Weight-valued geodesics, webs, and buildings

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Based on joint work with subsets of *Christian Gaetz, Oliver Pechenik, Stephan Pfannerer, Jessica Striker, and Haihan Wu*

Slides: https://www.jpswanson.org/talks/2026_ASU_geodesics.pdf

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Continuous metrics

Recall | $\mathbb{R}$-valued metric: $d: X \times X \to \mathbb{R}$ s.t.

1| (Non-negative-valued) $d(x,y) \geq 0$

2| (Symmetry) $d(x,y) = d(y,x)$

3| (Non-degeneracy) $d(x,y) = 0 \Leftrightarrow x = y$

4| (Triangle inequality) $d(x,y) + d(y,z) \leq d(x,z)$

- A geodesic $\gamma: [a,b] \to X$ satisfies $d(\gamma(\alpha), \gamma(\beta)) = |\alpha - \beta|$

Discrete metrics

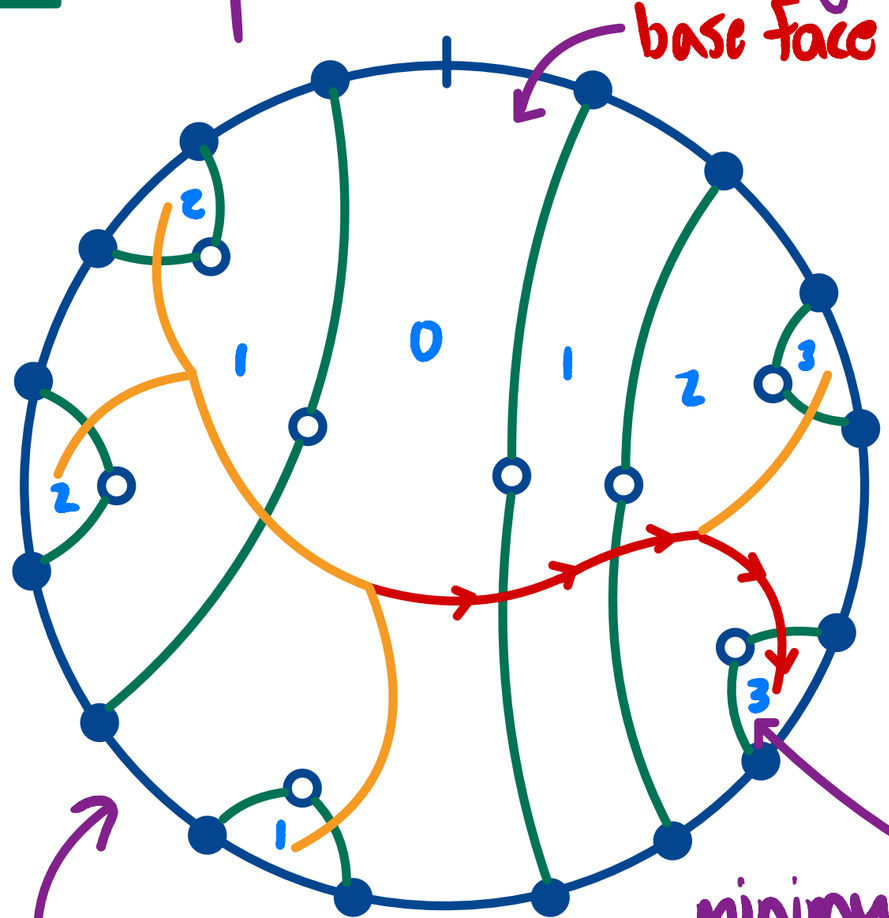
- Next: discrete, representation-theoretic analogues arising from work of Kapovich-Leeb-Millson, Fontaine-Kamnitzer-Kuperberg

- Key structure: combinatorial geodesics

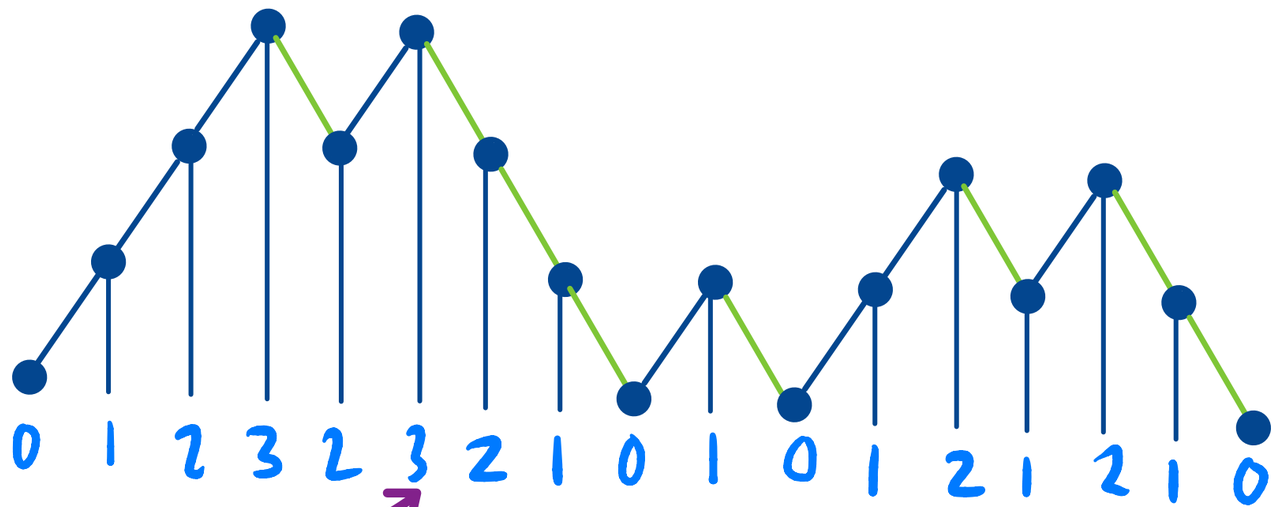
- Why? Crucial structure for affine building embeddings of web bases

Motivation

Ex Simple motivating example for sl_2:



1	2	3	5	9	11	12	14
4	6	7	8	10	13	15	16



minimum depth from base face

- Hence depths around the boundary encode the tableau

Weights

Recall · The $SL(r)$ dominant weights are Λ_+:

$$\lambda = (\lambda_1 \geq \dots \geq \lambda_r) \in \mathbb{Z}^r / \langle (1, \dots, 1) \rangle = \Lambda_r .$$

· The fundamental weights of $SL(r)$ are

$$\omega_i = (1^i, 0^{r-i})$$

$$\omega_i^* = (0^{r-i}, -1^i) \equiv \omega_{r-i} .$$

· If $\lambda, \mu \in \mathbb{Z}^r_{\text{doc}}$ where $|\lambda| = |\mu| + rk$ for some $k \in \mathbb{Z}$, say

$$\lambda \geq \mu \iff \forall m, \sum_{j=1}^{m} \lambda_j \geq \sum_{j=1}^{m} (\mu_j + k) .$$

KLM metrics on graphs

- Let G be a simple graph. Direct edges both ways, add self-loops.

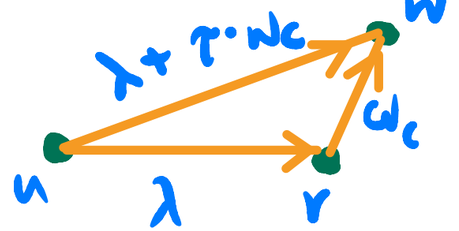
Def A KLM metric on G is d on such edges (u,v) s.t.

1| (Dominant-weight-valued) $d(u,v) \in \Lambda_+ = \mathbb{Z}^r_{dec}/(1^r)$

2| (Duality) $d(u,v) = d(v,u)^* = -rev(d(v,u))$

3| (Non-degeneracy) $d(x,y) = 0 \in \Lambda_+ \Leftrightarrow x = y$

4| (Relative triangle equality) If u,v,w is a triangle, $d(v,w) = w_c$, then $\exists \tau \in S_r$ s.t. $d(u,w) = d(u,v) + \tau \cdot d(v,w)$



KLM metrics on graphs

- Let

$$D_w(u,v) = \left\{ \sum_{i=1}^{k} d(\gamma_{i-1},\gamma_i) \;\middle|\; u=\gamma_0, v=\gamma_k,\ d(\gamma_{i-1},\gamma_i) \text{ fundamental} \right\} \subset \Lambda_+ .$$

- We have coherent geodesics from u to v if

$$\min D_w(u,v) = d(u,v)$$

exists uniquely.

