Webs, pockets, and buildings

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Based on joint work with subsets of *Christian Gaetz, Oliver Pechenik, Stephan Pfannerer, Jessica Striker, and Haihan Wu*

arXiv:2306.12501 (4-row)
arXiv:2402.13978 (2-column)
arXiv:2306.12506 (promotion permutations)

Slides: https://www.jpswanson.org/talks/2025_CanaDAM_pockets.pdf

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Outline

- SL_2-webs and SL_3-webs and bases
- Combinatorial geodesics
- SL_4 web basis and pockets
- Building embeddings

SL_2-Webs

Def An SL_2-web is

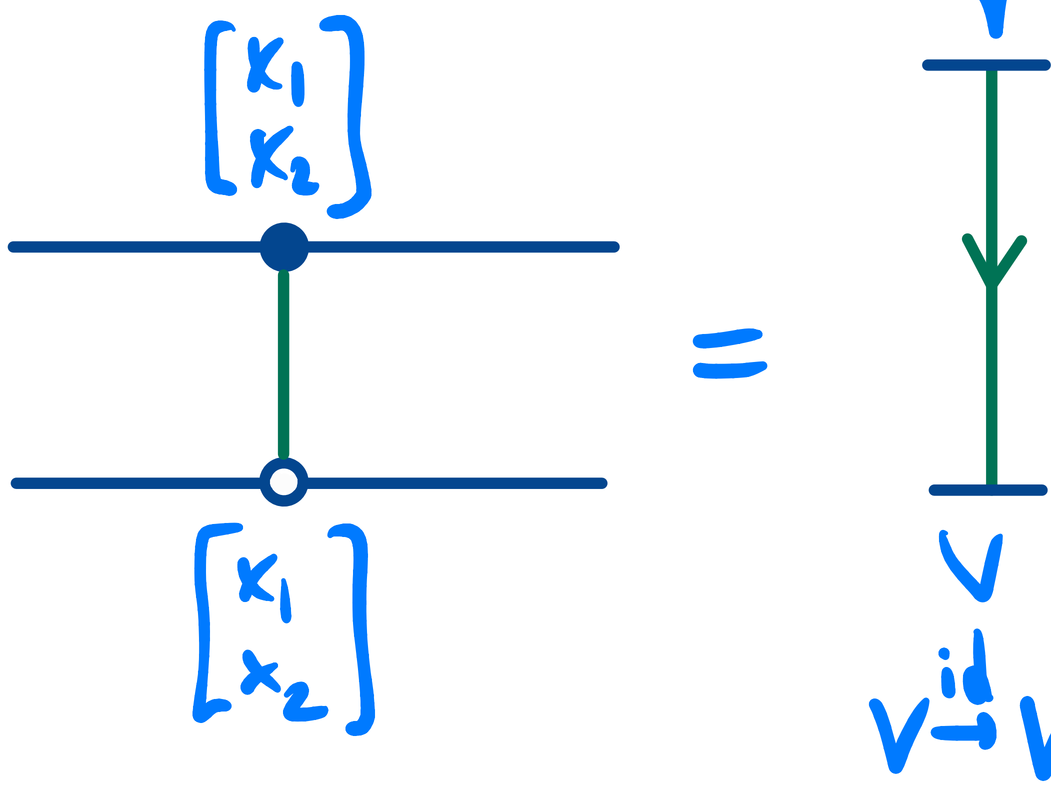
- a bipartite graph embedded in

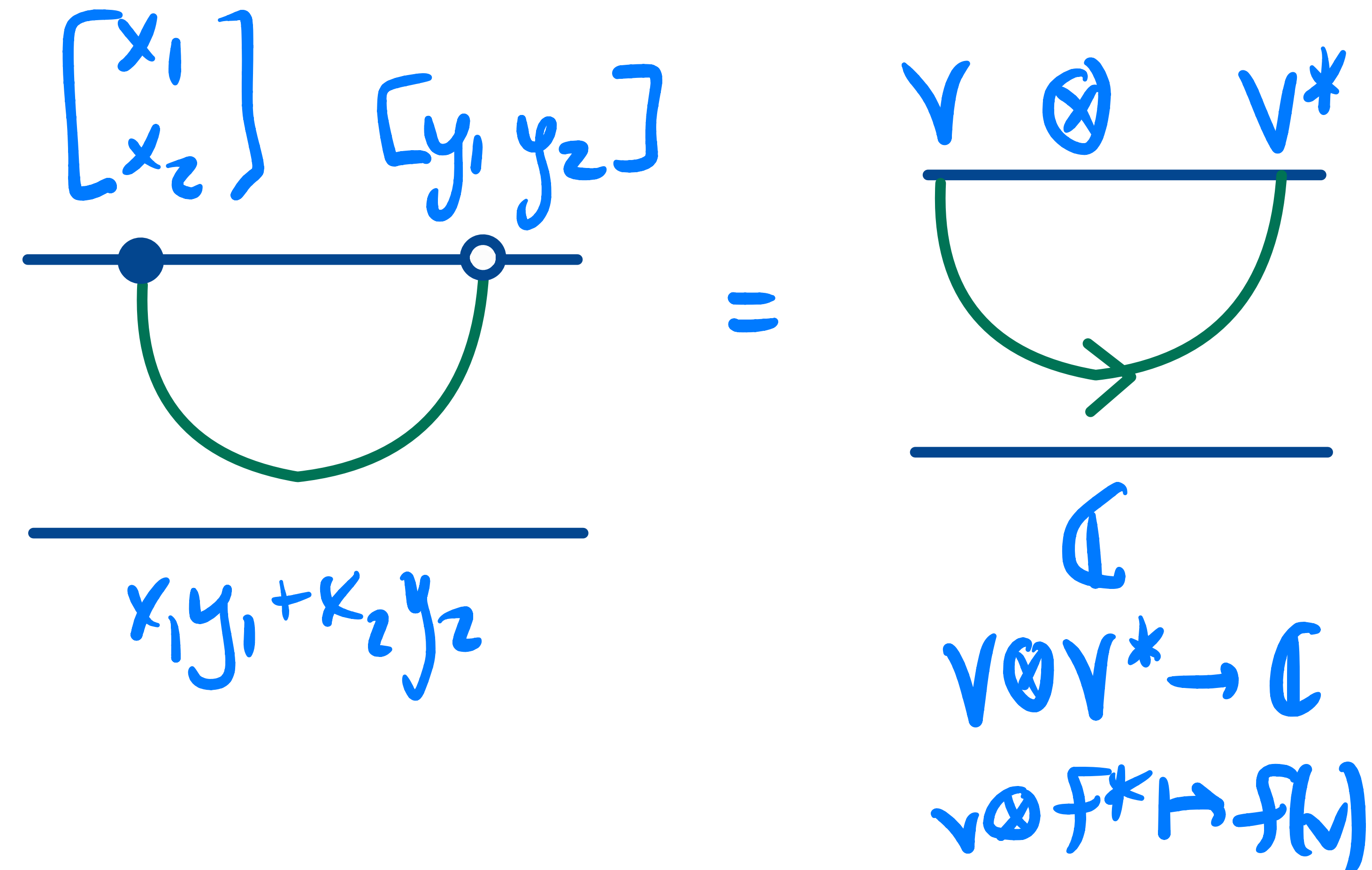
- with degree 2 internal vertices,
- and degree 1 boundary vertices.

