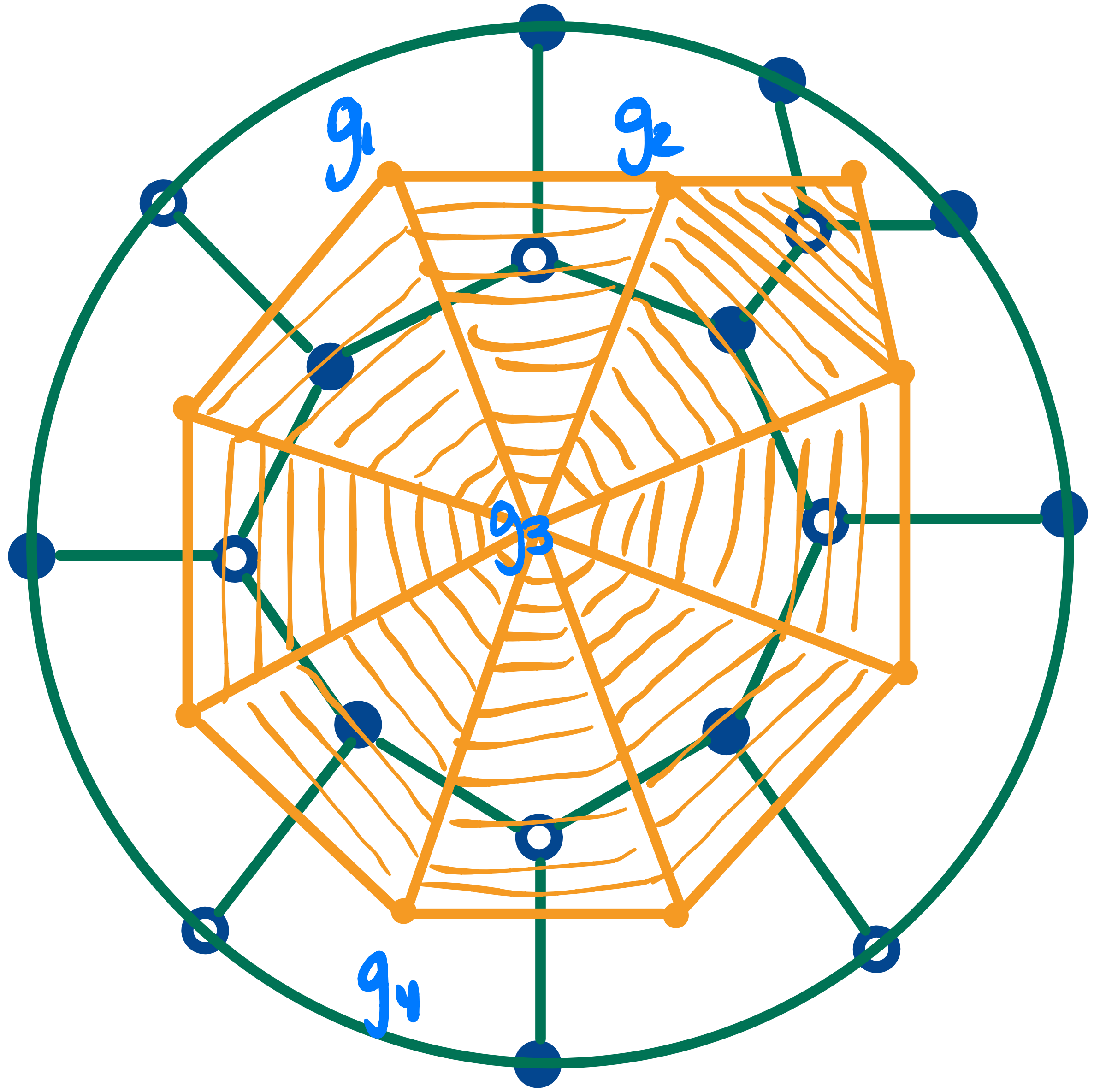
Ex $\Delta_2 =$ 

tree where every vertex has uncountable degree (Bruhat-Tits tree)

Affine Buildings

Thm ([FKK]) For W non-elliptic, $\exists$ isometric $D(W) \hookrightarrow \Delta_3$.

Ex There exist $g_1, g_2, g_3, g_4, \ldots \in Gr_3$ s.t.