Ex sl_2: $\Lambda_+ = \mathbb{Z}^2_{dec}/(1^2) \cong \mathbb{Z}_{\geq 0}$

$G = \mathbb{Z} =$ 

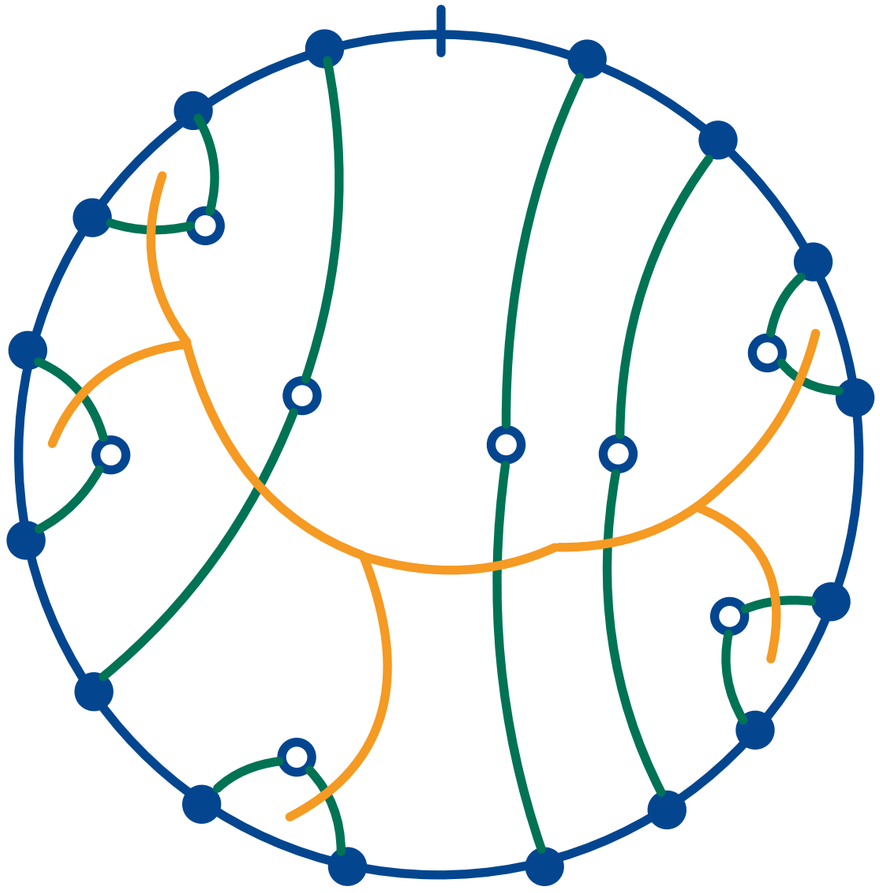
$d(n,n\pm1) = 1 \quad (= w_1)$

$d(n,m) = d(n,m\pm1) \pm 1$

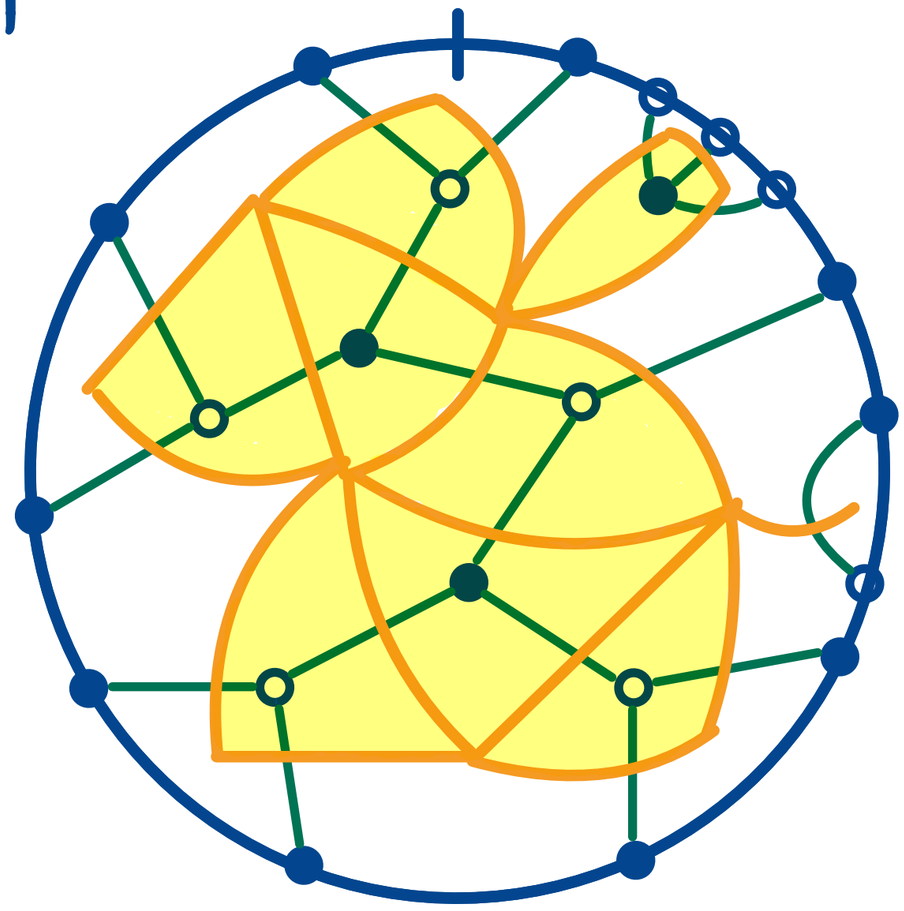
$d(n,m) = \min D_w(n,m) = |n-m|$

SL_3-web duals

Ex We're interested in dual graphs of webs:



vs.

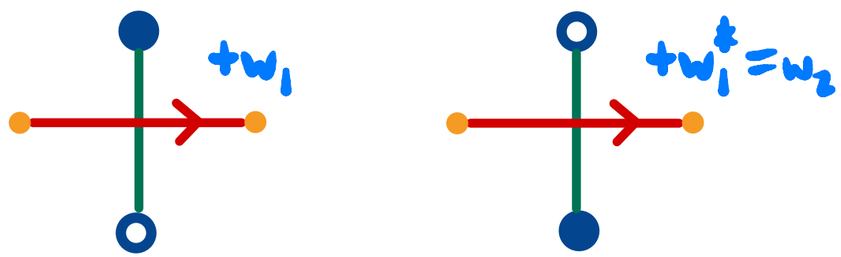


D(W) is 1D for SL(2)

D(W) is "2D" for SL(3)

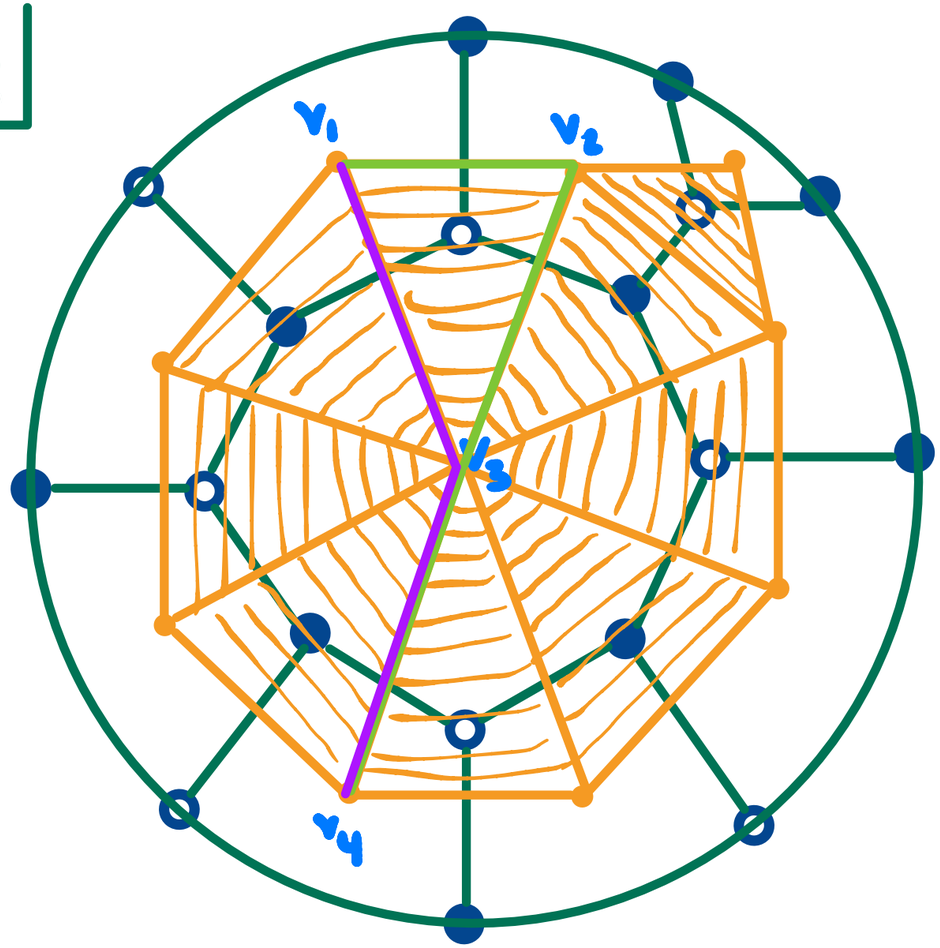
KLM Distances in Webs

Idea For an $SL(3)$ web, walk along the dual graph, picking up $SL(3)$ weights according to:



Call this the **distance** of the walk.