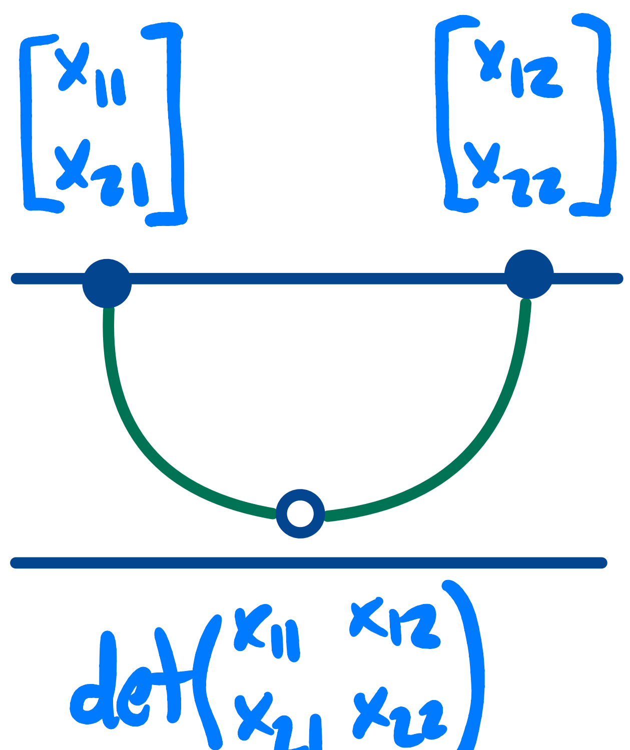
Fact Fully encodes category of SL_2-modules

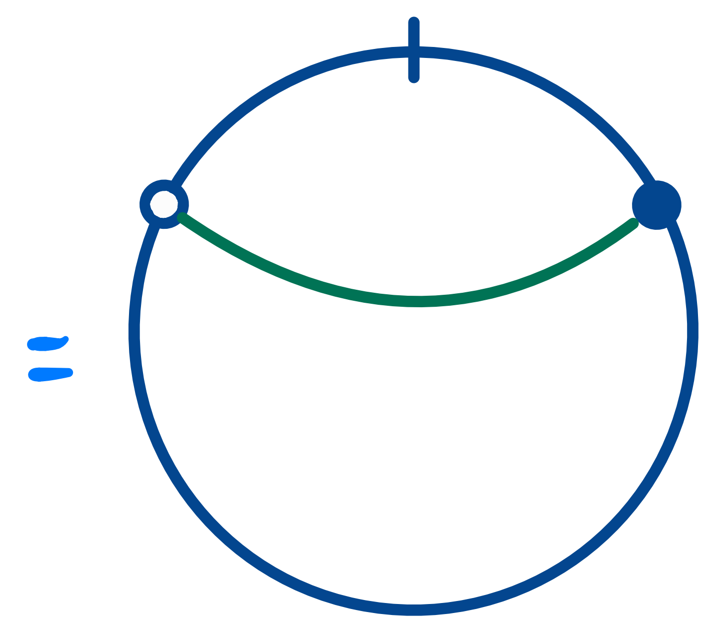
SL_2-Webs

Ex Let $V = \mathbb{C}^2$.









SL_2-Webs

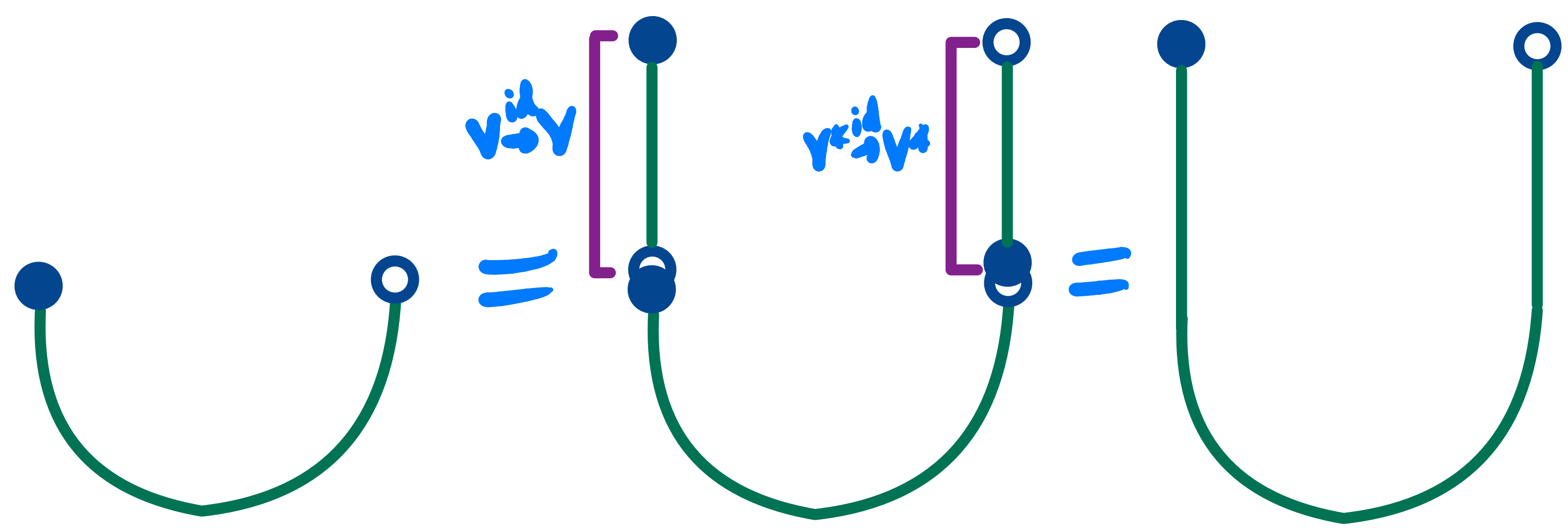
- Bipartite composition conventions

  ● $= V$ ○ $= V^*$ in domain

  ○ $= V$ ● $= V^*$ in codomain

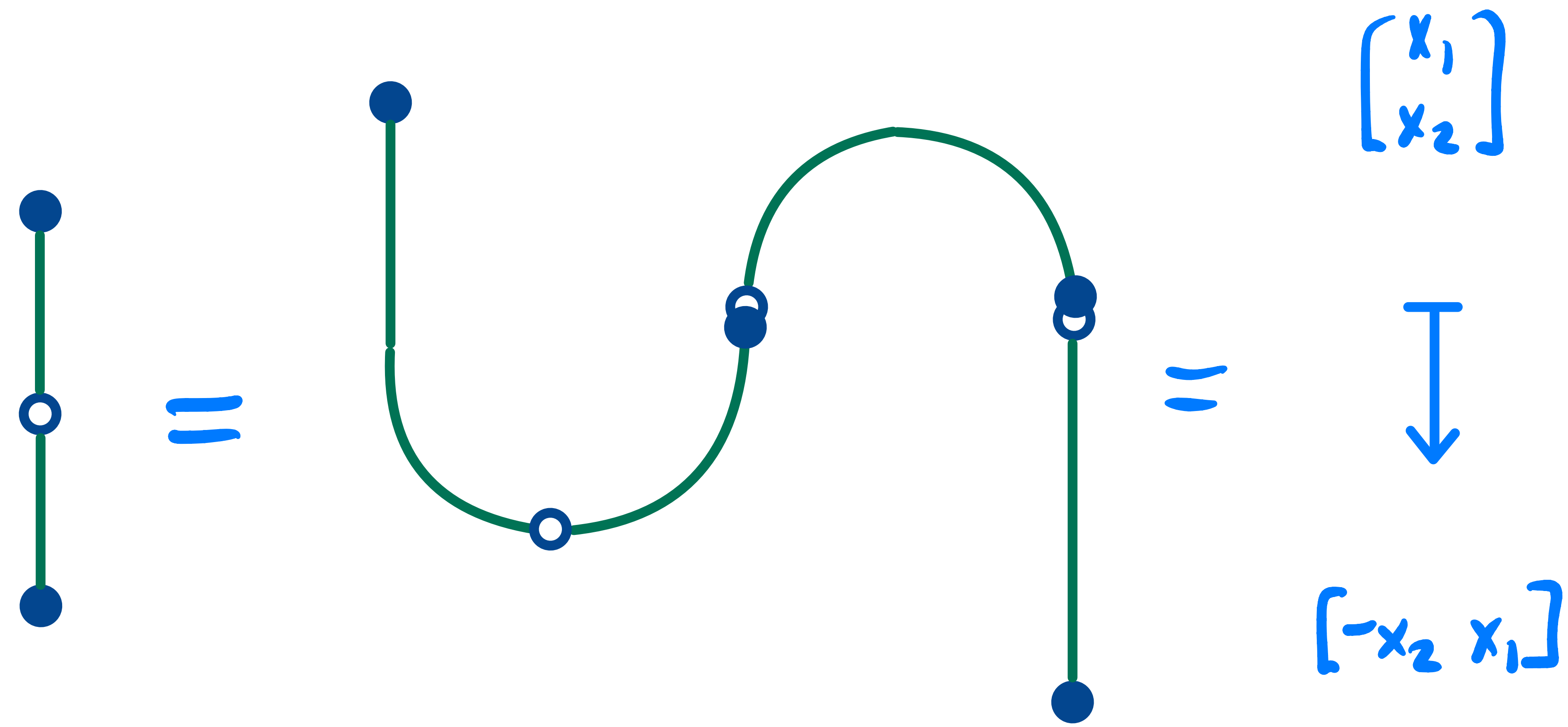
  — When composing, must match and cancel ●○ pairs:

Ex

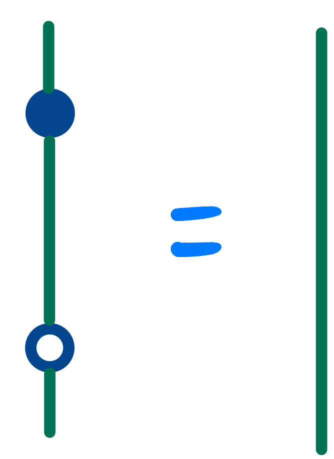


SL_2-Webs

Ex



Ex Have contraction relation:



SL_2-Webs

Facts

1. Have well-defined map

$$\{SL(2)\text{ webs}\}/\text{isotopy} \longrightarrow \left\{\begin{array}{c}SL(2)\text{ morphisms}\\ V^{\pm}\otimes V^{\pm}\otimes\cdots \to V^{\pm}\otimes\cdots\end{array}\right\}$$

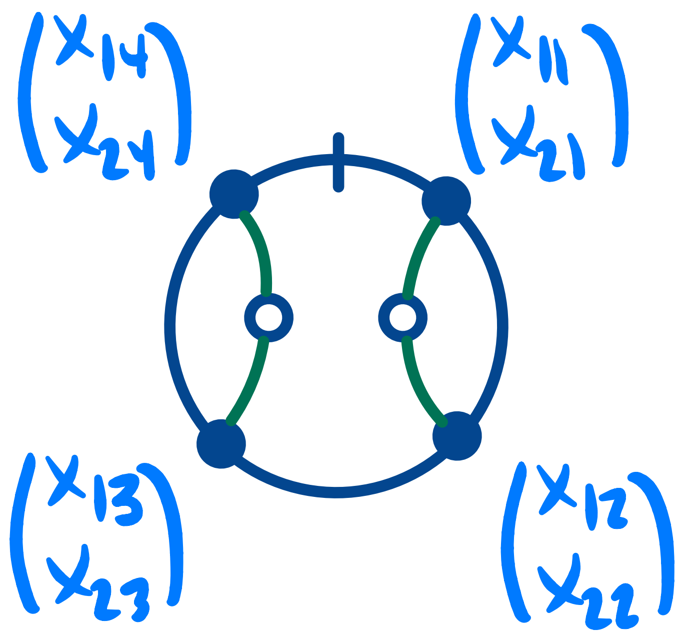
2. All of $Rep(SL_2)$ is encoded in disk case

(... surjective up to linear combinations, take Karoubi envelope...)

Q Generators and <u>relations</u>? <u>Bases</u>??

SL₂-Webs

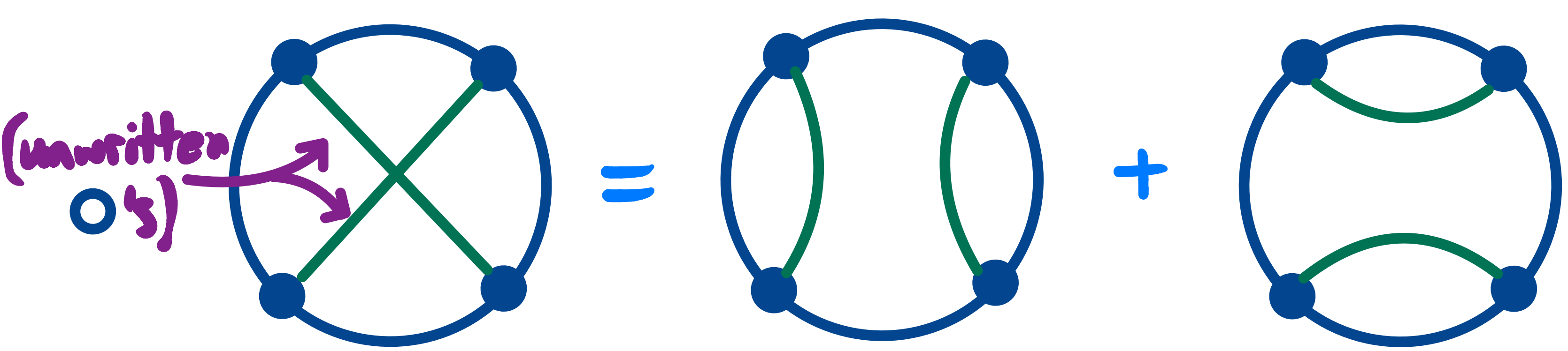
Ex



$$= \det\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \cdot \det\begin{pmatrix} x_{13} & x_{14} \\ x_{23} & x_{24} \end{pmatrix} \in \mathrm{Inv}(V^{\otimes 4})$$

$$= \mathrm{Hom}_{SL_2}(V^{\otimes 4}, \mathbb{C})$$

Ex Plücker relations:



$$(x_{11}x_{23} - x_{21}x_{13})(x_{12}x_{24} - x_{22}x_{14})$$

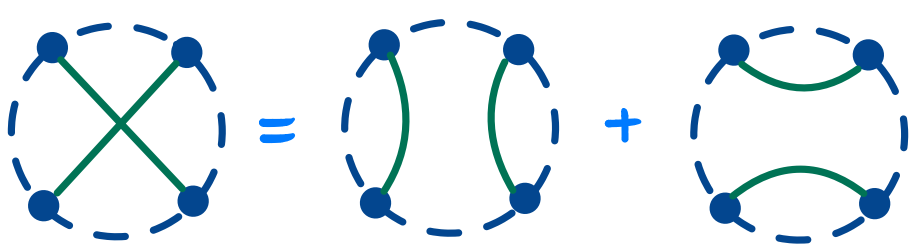
$$=$$

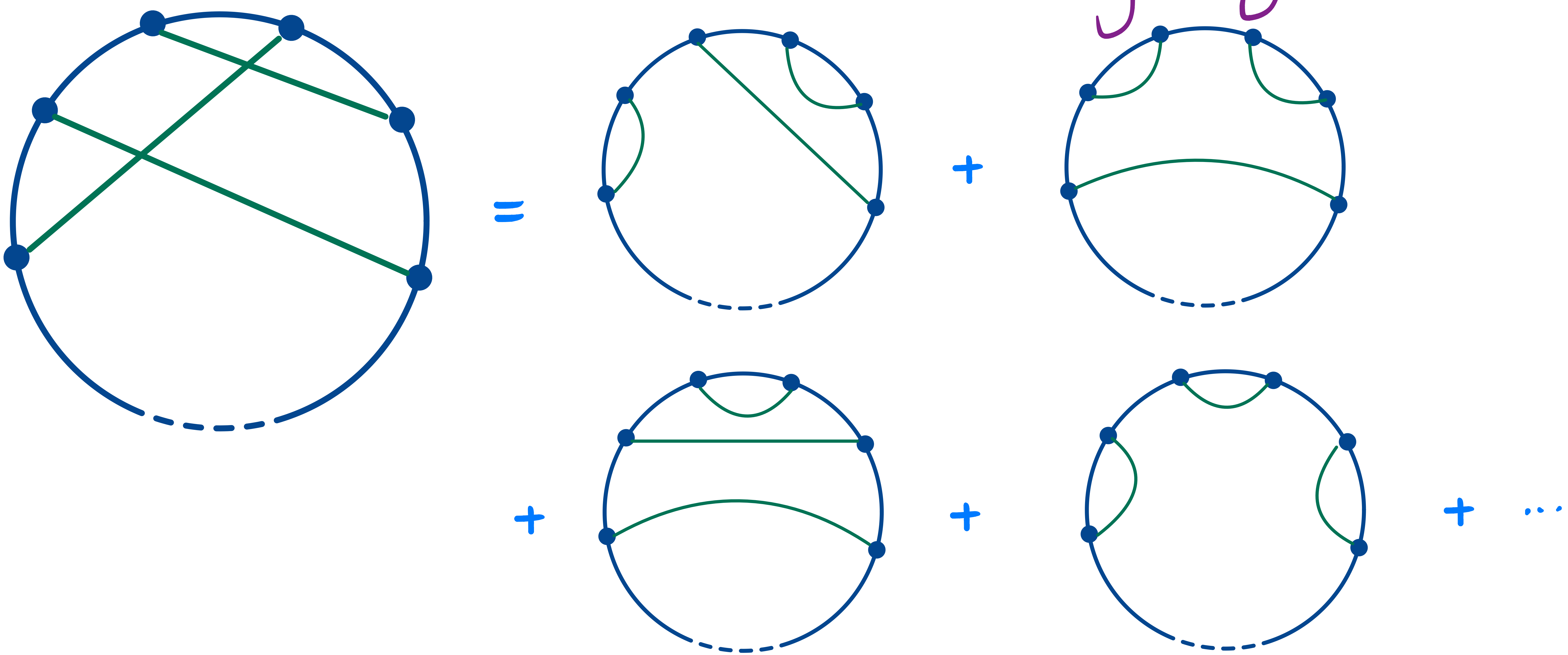
$$(x_{11}x_{22} - x_{21}x_{12})(x_{13}x_{24} - x_{23}x_{14})$$

$$+$$

$$(x_{11}x_{24} - x_{21}x_{14})(x_{12}x_{23} - x_{22}x_{13})$$

Temperly-Lieb basis

- Using  , can reduce any matching diagram to a linear combination of matching diagrams:



Temperley-Lieb basis

Thm The <u>noncrossing</u> 2-row webs are a basis for

$$\mathrm{Inv}_{SL_2}(V_1 \otimes \cdots \otimes V_n) \qquad (V_i \in \{V, V^*\})$$

called the Temperley-Lieb basis.

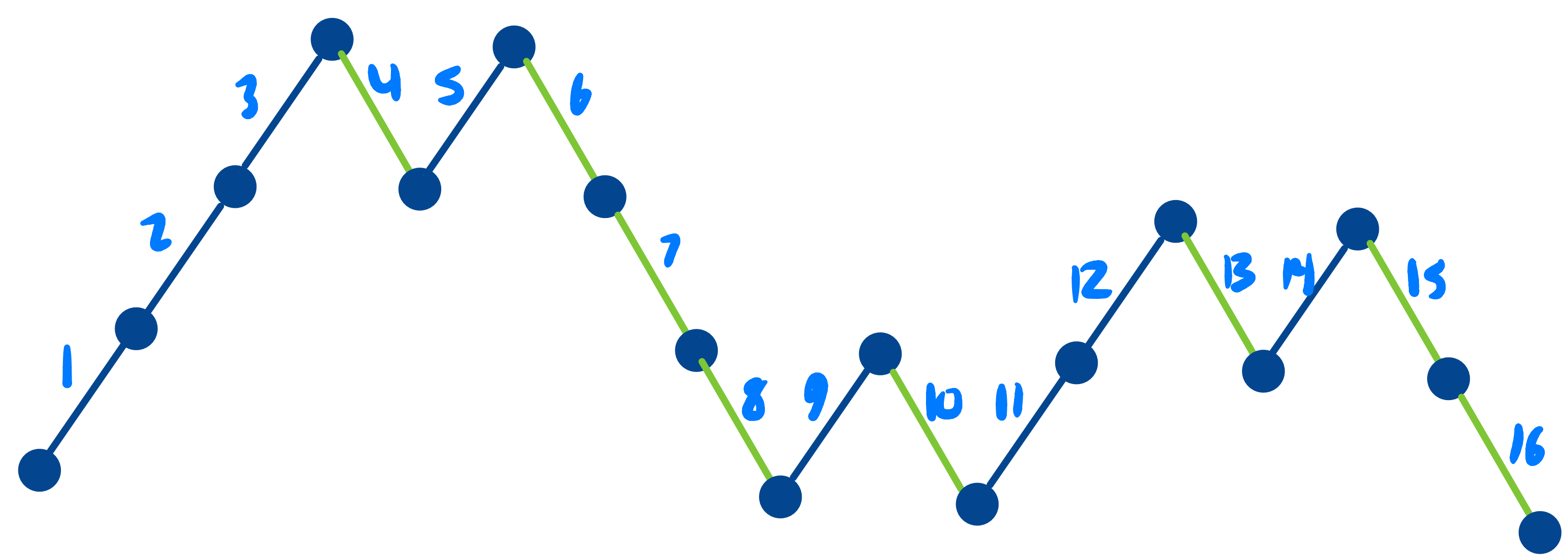
Pf
- <u>Spanning</u>: diagrams span by classical invariant theory, noncrossing by uncrossing rule.
- <u>Independence</u>: by Pieri rule,

$$\dim \mathrm{Inv}_{SL_2}(V^{\otimes n}) = \#SYT(2 \times \tfrac{n}{2}). \text{ Count!}$$

→

Temperley-Lieb basis

Some Catalan bijections:



Dyck path

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16

↦ ((()()))()(()()) ↦ 1112122212112122

lattice word

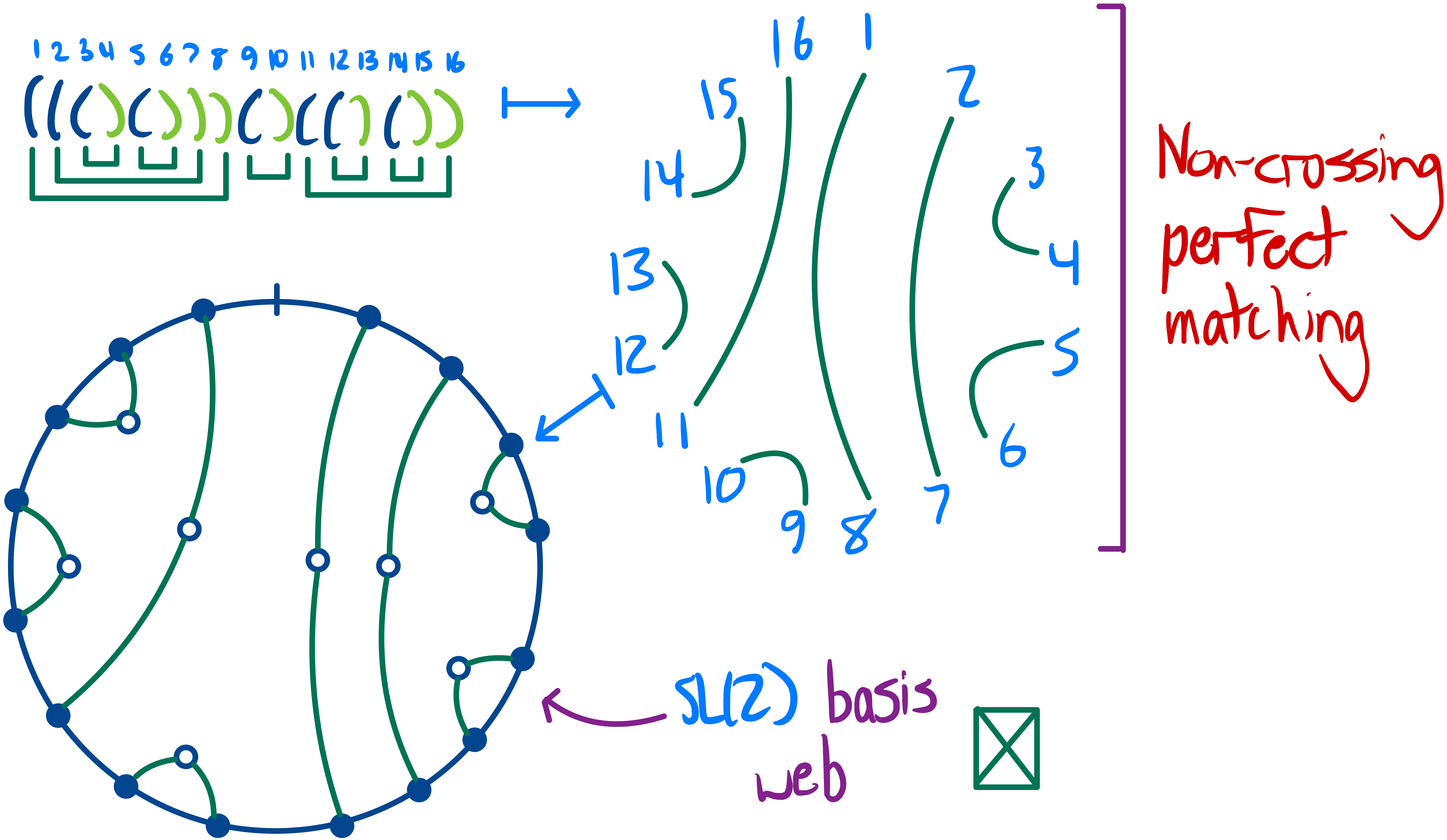
↦

1	2	3	5	9	11	12	14
4	6	7	8	10	13	15	16

Standard Young tableau

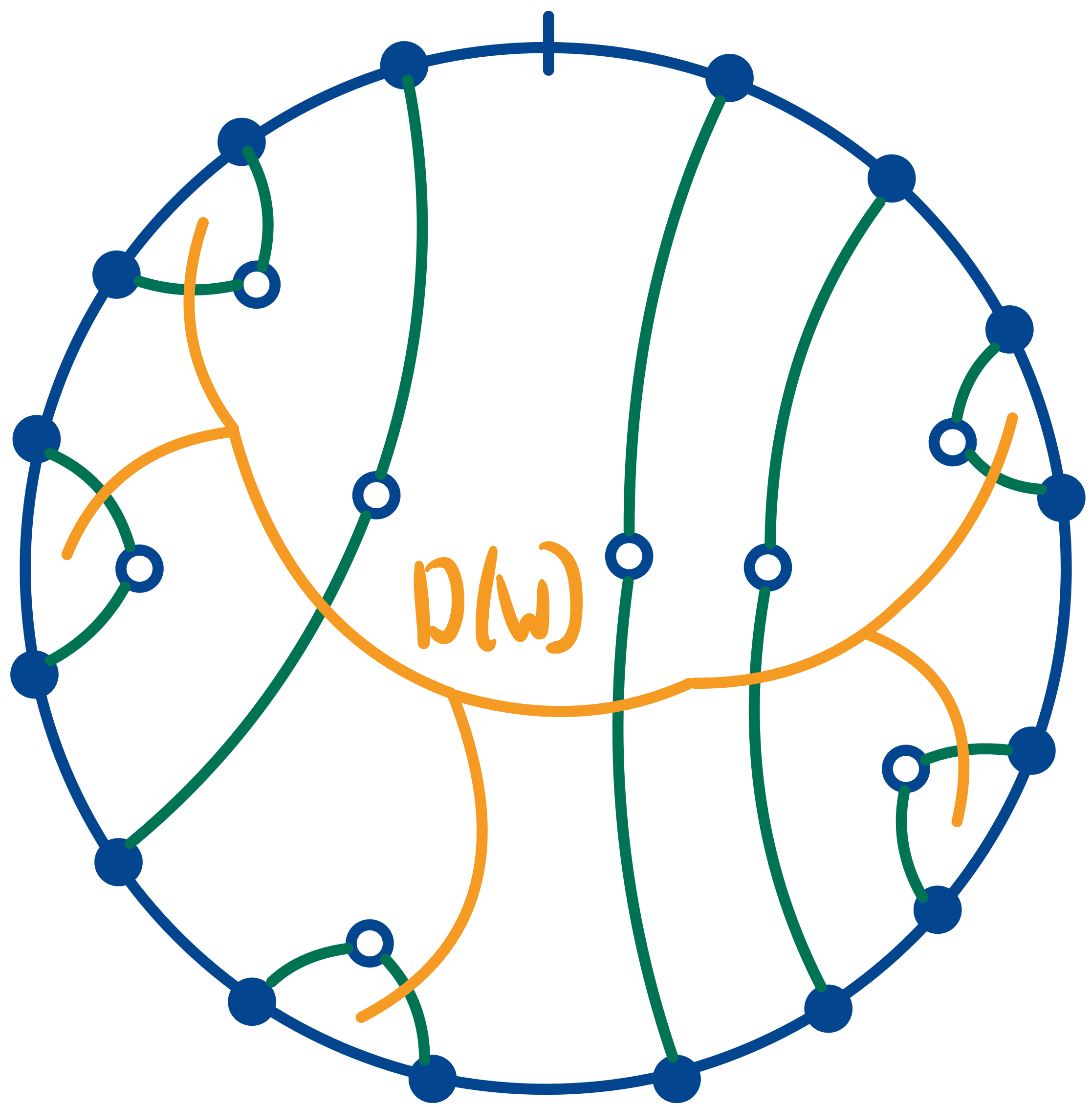
→

Temperly-Lieb basis



Duals and trees

Obs The dual graph D(W) of an SL(2) basis web W is a tree:



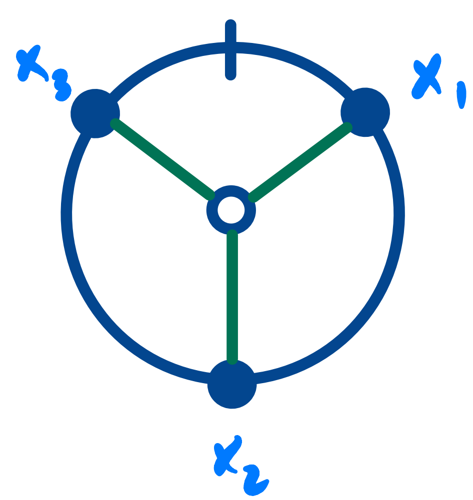
e.g. # faces of W = # vertices of D(W)

SL_3-webs

- Let $V = \mathbb{C}^3$, $V_i \in \{V, V^*\}$.

  Q Is there a nice web basis for $\mathrm{Inv}_{SL_3}(V_1 \otimes \cdots \otimes V_n)$?

- Use bipartite planar graphs in a disk built from



$$= \det\begin{pmatrix} | & | & | \\ x_1 & x_2 & x_3 \\ | & | & | \end{pmatrix}$$ (so trivalent).