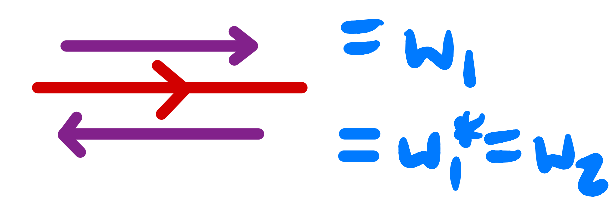
$d(g_1, g_2) = w_1$

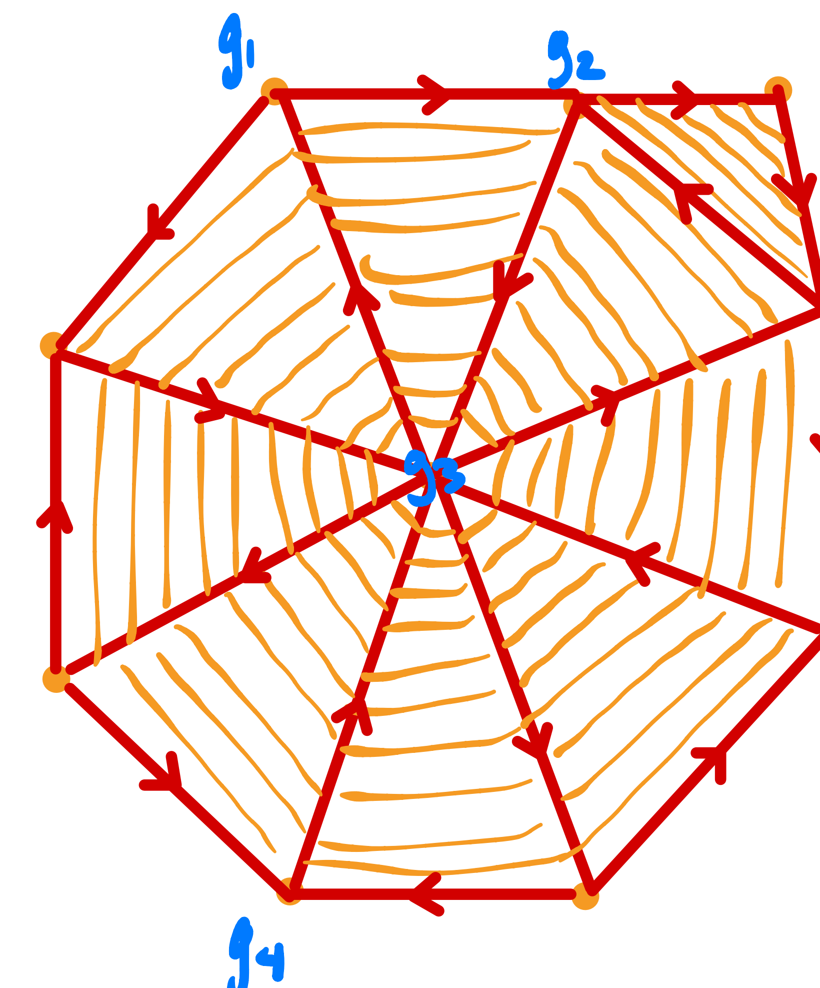
$d(g_2, g_3) = w_1$

$\vdots$

$d(g_4, g_1) = 2w_1$

$\vdots$

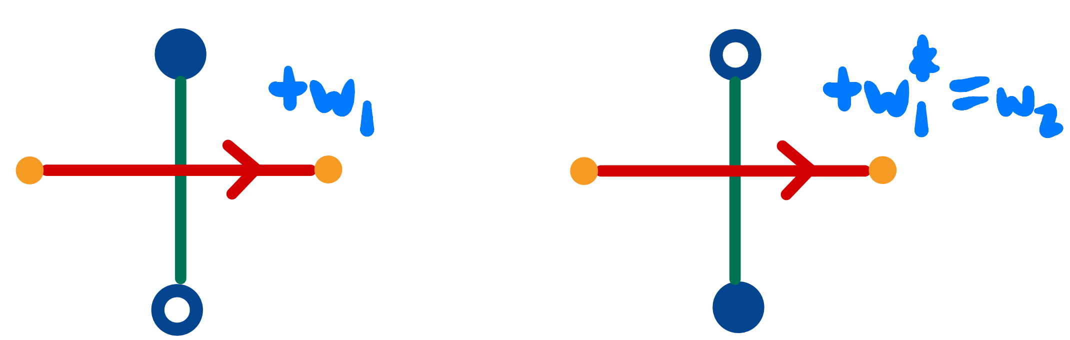




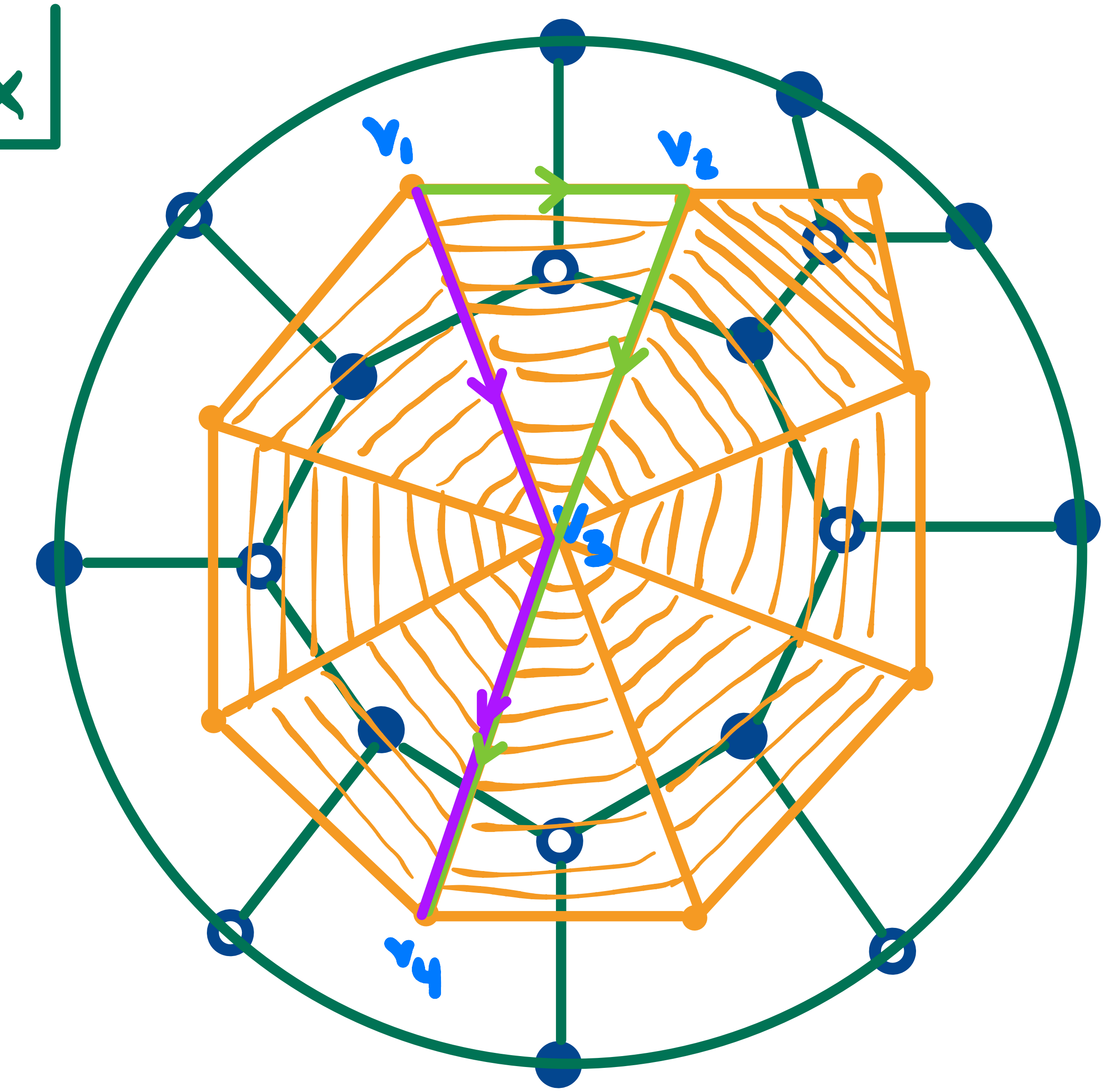
in Δ_3

Miniscule walks

Idea: For an SL(3) web, walk along the dual graph, adding distances at each step. Picks up SL(3) weights according to:



Ex



$$d(v_1 \xrightarrow{w_1} v_2 \xrightarrow{w_1} v_3 \xrightarrow{w_2} v_4) = 2w_1 + w_2 = (3,1,0)$$

vs.

$$d(v_1 \xrightarrow{w_2} v_3 \xrightarrow{w_2} v_4) = 2w_2 = \underbrace{(2,2,0)}_{\text{smaller!}}$$

Coherent geodesics

Suppose X has KLM distance $d : X \times X \to \Lambda_+$.

- A minscule walk from $a \in X$ to $b \in X$ is

$$a = \gamma_0, \gamma_1, \dots, \gamma_k = b \text{ s.t. } \forall i,\ d(\gamma_{i-1}, \gamma_i) \text{ miniscule}$$

- Let

$$\delta(a,b) = \left\{ \sum_{i=1}^{k} d(\gamma_{i-1}, \gamma_i) \,\middle|\, \gamma \text{ miniscule walk } a \to b \right\} \subset \Lambda_+.$$

- A KLM geodesic is a path achieving a minima of $\delta(a,b)$
- X has coherent geodesics if $d(a,b)$ is always the unique minimum of $\delta(a,b)$.

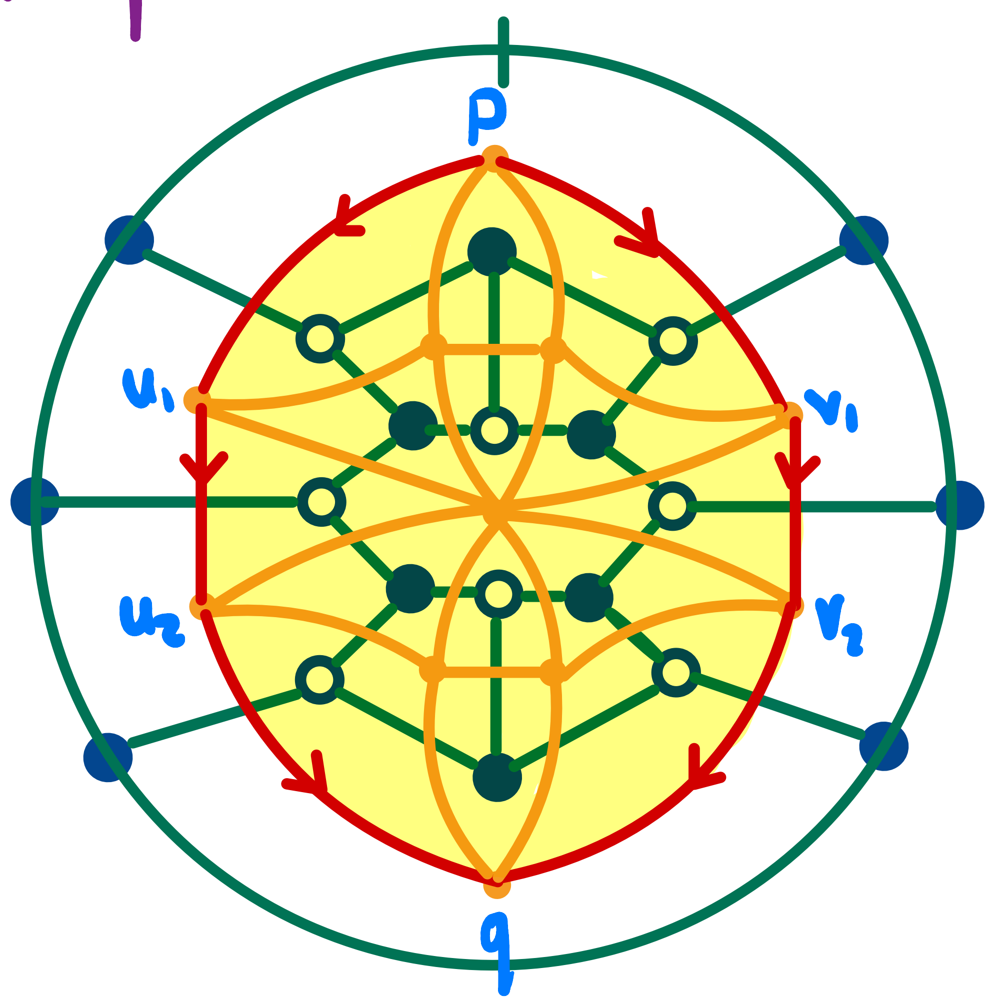
$\boxed{\text{Fact}}$ Gr_- works

Geodesics

Thm ([Kuperberg '96], FKK)

If W is a non-elliptic $SL(3)$ web, $D(W)$ has coherent geodesics.

Ex Non-example:



$$d(p \to u_1 \to u_2 \to q) = (0,0,-3) \equiv (3,3,0)$$

$$d(p \to v_1 \to v_2 \to q) = (3,0,0) \equiv (4,1,1)$$

incomparable!

(has squares)

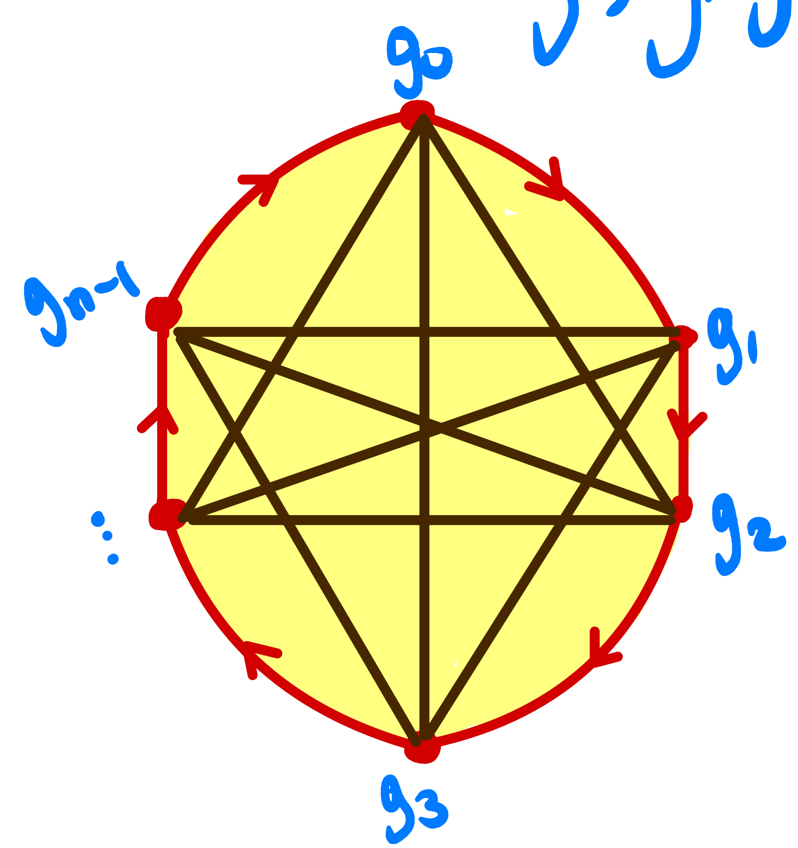
Satake Components

Fact Δ_3 is very infinite, complicated!