Ex



$$d(v_1 \xrightarrow{w_1} v_2 \xrightarrow{w_1} v_3 \xrightarrow{w_2} v_4) = 2w_1 + w_2 = (3,1,0)$$

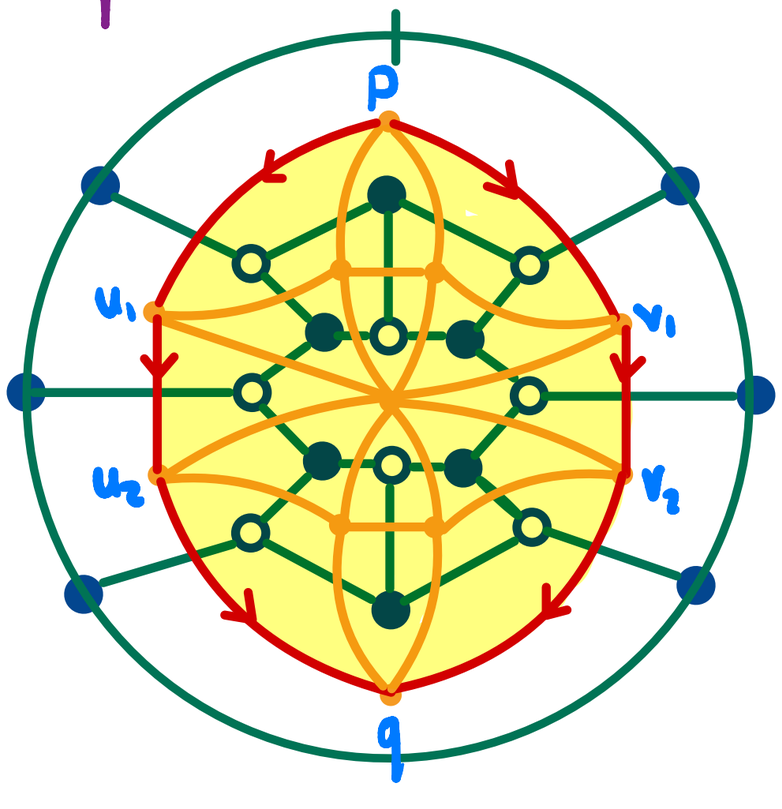
vs.

$$d(v_1 \xrightarrow{w_2} v_3 \xrightarrow{w_2} v_4) = 2w_2 = \underbrace{(2,2,0)}_{\text{smaller!}}$$

Geodesics

Thm ([Kuperberg '96], [Fontaine-Kamnitzer-Kuperberg '13])
If W is a non-elliptic $SL(3)$ web, it has coherent geodesics.

Ex Non-example:



$d(p \to u_1 \to u_2 \to q) = (3,3,0)$

$d(p \to v_1 \to v_2 \to q) = (3,0,0)$
$\equiv (4,1,1)$

incomparable!

$\Rightarrow$ not coherent!

(has squares)

Buildings

Q Why do they have such distance structure?

A They embed in the affine building $\Delta_3 = \Delta(SL_3^\vee)$.

big simplicial complex built from the affine Grassmannian

- [FKK] proved this for $D(W)$ w/ W non-elliptic

Affine Grassmannians

Def The affine Grassmannian of $SL(r)^\vee$ is

$$Gr = Gr_r = Gr(PGL(r)) = GL_r(\mathbb{C}((t)))/t^{\mathbb{N}}GL_r(\mathbb{C}[[t]]).$$

- Have points $[t^\lambda] = [\mathrm{diag}(t^{\lambda_1}, \dots, t^{\lambda_r})]$
- **Fact** Double coset reps $H\backslash G/H \xleftrightarrow{\sim} \{t^\lambda \mid \lambda \in \Lambda_+\}$
- Gives KLM distance $d: Gr \times Gr \to \Lambda_+$ w/coherent geodesics

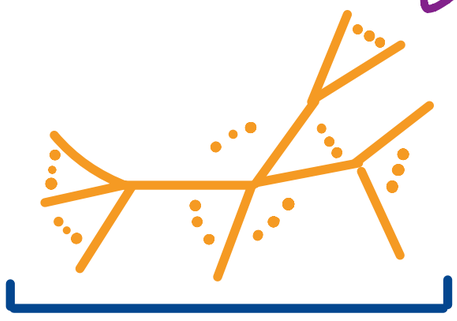
Ex $H(t^\lambda)^{-1} t^\mu H = H t^{\mathrm{sort}(\mu-\lambda)} H \Rightarrow d([t^\lambda],[t^\mu]) = \mathrm{sort}(\mu-\lambda)$

Affine Buildings

Def The affine building on G_{Γ_r} is the simplicial complex Δ_r whose vertices are the points of G_{Γ_r} and whose simplices are collections of points all of whose distances are fundamental weights.

Ex $t^{110}H$ (w_1), $t^{100}H$ (w_1), $t^{000}H$ (w_1) $\in \Delta_3$

Ex $\Delta_2 =$

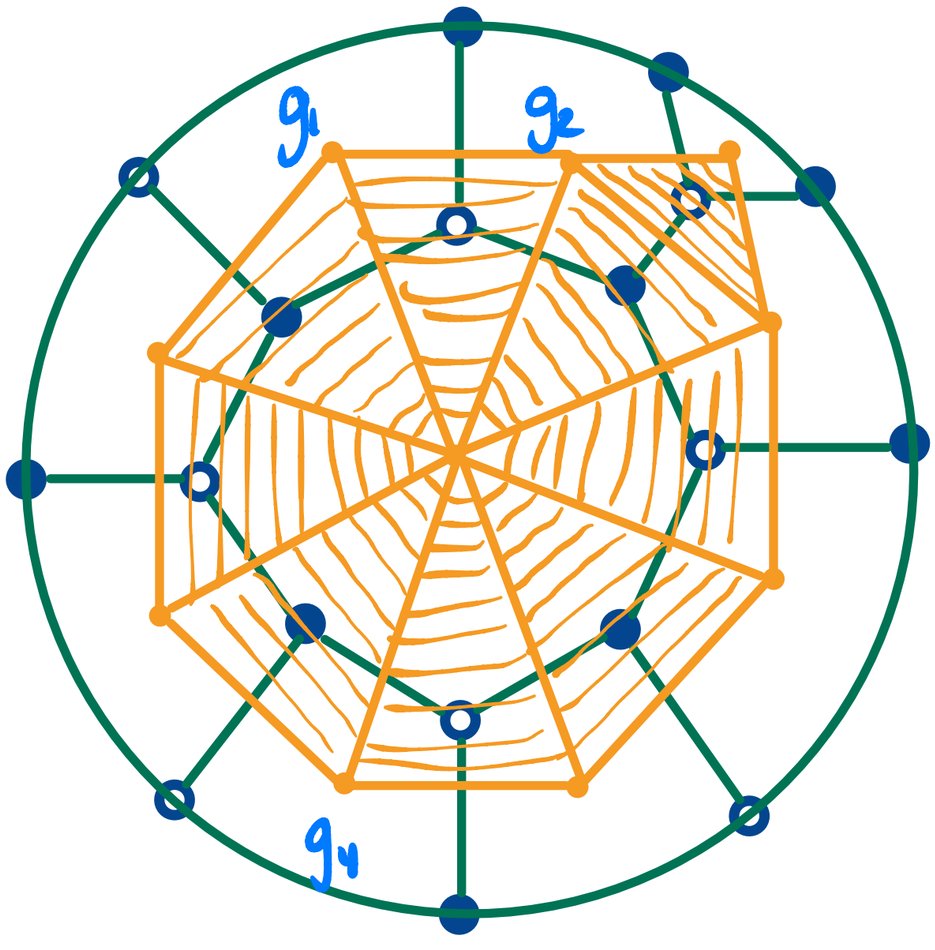


tree where every vertex has uncountable degree (Bruhat-Tits tree)

Affine Buildings

Thm ([FKK]) For W non-elliptic, $\exists$ isometric $D(W) \hookrightarrow \Delta_3$.