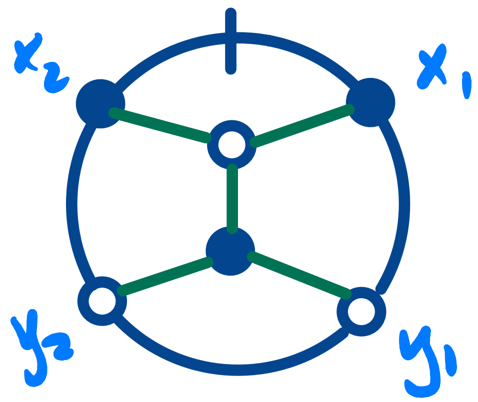
(+ dual)

Have univalent boundary vertices and connected to boundary.

SL_3-webs

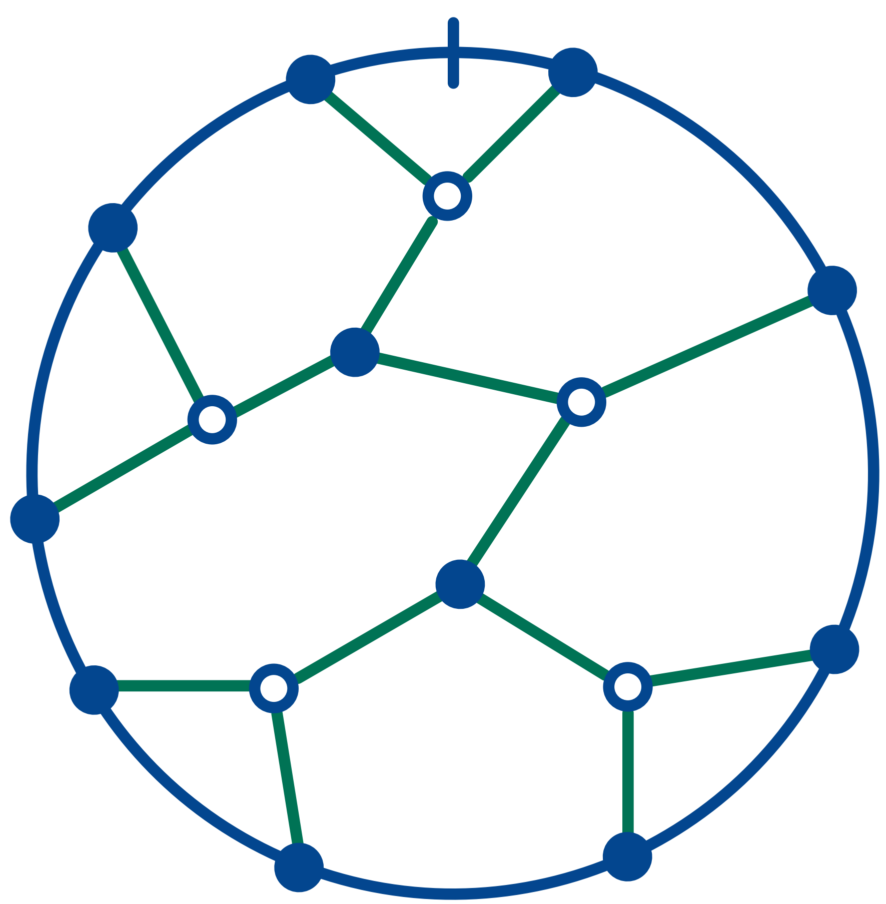
SL_3-webs

Ex



$$= \det\begin{pmatrix} | & | & | \\ x_1 & x_2 & y_1 \wedge y_2 \\ | & | & | \end{pmatrix}$$

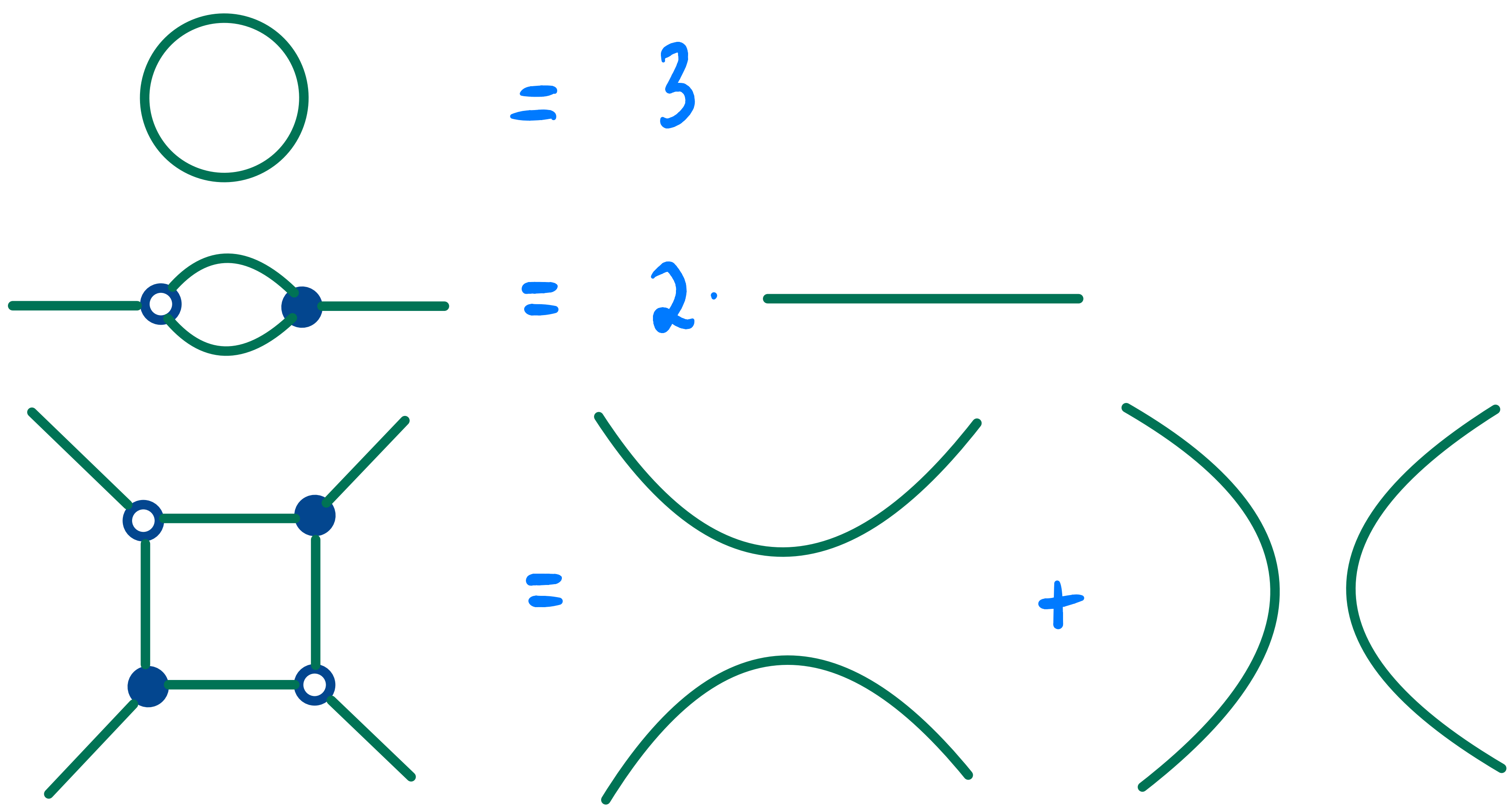
Ex



= … a polynomial obtained by summing over <u>proper edge 3-colorings</u>…

SL_3-webs

<u>Thm</u> (Kuperberg '94) The generating SL_3-web relations are



Non-elliptic web basis

Thm (Kuperberg '94)

Call an SL_3-web **non-elliptic** if it has no 2-faces or 4-faces.