Q What do loops in Δ_3 look like?

- For any $T \in SYT(3 \times \frac{n}{3})$, have

$$D(W) \hookrightarrow \Delta_3 \text{ where } d(g_i, g_{i+j}) = \underbrace{\mathrm{prom}^i(T)_{\leq j}}_{\text{partition w/ boxes } 1, \ldots, j}$$



Satake Components

- The geometric Satake correspondance relates

$$Gr_r \text{ and } SL_r\text{-mod}$$

(...equivalence of the category of equivariant perverse sheaves on Gr_r and $\mathrm{rep}^u(SL_r)$ as symmetric and pivotal tensor categories...)

$\boxed{\text{Def}}$ Let $F_n = \{$loops in 1-skeleton of Δ_r of length n starting at $[t^0]$ w/steps of distance $w_1\}$

— This is a Satake Fiber.
It is a projective variety.

Satake Components

Fact • $\mathrm{Inv}_{SL_r}(V^{\otimes n}) \cong H_{top}(F_n, \mathbb{C})$ $\quad (V = \mathbb{C}^r)$

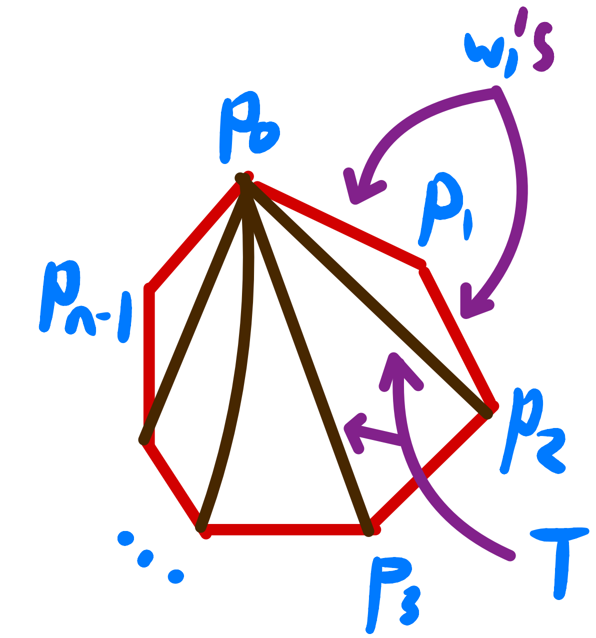
• Hence top-dim'l irreducible components of F_n correspond to a basis of $\mathrm{Inv}_{SL_r}(V^{\otimes n})$, the **Satake basis**

Q What are those components?

Satake Components

Def For $T \in SYT(r \times c)$, let $n = rc$ and set

$$F_T = \{(p_0, p_1, \dots, p_{n-1}) \in F_n \mid p_0 = [t^0],\ d(p_0, p_i) = T_{\le i}\ \forall i\}$$



— These are fan configurations

Thm ([Haines '06]; see [FKK, §4])
The irreducible components of F_n are $\{\overline{F_T} \mid T \in SYT(r \times c)\}$.

Sotable Components

Def Geometric rotation on F_n is given by

$$\mathrm{rot}(p_0, p_1, \dots, p_{n-1}) = (p_1^{-1}p_1, p_1^{-1}p_2, \dots, p_1^{-1}p_{n-1}, p_1^{-1}p_0)$$

Thm (Fontaine–Kamnitzer)

- For every $T \in SYT(r \times c)$, we have $\mathrm{rot}(\overline{F_T}) = \overline{F_{\mathrm{Prom}(T)}}$.
- Moreover, the subset

$$U_T = \bigcap_i \mathrm{rot}^i F_{\mathrm{Prom}^i(T)} = \{(p_0, p_1, \dots, p_{n-1}) \in F_n \mid d(p_i, p_{i+j}) = \mathrm{Prom}^i(T)_{\leq j}\}$$

is dense in F_T.

Satake Components

Thm (FKK) For $T \in SYT(3 \times \frac{n}{3})$, every loop in U_T extends uniquely to an embedding $p: D(w) \hookrightarrow \Delta_3$ s.t.

$$d(F,G) = w_i \Rightarrow d(p(F), p(G)) = w_i.$$

Moreover, p is isometric.

Rem
- Hence a generic loop in Δ_3 uniquely "fills in" to an isometrically embedded copy of some $D(w)$!
- Gives "finite models" of Δ_3's geometry!

SL(4) building geometry

Q Can this extend to SL(4)?

- Immediate problem: Δ_4 is 3D, but $D(W)$ is 2D!

Hourglass plabic graphs

Thm ([Gaetz–Pechenik–Pfannerer–Striker–S. '25])

There is a (rotation-invariant) $SL(4)$ web basis consisting of top, fully reduced, 4-row **hourglass plabic graphs**.

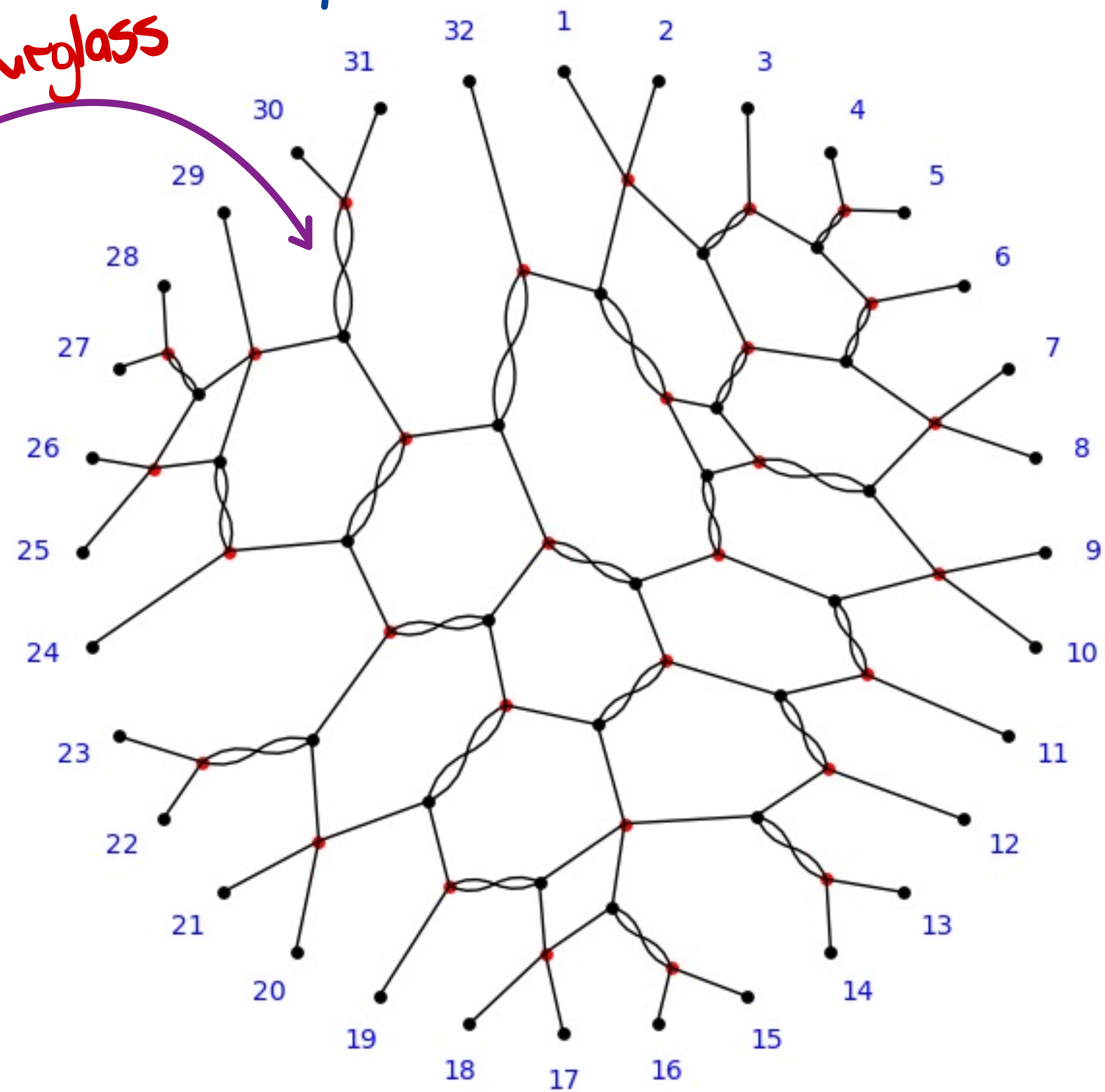
HPG basis example

Ex

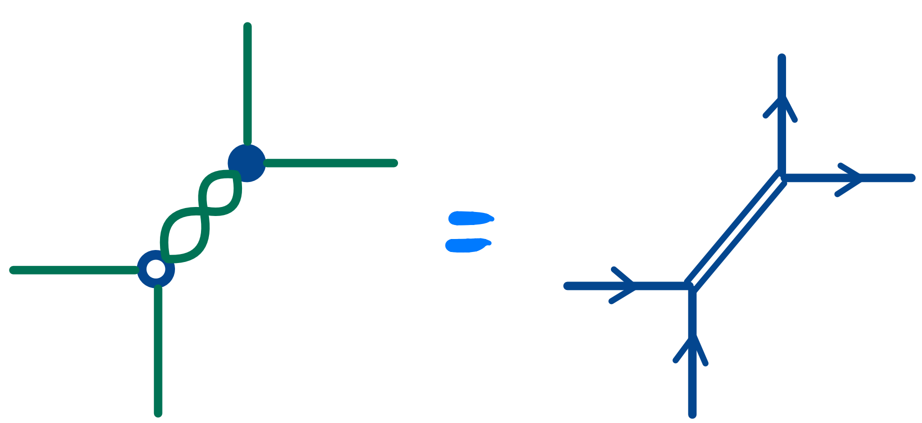
1	3	4	7	8	17	19	23
2	5	6	9	14	18	21	24
10	12	13	15	16	25	26	28
11	20	22	27	29	30	31	32