Ex There exist $g_1, g_2, g_3, g_4, \ldots \in Gr_3$ s.t.



$d(g_1, g_2) = w_1$

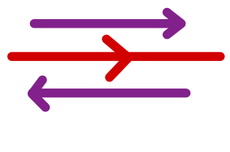
$d(g_2, g_3) = w_1$

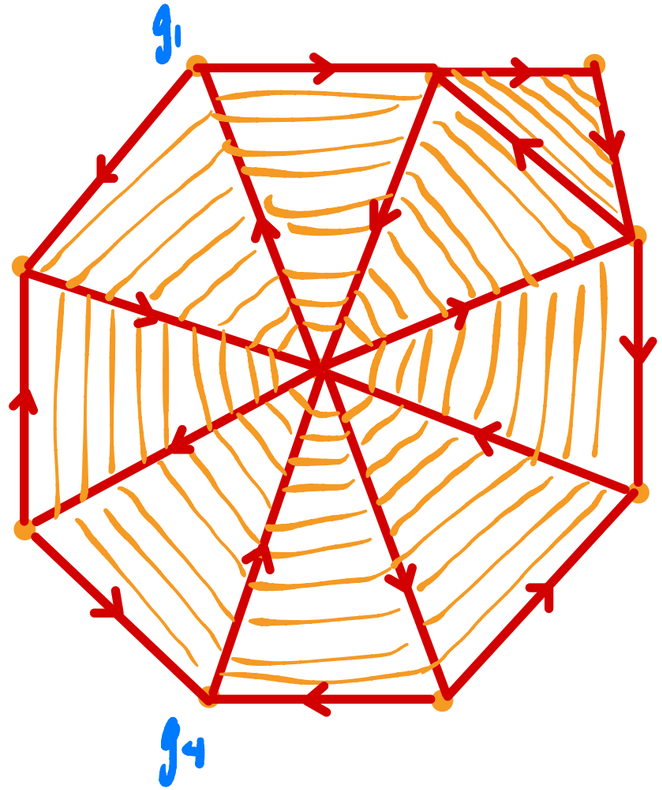
$\vdots$

$d(g_4, g_1) = 2w_1$

$\vdots$

geodesic distance





in Δ_3

Hourglass plabic graphs

Thm ([Gaetz-Pechenik-Pfannerer-Striker-S. '26])

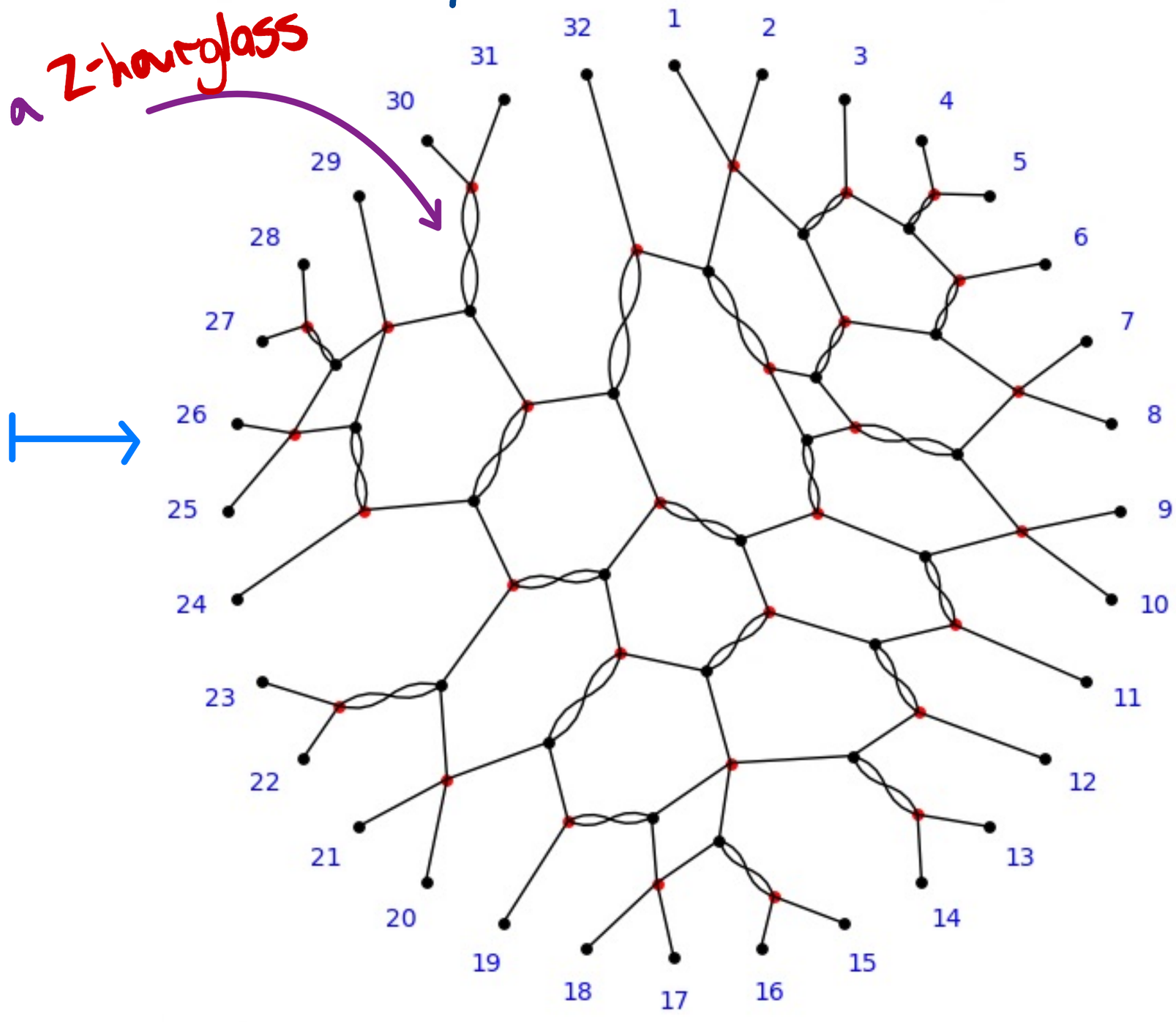
There is a (rotation-invariant) $SL(4)$ web basis consisting of top, fully reduced, 4-row hourglass plabic graphs.

Cor Effective equality testing in $SL(4)$-mod!

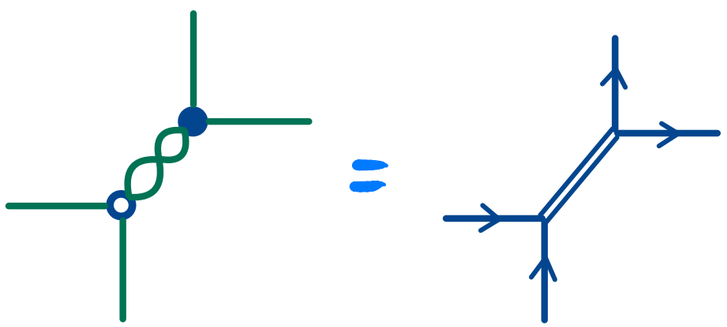
HPG basis example

Ex

1	3	4	7	8	17	19	23
2	5	6	9	14	18	21	24
10	12	13	15	16	25	26	28
11	20	22	27	29	30	31	32



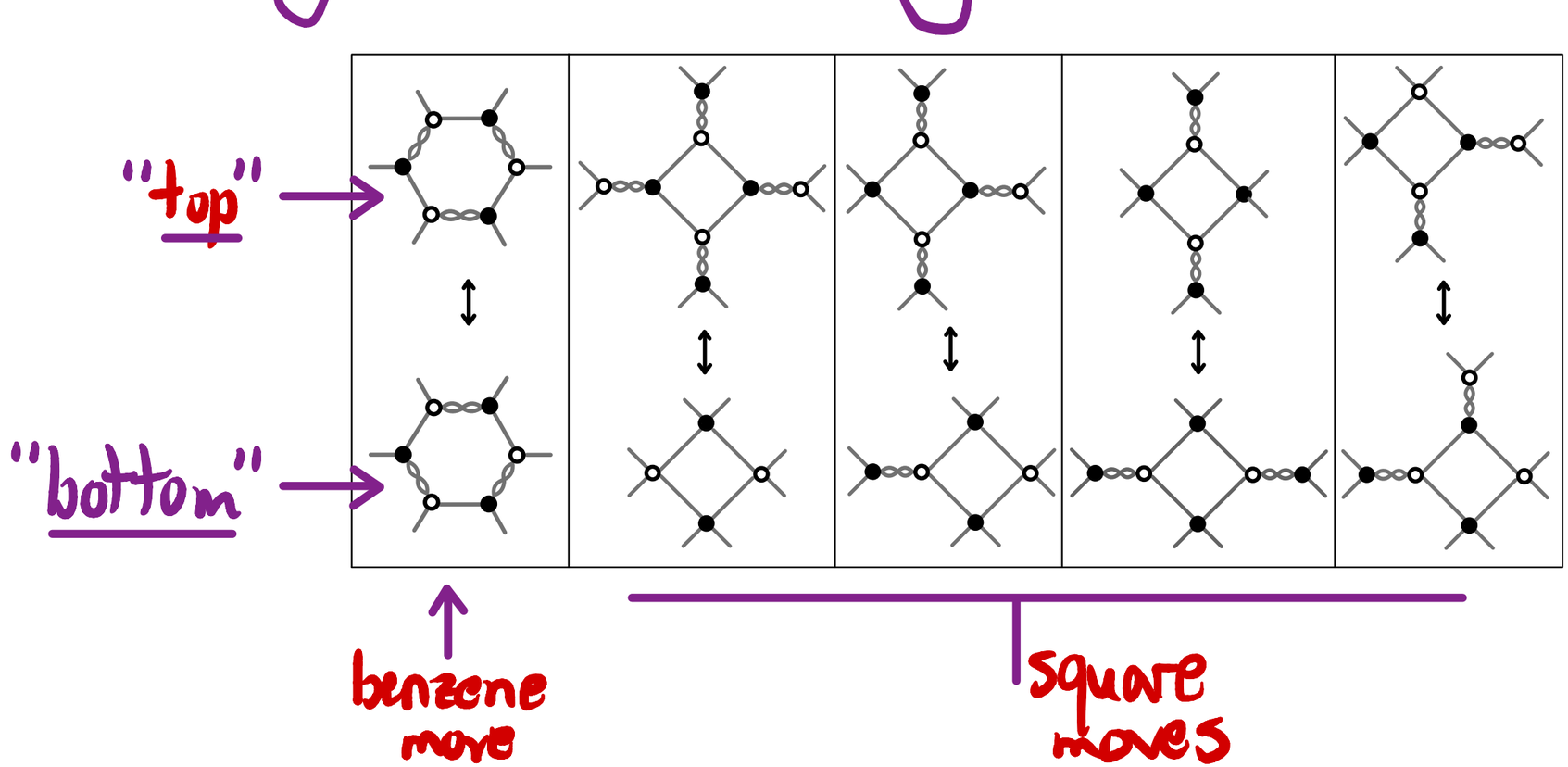
Note



r=4 moves

Thm (GPPSS '26) Two contracted, fully reduced 4-HPG's have the same trip permutations

$\Leftrightarrow$ they are related by moves:



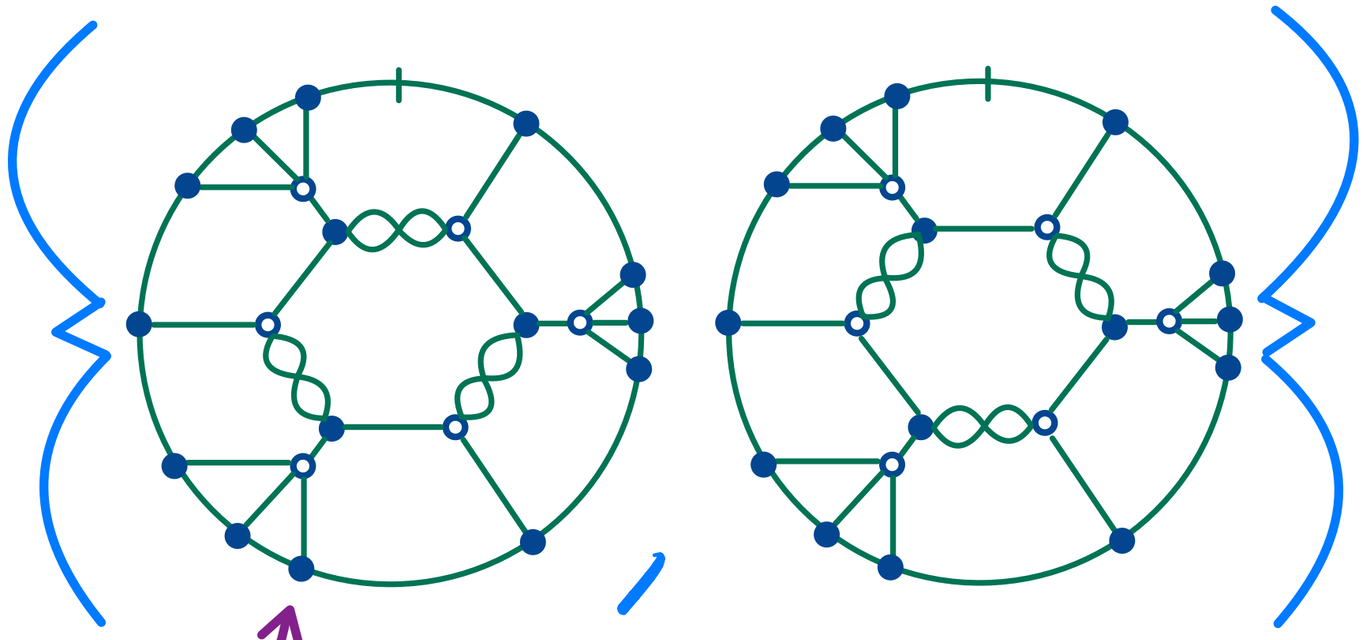
r=4 moves

Ex

$T =$

1	2	6
3	5	10
4	7	11
8	9	12

$\longmapsto$



Note • There is a notion of $\mathbb{Sl}_4$-valued distance

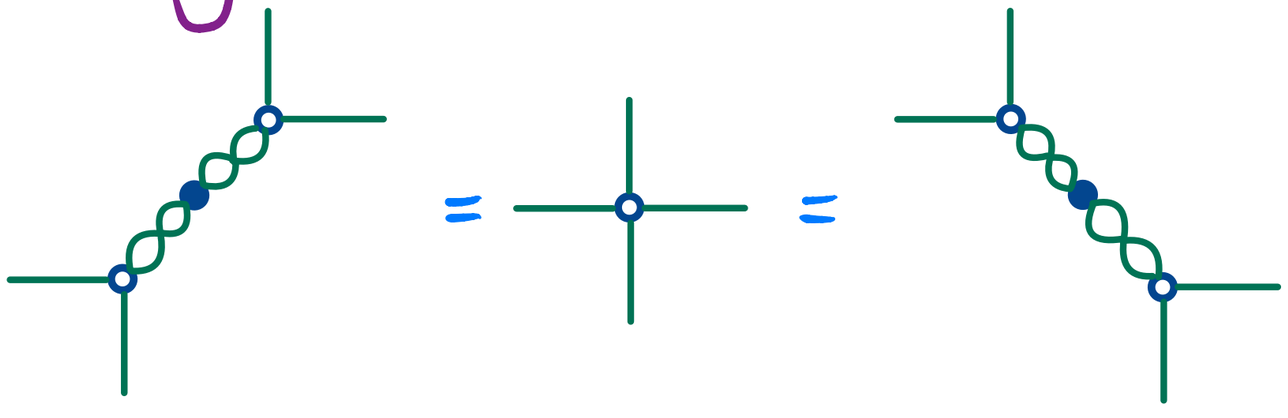
• Without moves, can fail to be coherent!

Pockets

Def (Gaetz-Striker-S.-Wu)

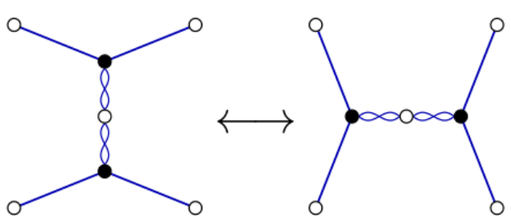
Let $\mathcal{C}$ be the move-class of some fully reduced 4-HPG W. Then there is a 3D flag simplicial complex $P(\mathcal{C})$ called the <u>pocket</u> of $\mathcal{C}$, which $D(W)$ embeds into.