$\longmapsto$

a Z-hourglass



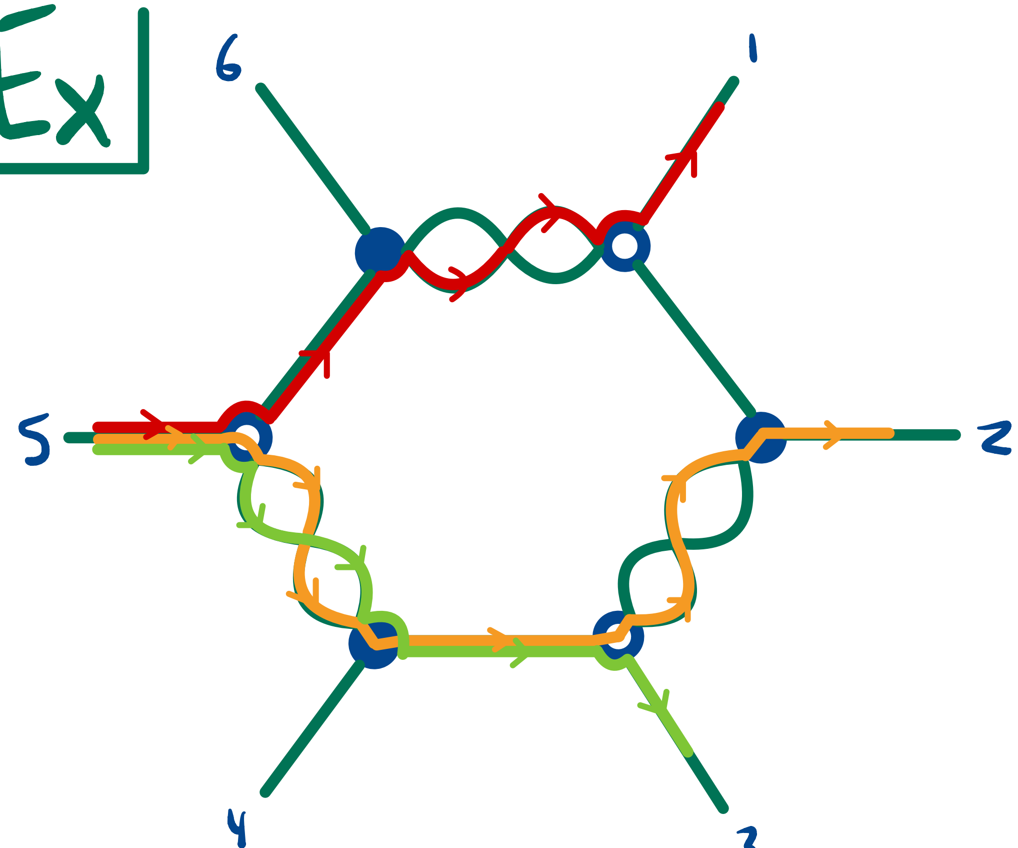
Note



Trip permutations

Def (GPPSS '26) An r-hourglass plabic graph has **trip permutations** $\mathrm{trip}_1, \ldots, \mathrm{trip}_{r-1}$ where trip_i takes the <u>ith left at white</u> and <u>ith right at black</u>:

Ex



$(r=4)$

$\longrightarrow = \mathrm{trip}_1 = (135)(642)$

$\longrightarrow = \mathrm{trip}_2 = (14)(25)(36)$

$\longrightarrow = \mathrm{trip}_3 = (531)(246)$

Note

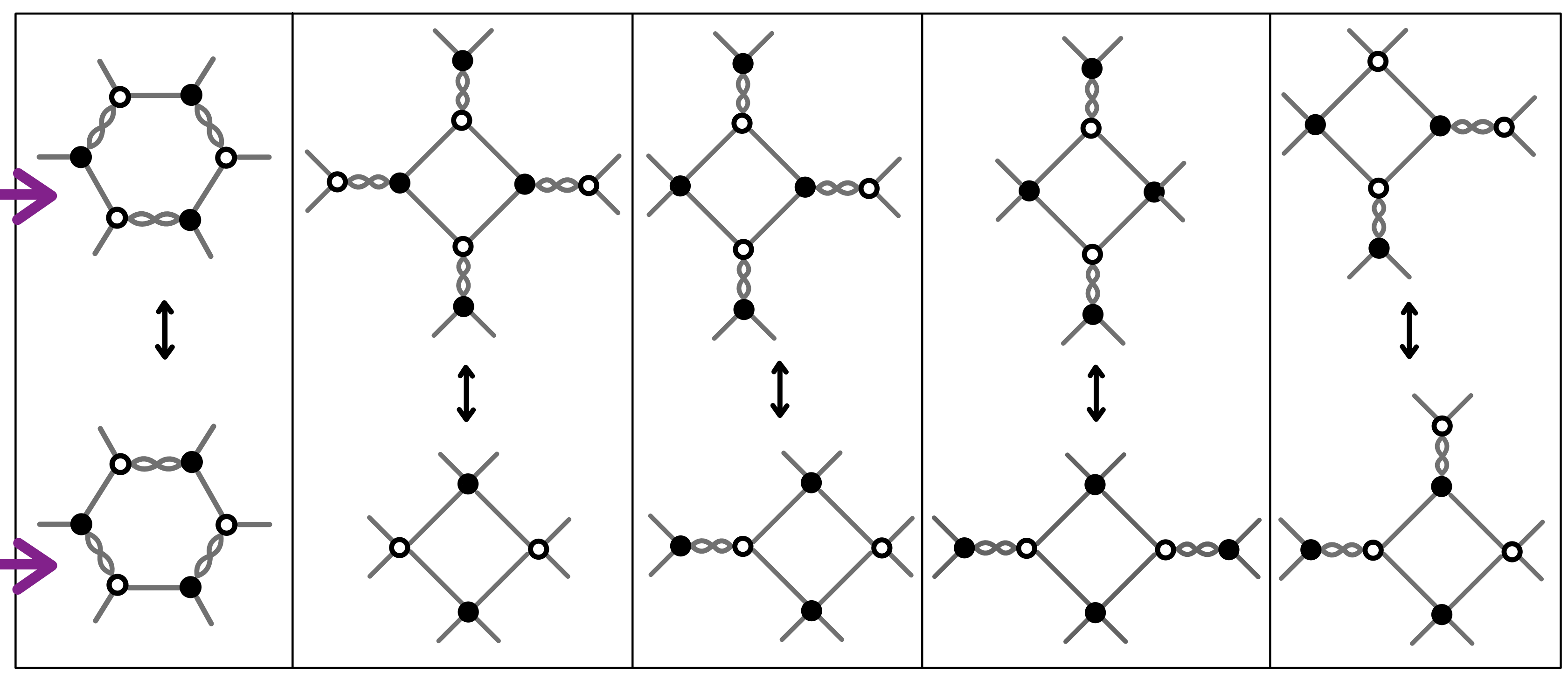
$\mathrm{trip}_i = \mathrm{trip}_{r-i}^{-1}$!