- Requires expanding sources/sinks via I=H moves:

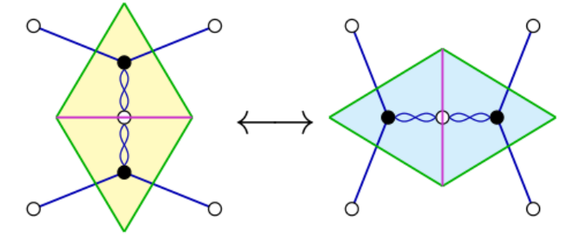


Pockets

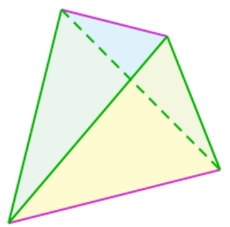
- Build a pocket from a move-class by gluing D(w)'s with tetrahedra between moves:



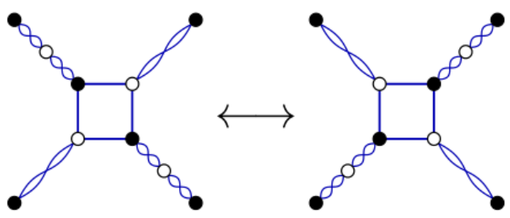
IH move between webs



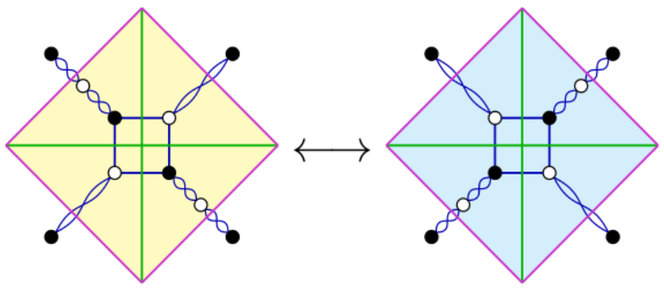
Dual diskoids



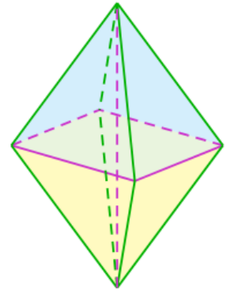
tetrahedron



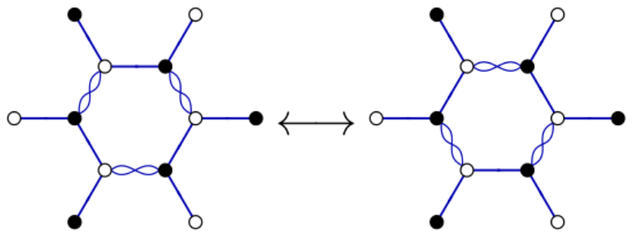
Square move between webs



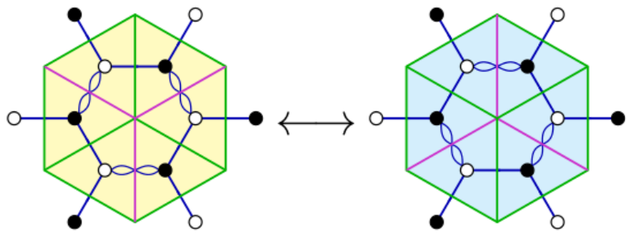
Dual diskoids



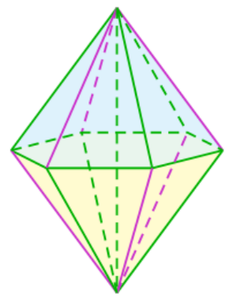
octahedron



Benzene move between webs



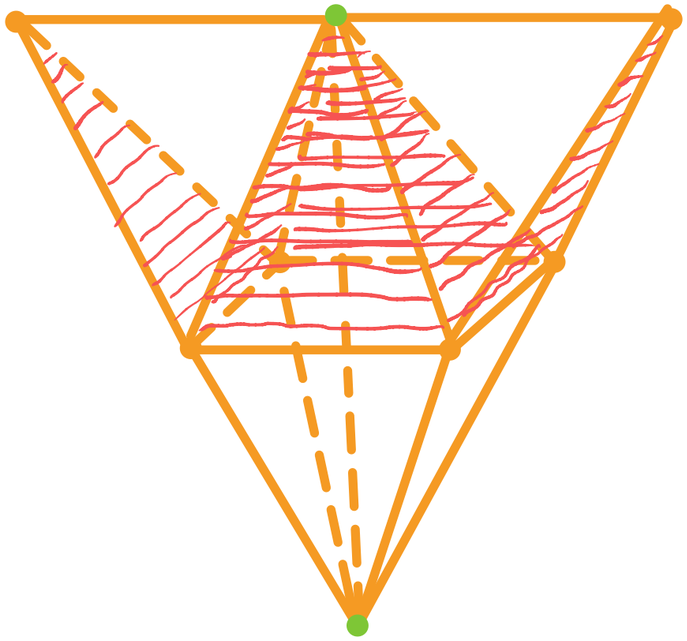
Dual diskoids

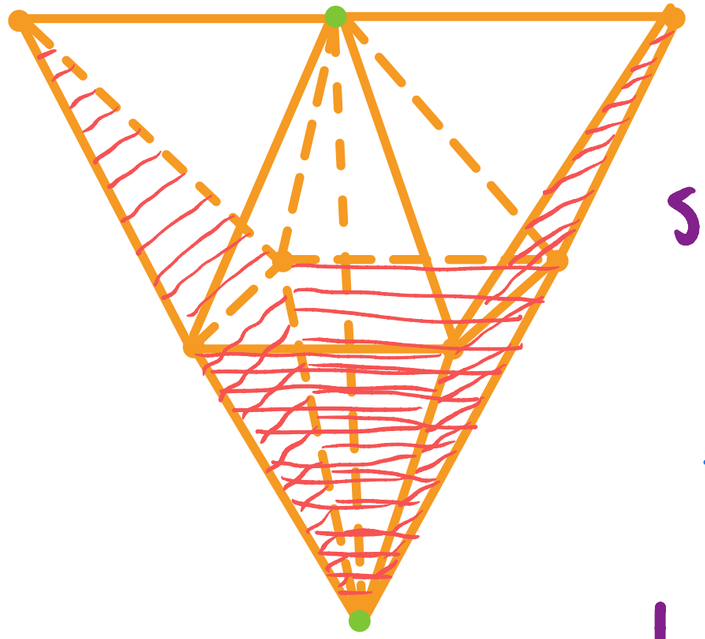


dodecahedron

Pockets

Ex





Pocket of square + benzene + IH

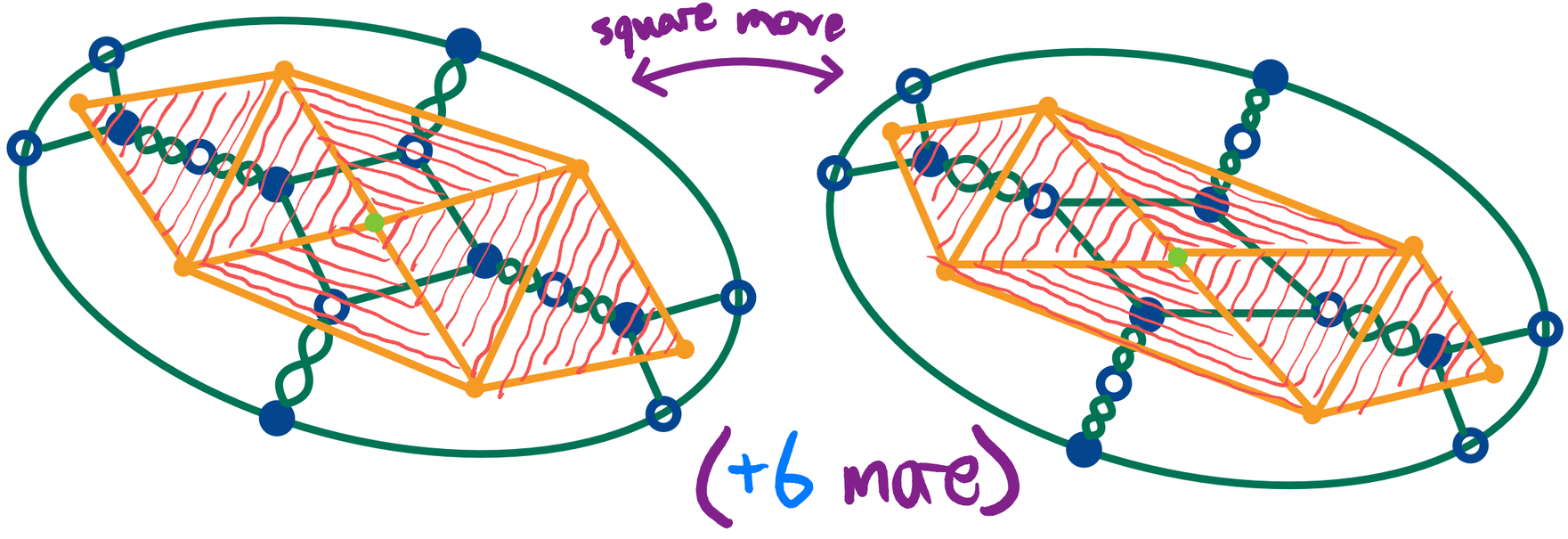
class of $\overline{4}\overline{6}1,23\overline{3}\overline{2}\overline{5}3,45\overline{7}$

has 6 tetrahedra

14 triangles

19 edges

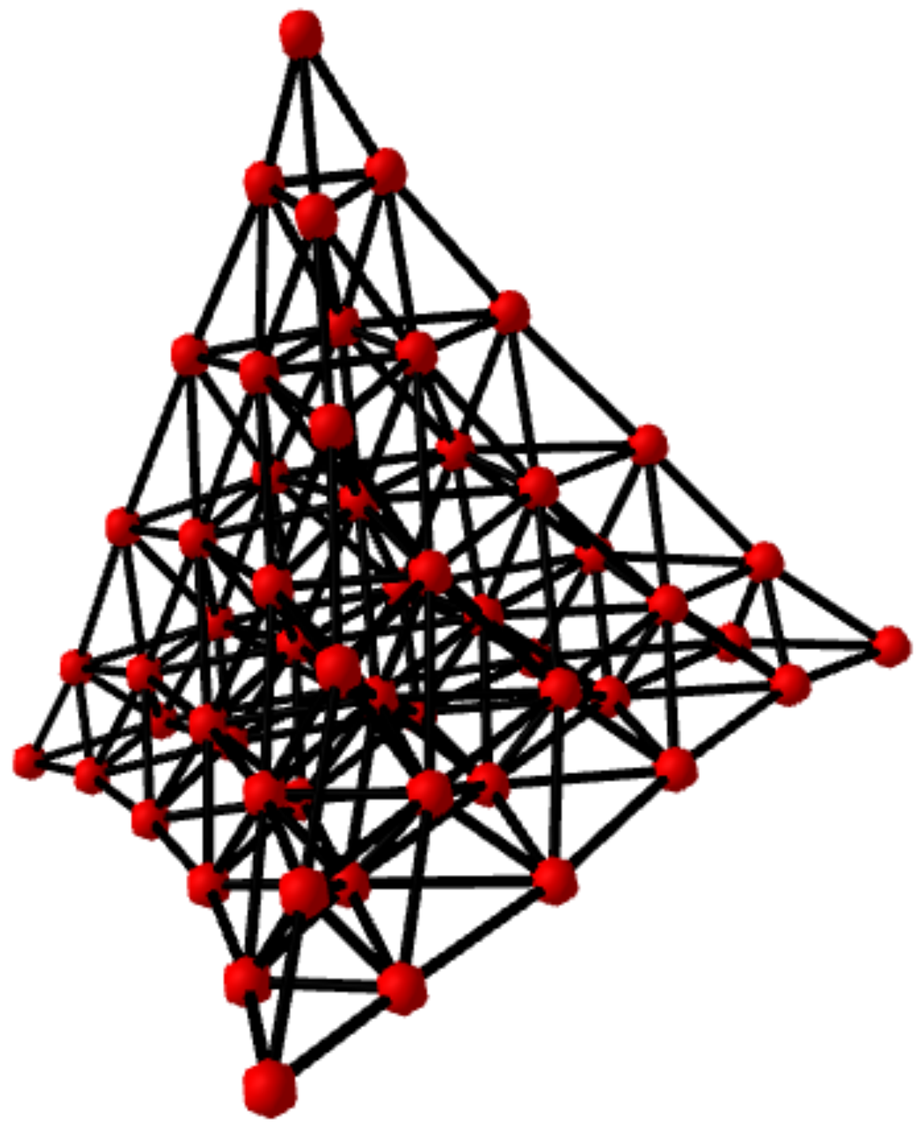
8 vertices



*

Podiets

Ex Podiet of 5×5 ASM:



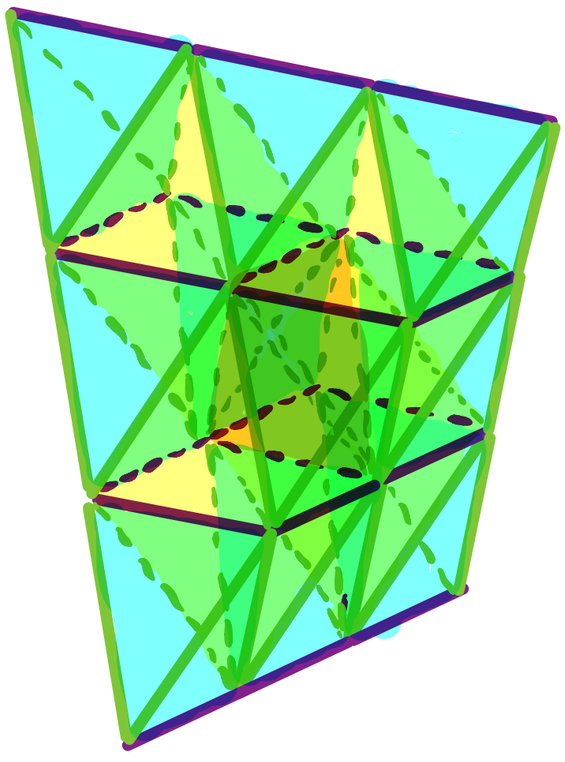
$L = 1^5 2^5 3^5 4^5$

$$T = \begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 & 20 \end{matrix}$$

Note Related to height functions, octahedral recurrence, tilings of the Aztec diamond, distributive lattices

Polets

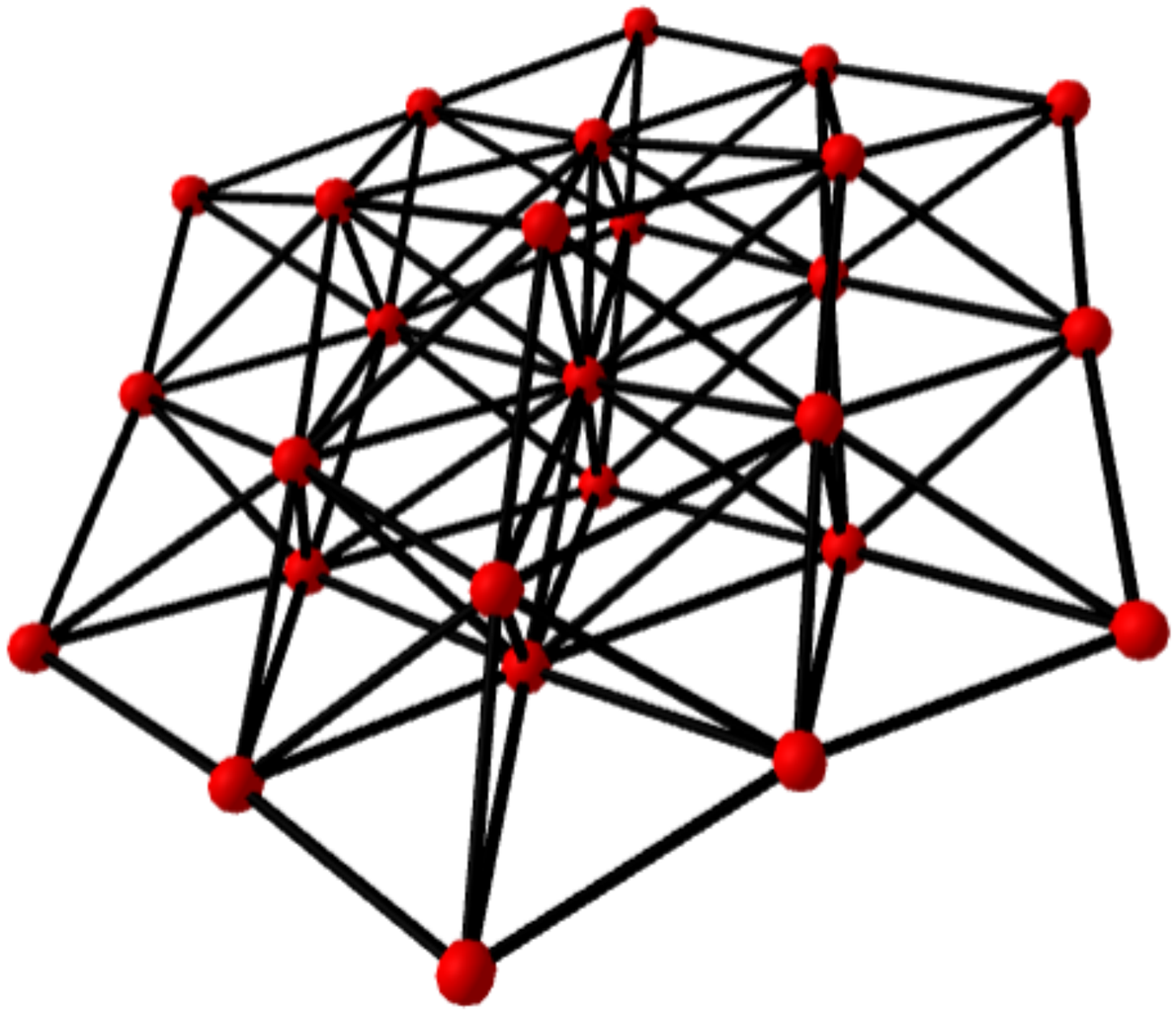
Ex Polet of 3×3 ASM: tetrahedral-octahedral honeycomb