$r=4$ moves

Thm (GPPSS '26) Two contracted, fully reduced 4-HPG's have the same trip permutations $\Leftrightarrow$ they are related by moves:

"top"

"bottom"



benzene move

square moves

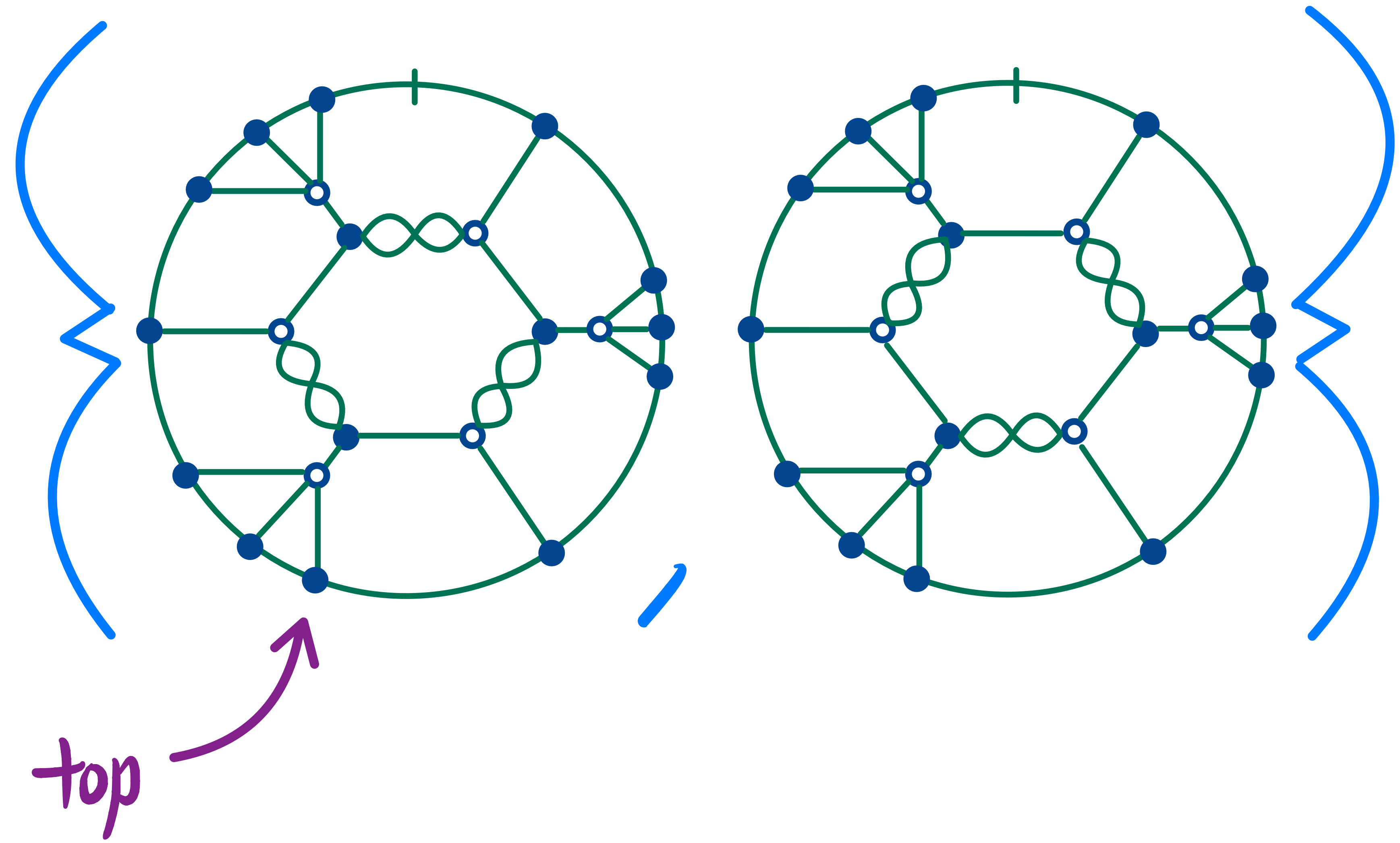
r=4 moves

Ex

$T =$

1	2	6
3	5	10
4	7	11
8	9	12

$\mapsto$

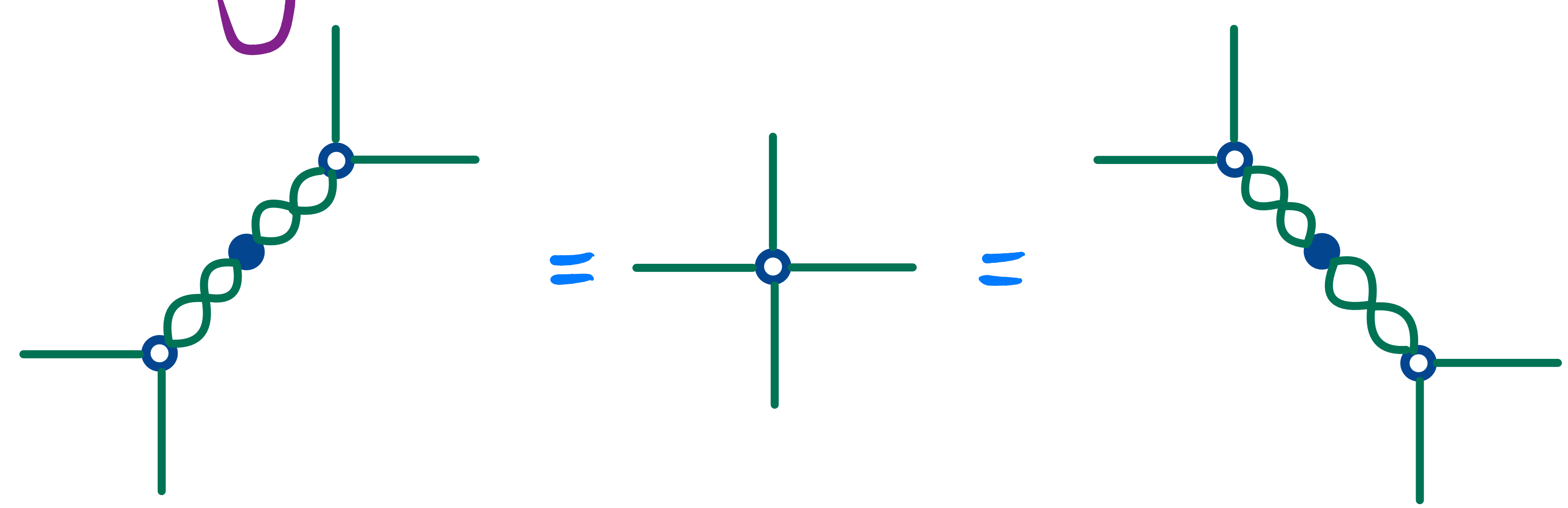


Pockets

Def (Gaetz–Striker–S.–Wu)

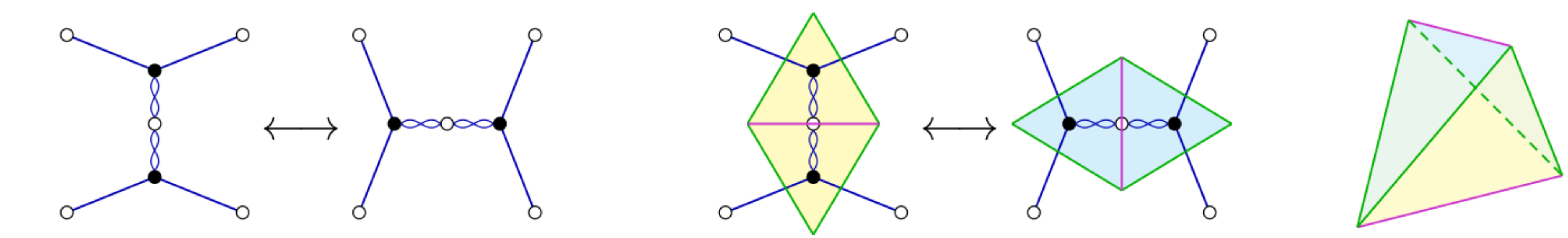
Let $\mathcal{C}$ be the move-class of some fully reduced 4-HPG W. Then there is a 3D flag simplicial complex $P(\mathcal{C})$ called the **pocket** of $\mathcal{C}$, which $D(W)$ embeds into.

- Requires expanding sources/sinks via I=H moves:

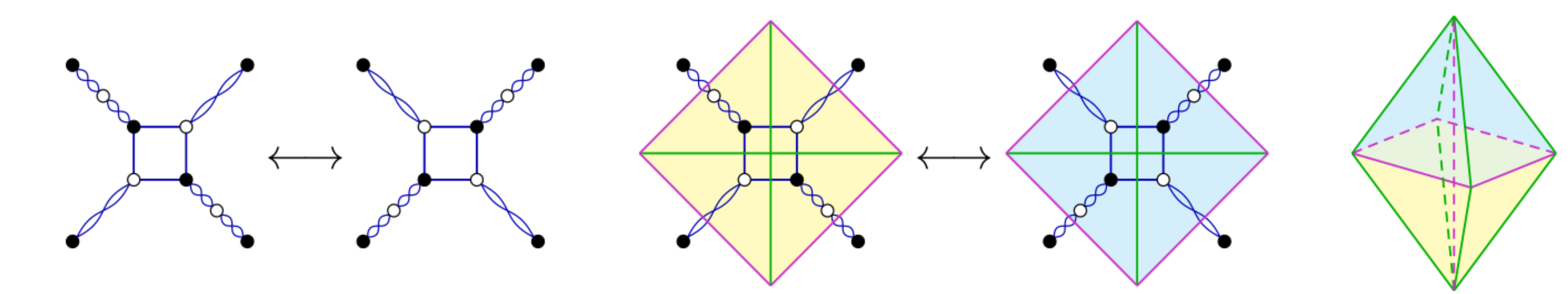


Pockets

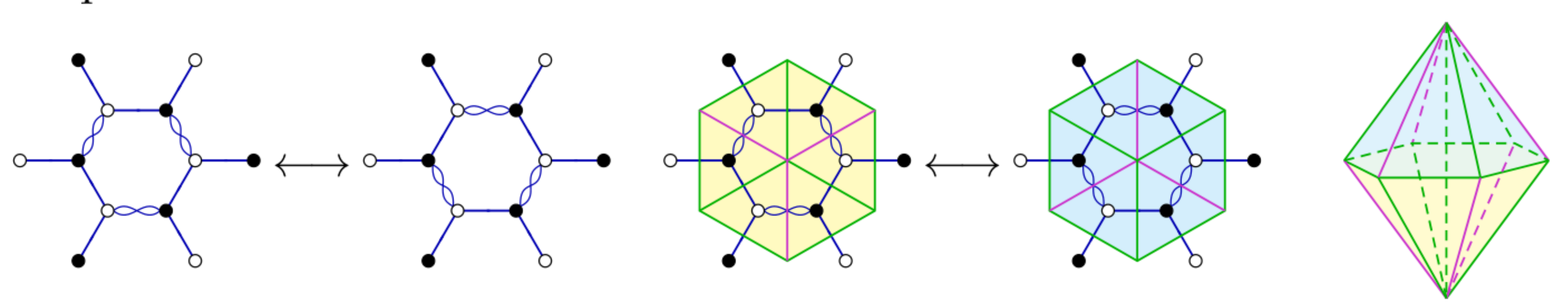
- Build a pocket from a move-class by gluing D(w)'s with tetrahedra between moves:



IH move between webs | Dual diskoids | tetrahedron



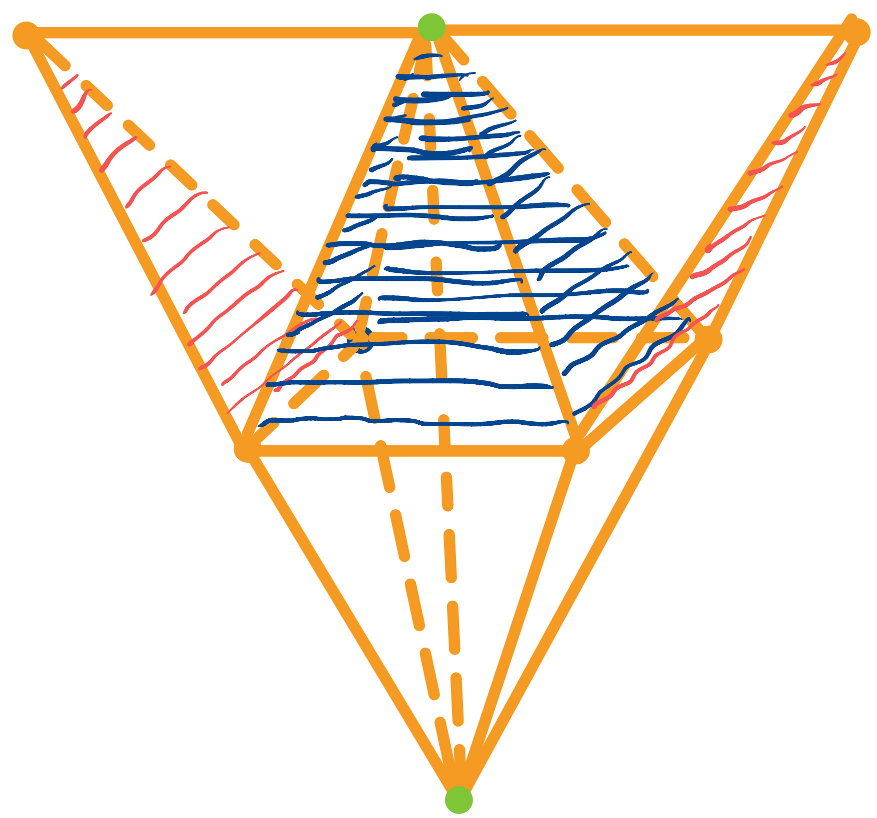
Square move between webs | Dual diskoids | octahedron

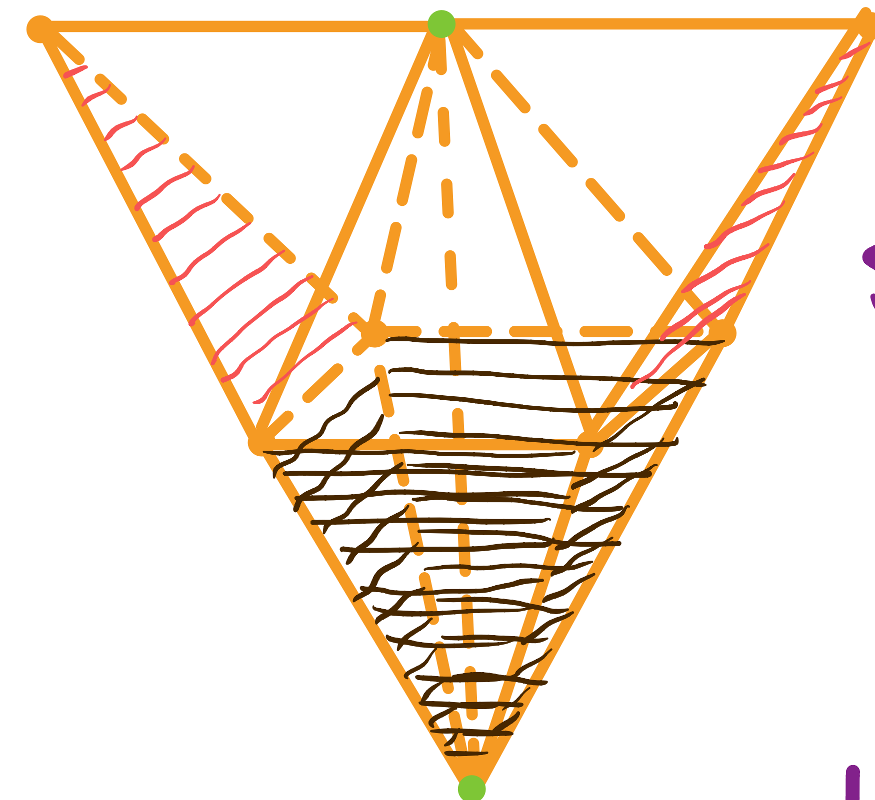


Benzene move between webs | Dual diskoids | dodecahedron

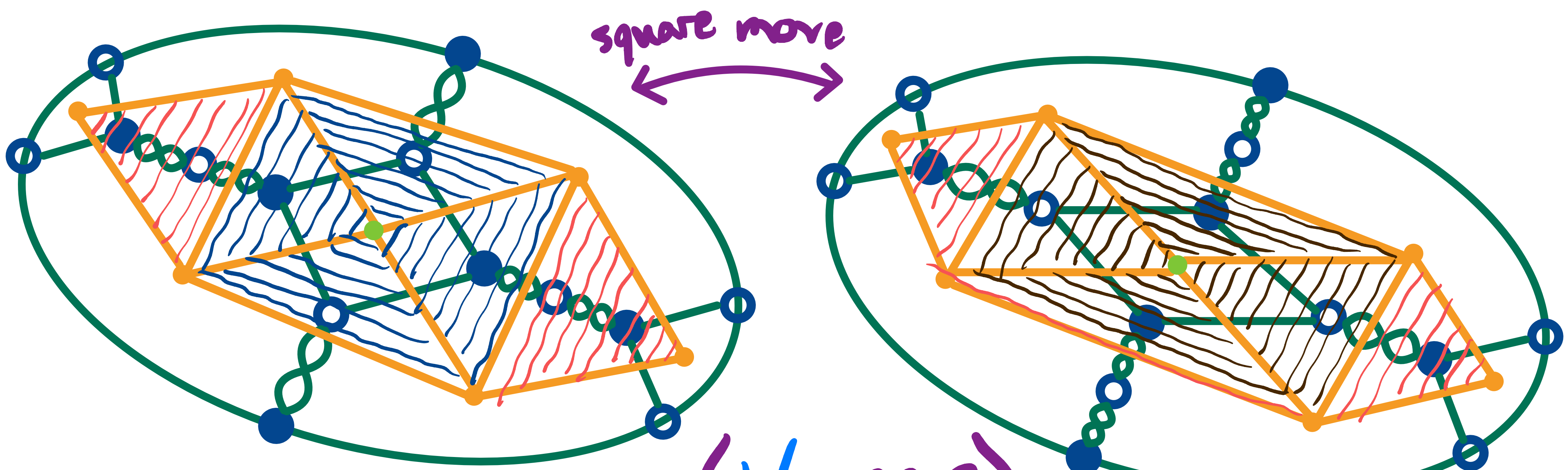
Pockets

Ex





Pocket of square+benzene+IH class of $\bar{4}\{1,2\}\bar{3}\,\bar{2}\{3,4\}\bar{1}$ has 6 tetrahedra, 14 triangles, 19 edges, 8 vertices



(+6 more)

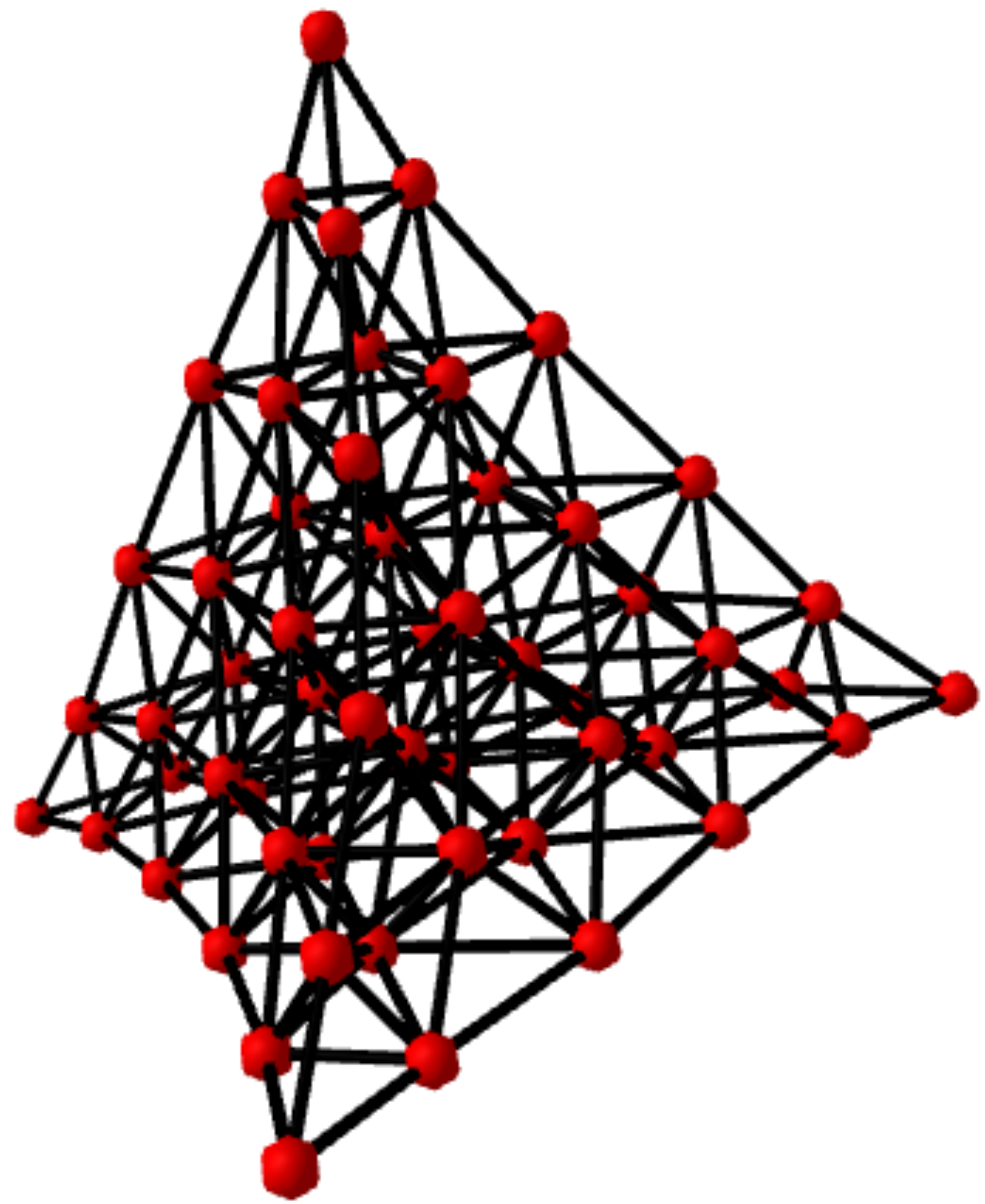
*

Pockets

Ex Pocket of 5×5 ASM:

$L = 1^5 2^5 3^5 4^5$

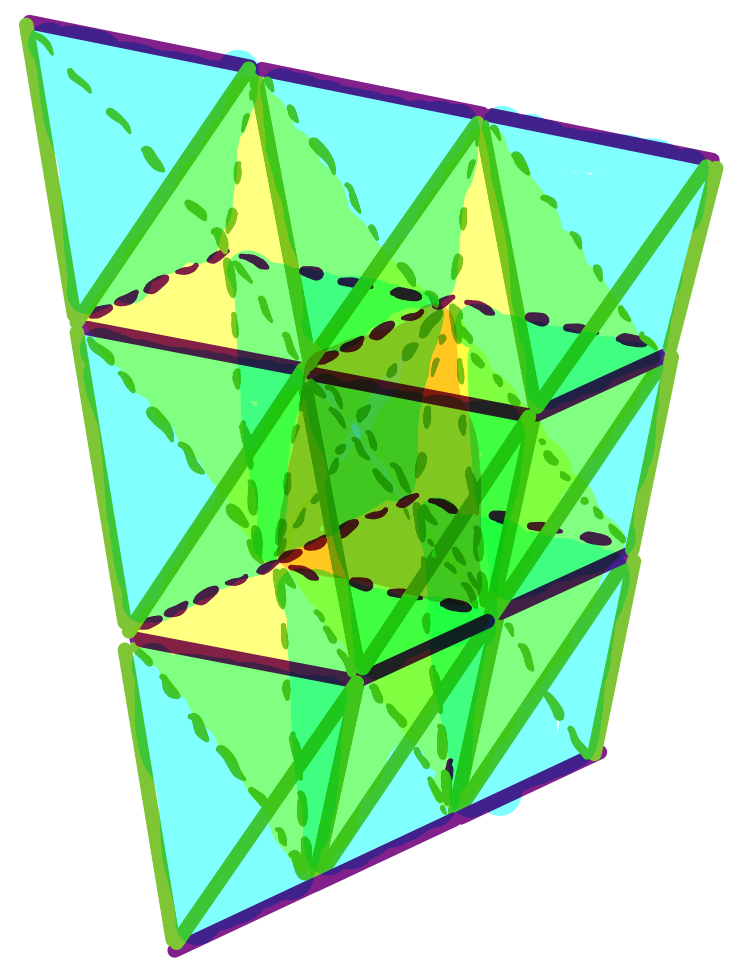
$\left(T = \begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 & 20 \end{matrix} \right)$



Note Related to height functions, octahedral recurrence, tilings of the Aztec diamond, distributive lattices

Pockets

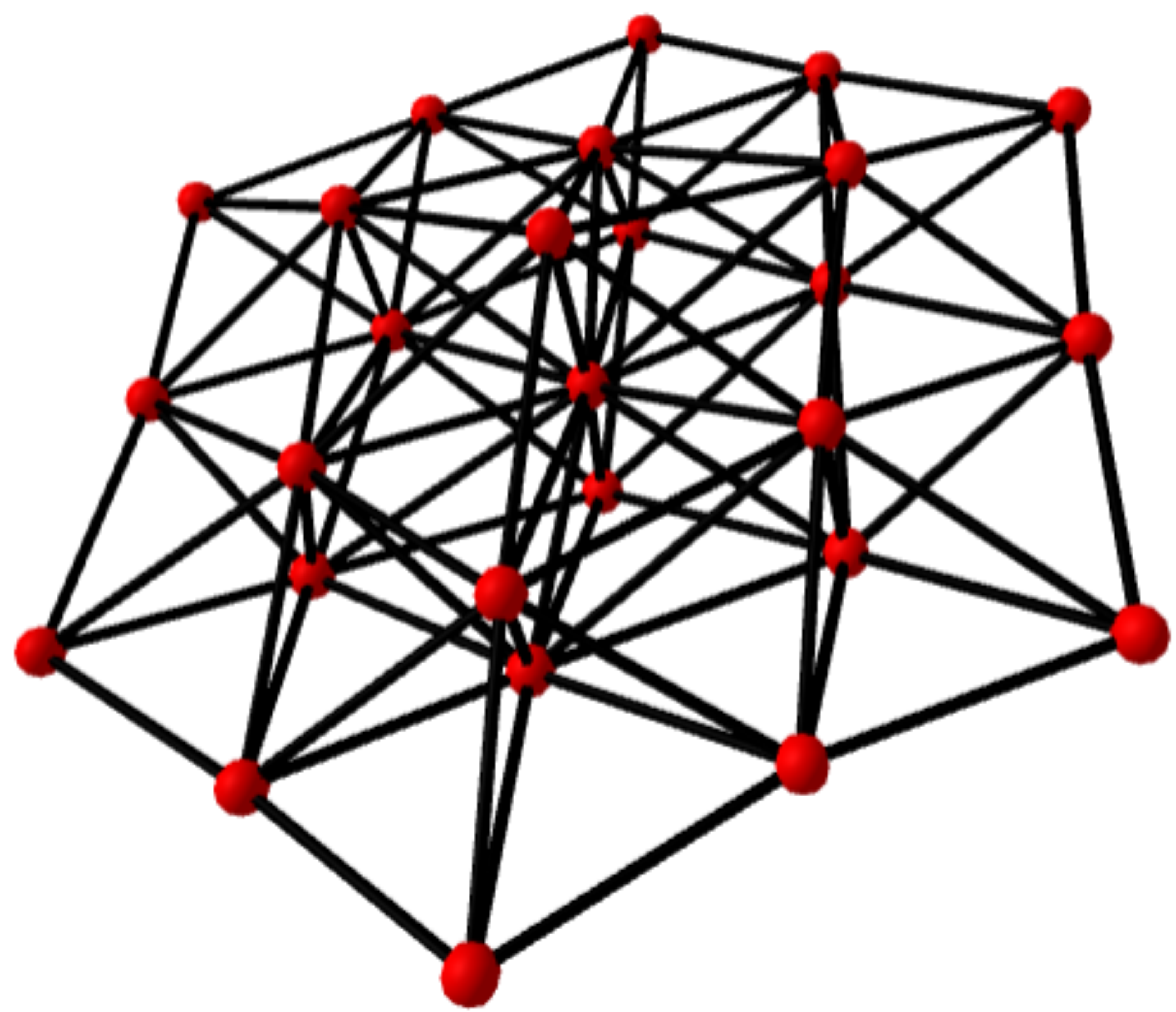
$\boxed{\text{Ex}}$ Pocket of 3×3 ASM: tetrahedral-octahedral honeycomb



(vertical edges omitted)