(vertical edges omitted)

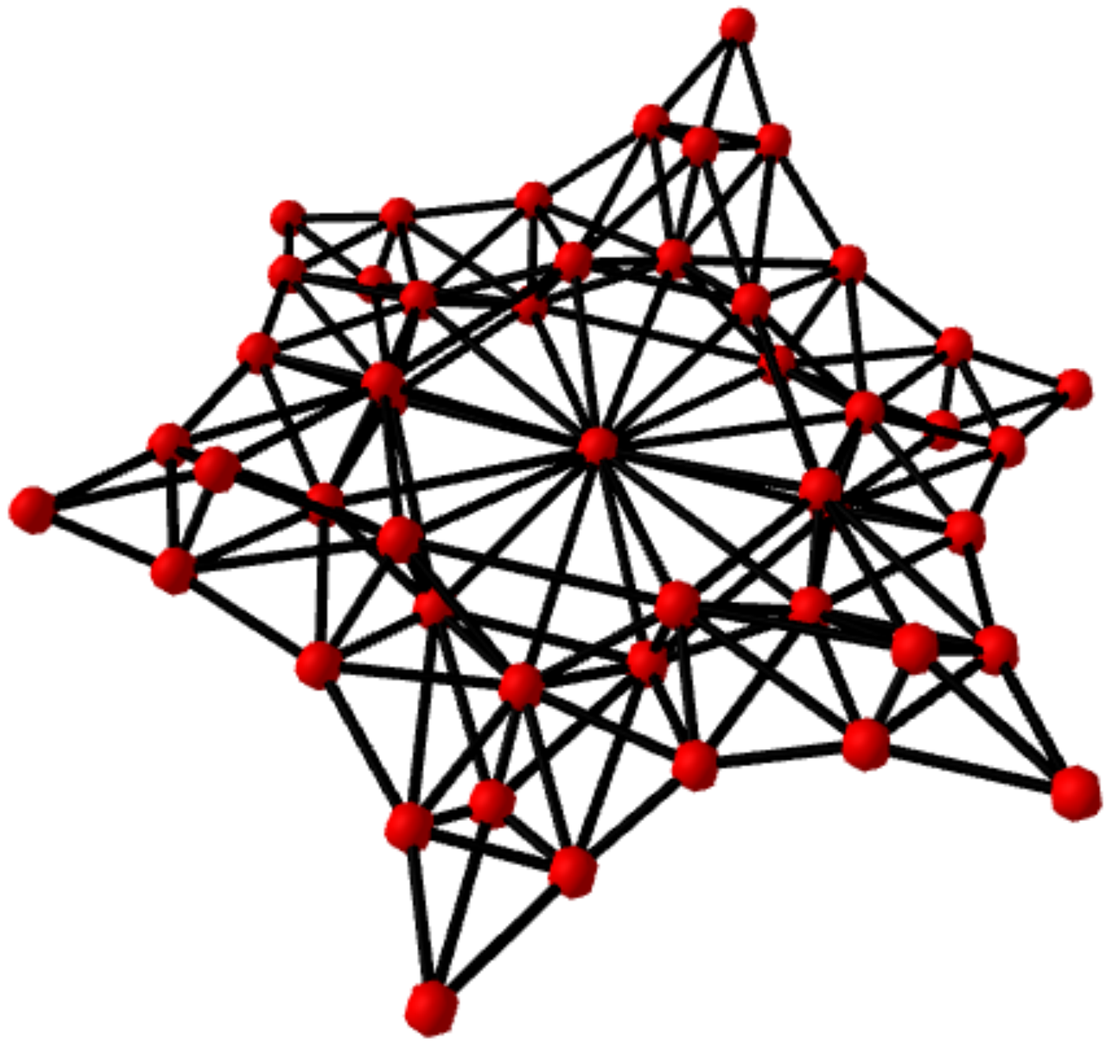
Products

Ex Product of $2 \times 2 \times 2$ PP:



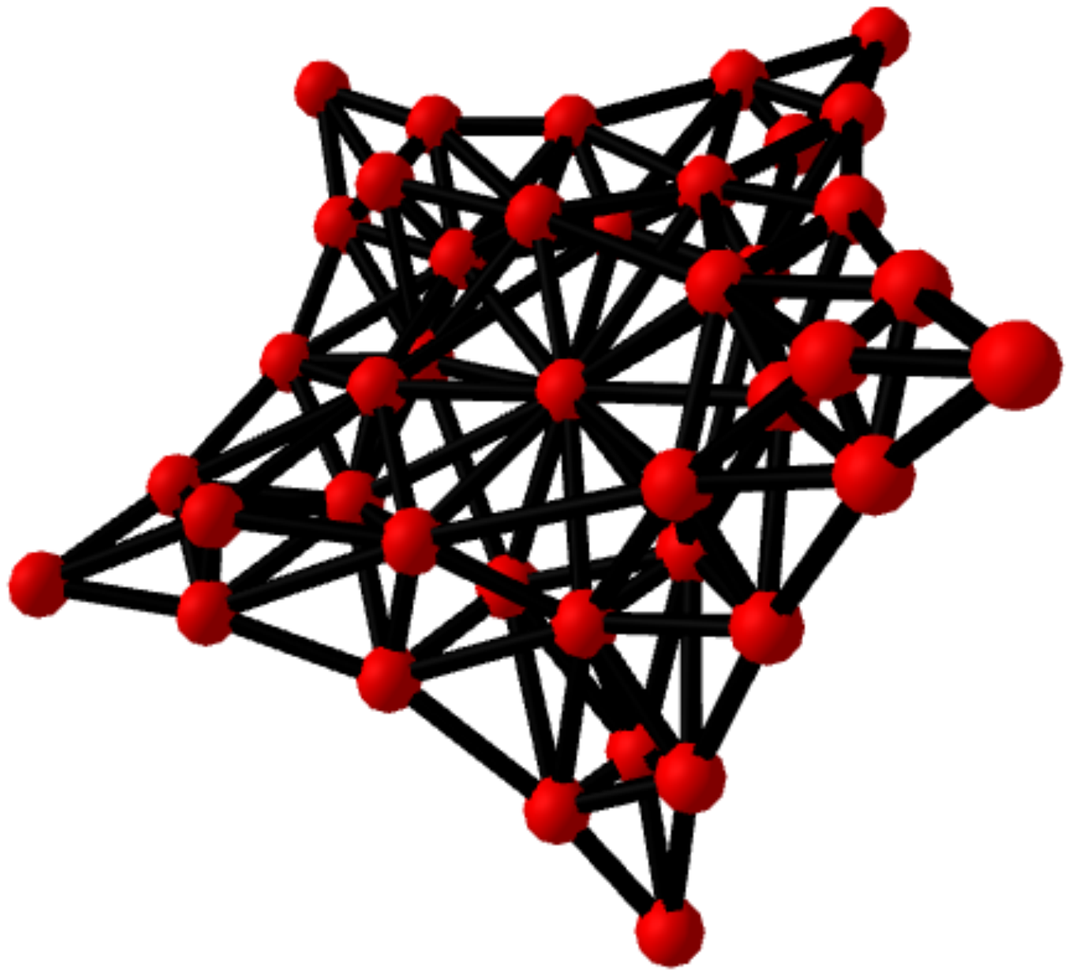
Pockets

Ex Pocket of a "chained hexagon":



Pockets

Ex Pocket of a "chained pentagon":



Note Not embeddable in $\mathbb{R}^3$

Geodesics

Thm (Gaetz-Striker-S.-Wu)

- Pockets have an $SL(4)$ KLM distance.
  - It has coherent geodesics.
  - The corresponding $T \in SYT(4 \times c)$ is encoded in boundary distances to base face.
  - Geodesics avoid double-crossing trips.

Note Sometimes geodesics must move through the pocket rather than a single dual disloid!

Affine Buildings

Thm (Gaetz-Striker-S.-Wu)

For a poset P, $\exists$ isometric $P \hookrightarrow \Delta_4$

Rem
- Moves are a "feature" rather than a bug!
  - Suggests higher rank web bases will have even more moves
  - Seeking "top" reps has geometric meaning
  - Posets are generally non-affine (not in a single apartment)

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THANKS!