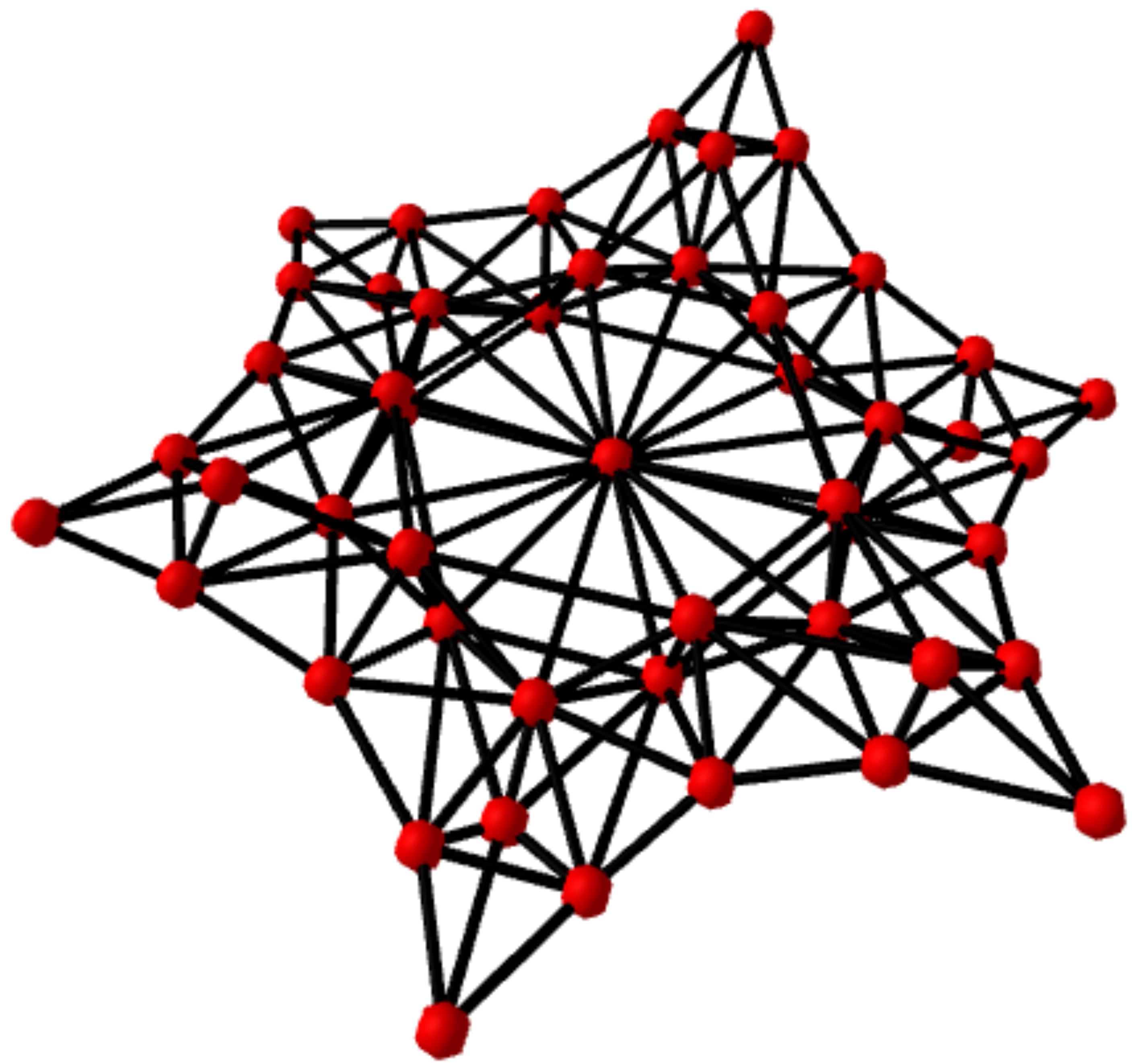
Pockets

Ex Pocket of $2\times2\times2$ PP:



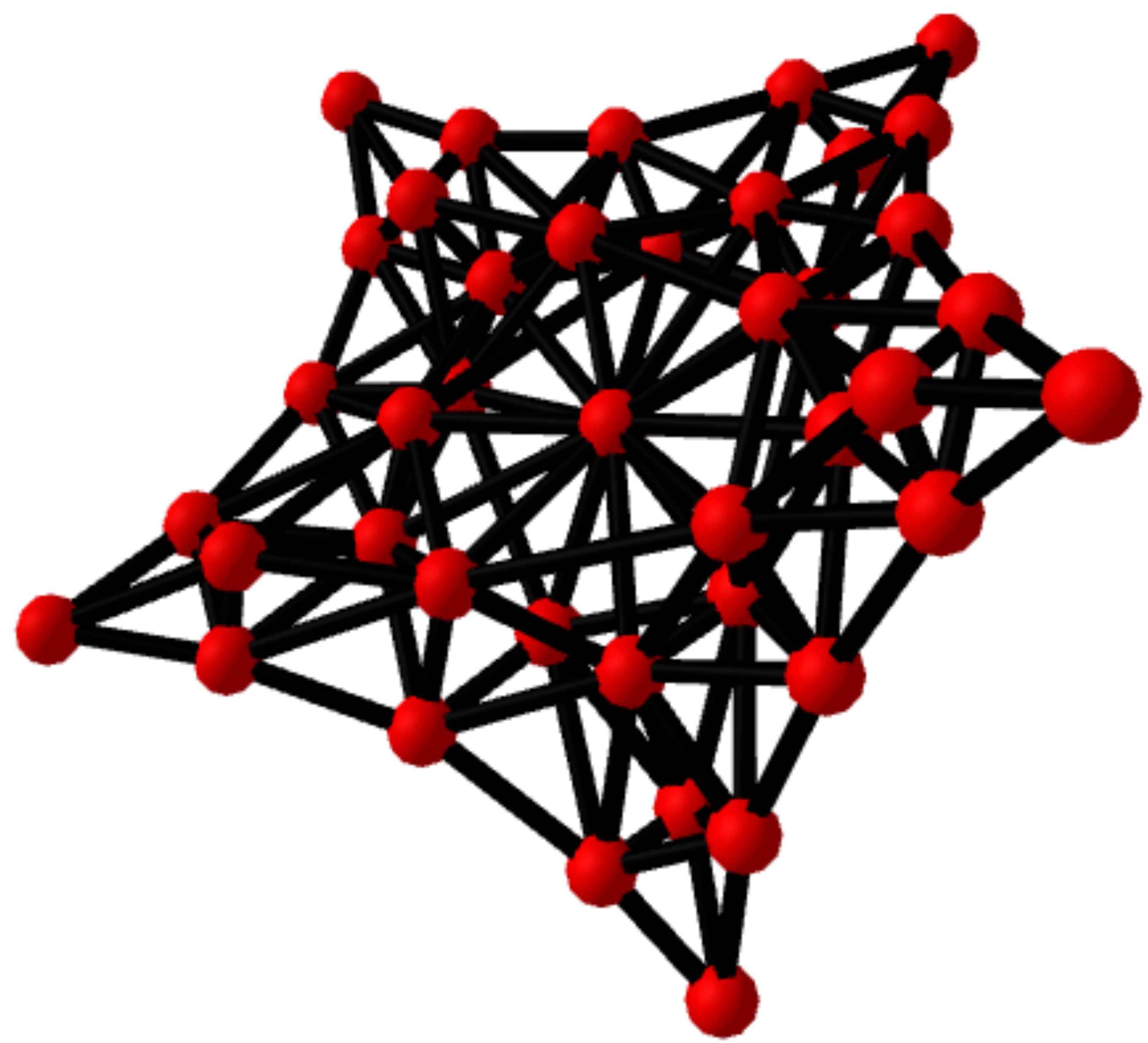
Pockets

Ex Pocket of a "chained hexagon":



Pockets

Ex Pocket of a "chained pentagon":



Note Not embeddable in $\mathbb{R}^3$

Pocket topology

Thm (GSSW)

- A pocket $P = P(\ell)$ is simply-connected, contractible, and CAT(0).
- P is a singular interval bundle over $\mathbb{D}$.
- P embeds in $\mathbb{R}^3$ in "nice" cases, though there can be embedded Klein bottles requiring $\mathbb{R}^4$!

Pocket dynamics

- Pockets remember the base face and boundary loop.
  - "Rotation" moves base face mark along
  - "Reflection" reverses boundary loop

$\boxed{\text{Thm}}$ (GSSW) The map from $SYT(4\times c)$ to pockets intertwines prom, evac with rotation, reflection.

Pocket geodesics

- We define a separation-style labeling on pockets using $trip_1, trip_2, trip_3$.
  - Agrees w/ growth labels from each $D(w) \hookrightarrow P$
  - Includes rules for labels on vertical edges
  - Proper!

Pocket geodesics

*Current proof requires one face to be on boundary

Thm (GSSW)

- Pockets have an SL(4) KLM distance.*
  - It has coherent geodesics.
  - The corresponding $T \in SYT(4 \times c)$ is encoded in boundary distances to base face.
  - Geodesics avoid double-crossing trips.

Note Sometimes geodesics must move through the pocket rather than a single dual diskoid!

Pocket embeddings

Thm (GSSW)

For a pocket P, $\exists$ isometric $P \hookrightarrow \Delta_4$

Rem
- Moves are a "feature" rather than a bug!
- Suggests higher rank web bases will have even more moves
- Seeking "top" reps has geometric meaning
- Pockets are generally non-affine (not in a single apartment)

Pocket embeddings

Thm (GSSW) For $T \in SYT(4 \times \frac{n}{4})$, every loop in $\mathcal{U}_T$ extends uniquely to an embedding $p : P(T) \hookrightarrow \Delta_4$ s.t.

$$d(F,G) = w_i \Rightarrow d(p(F), p(G)) = w_i.$$

Moreover, p is isometric.

– In this sense, pockets are generic finite models of the geometry of Δ_4!

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THANKS!