The non-elliptic webs form a basis of

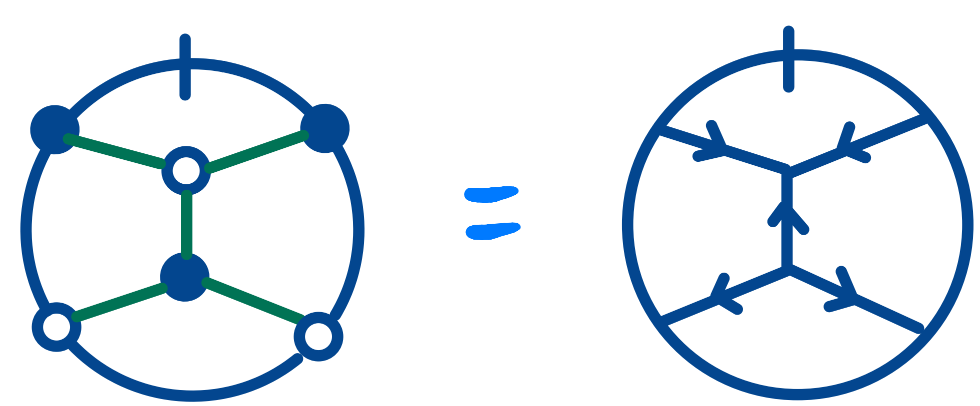
$$\mathrm{Inv}_{SL_3}(V_1 \otimes \cdots \otimes V_n). \qquad (V_i \in \{V, V^*\})$$

Non-elliptic web basis

Pf] • Spanning: similar to SL_2 case.

• Independence: bijection to $SYT(3 \times \frac{n}{3})$ using growth rules. ($\exists$ other approaches)

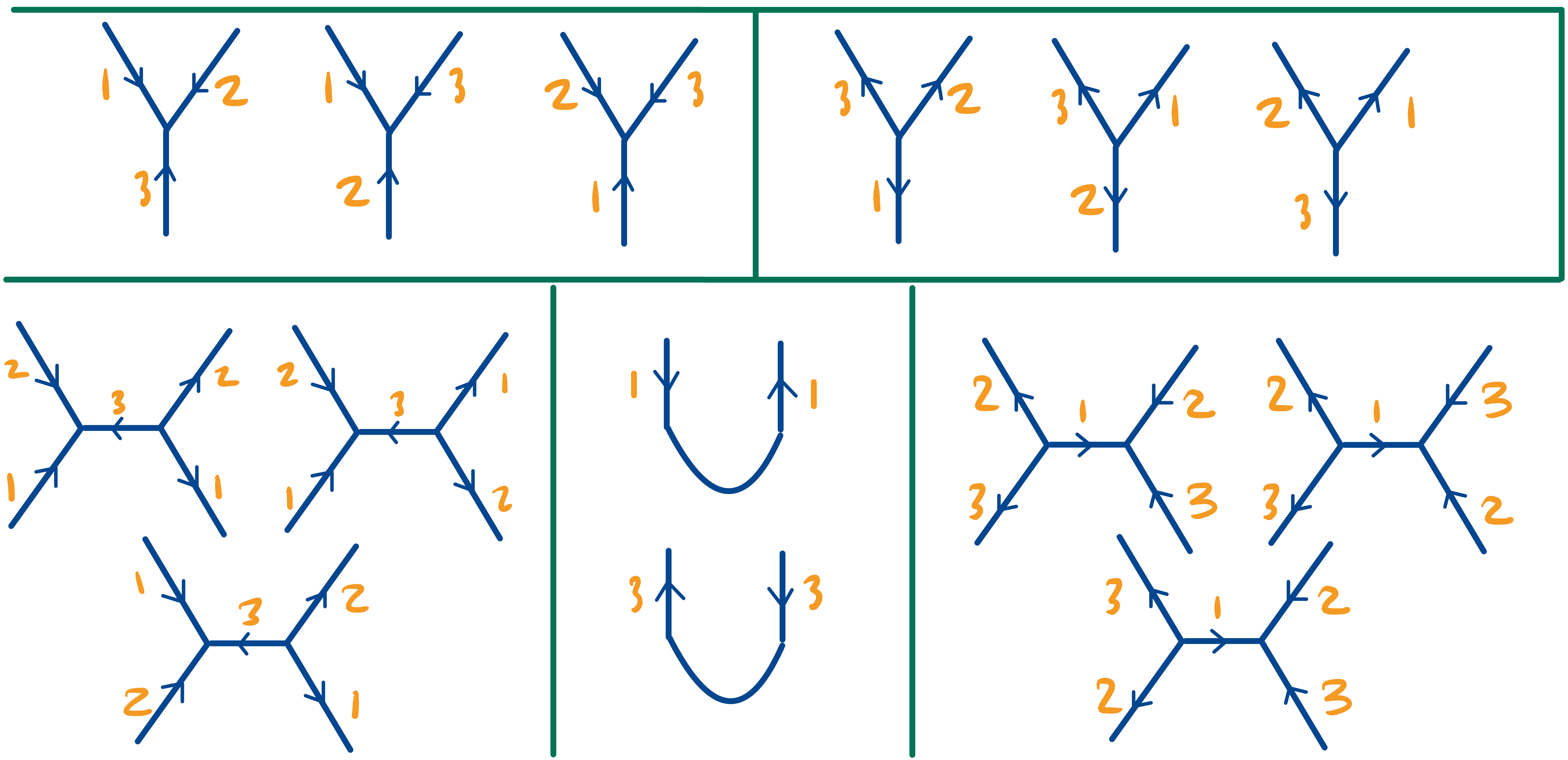
Will use directed notation here:



→

SL_3-growth rules

Kuperberg-Khovanov growth rules:



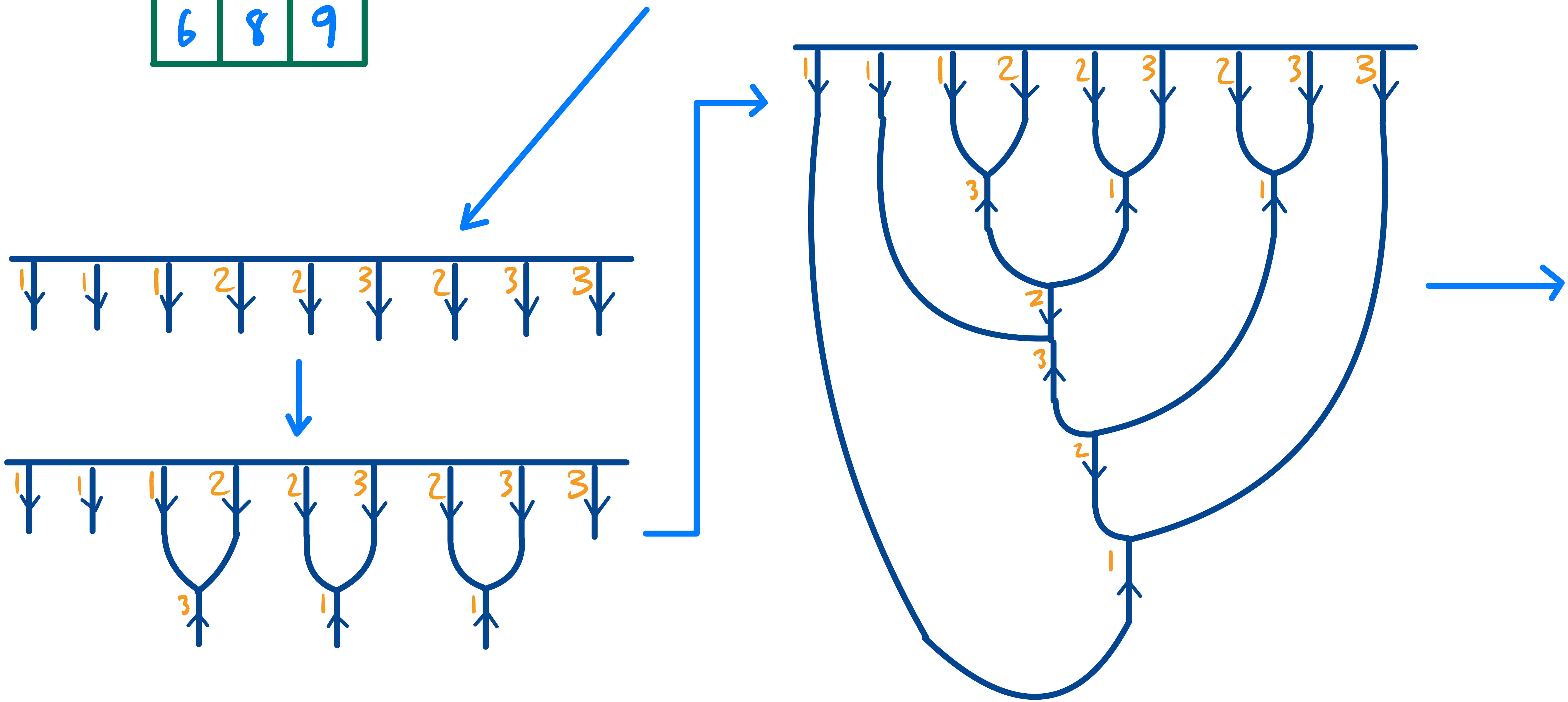
SL_3-growth rules

Ex

$T =$

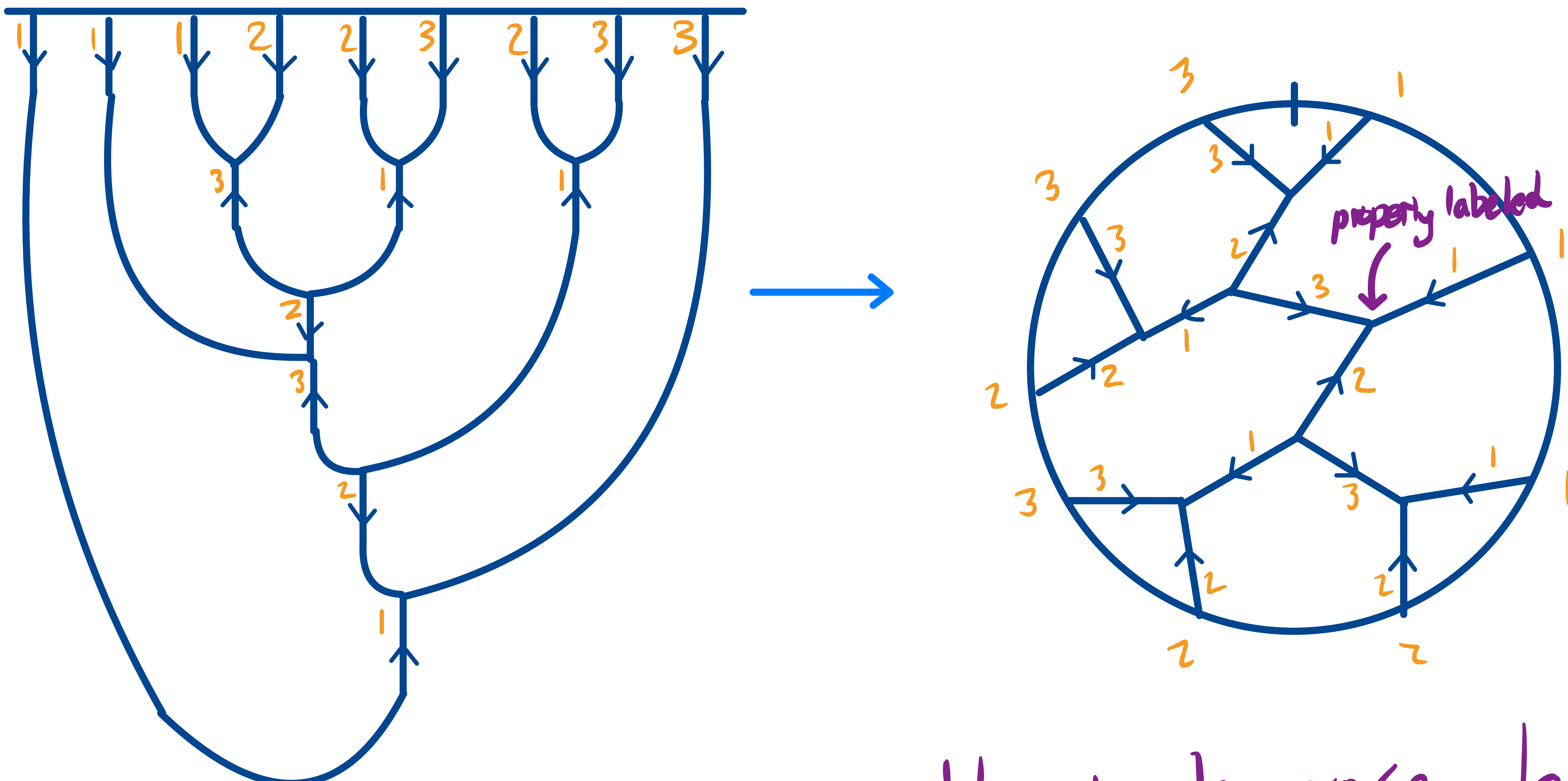
1	2	3
4	5	7
6	8	9

→ 111223233



SL_3-growth rules

$(T \longrightarrow 111223233)$



Now just erase labels!

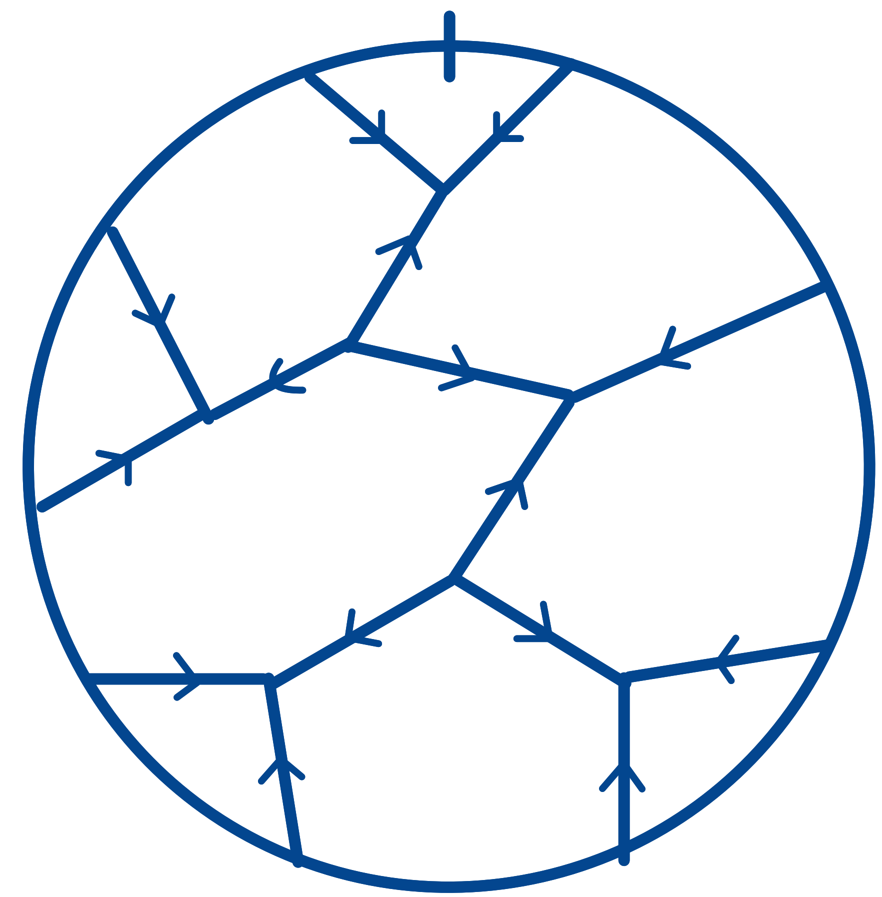
SL_3-growth rules

In all:

$T =$

1	2	3
4	5	7
6	8	9

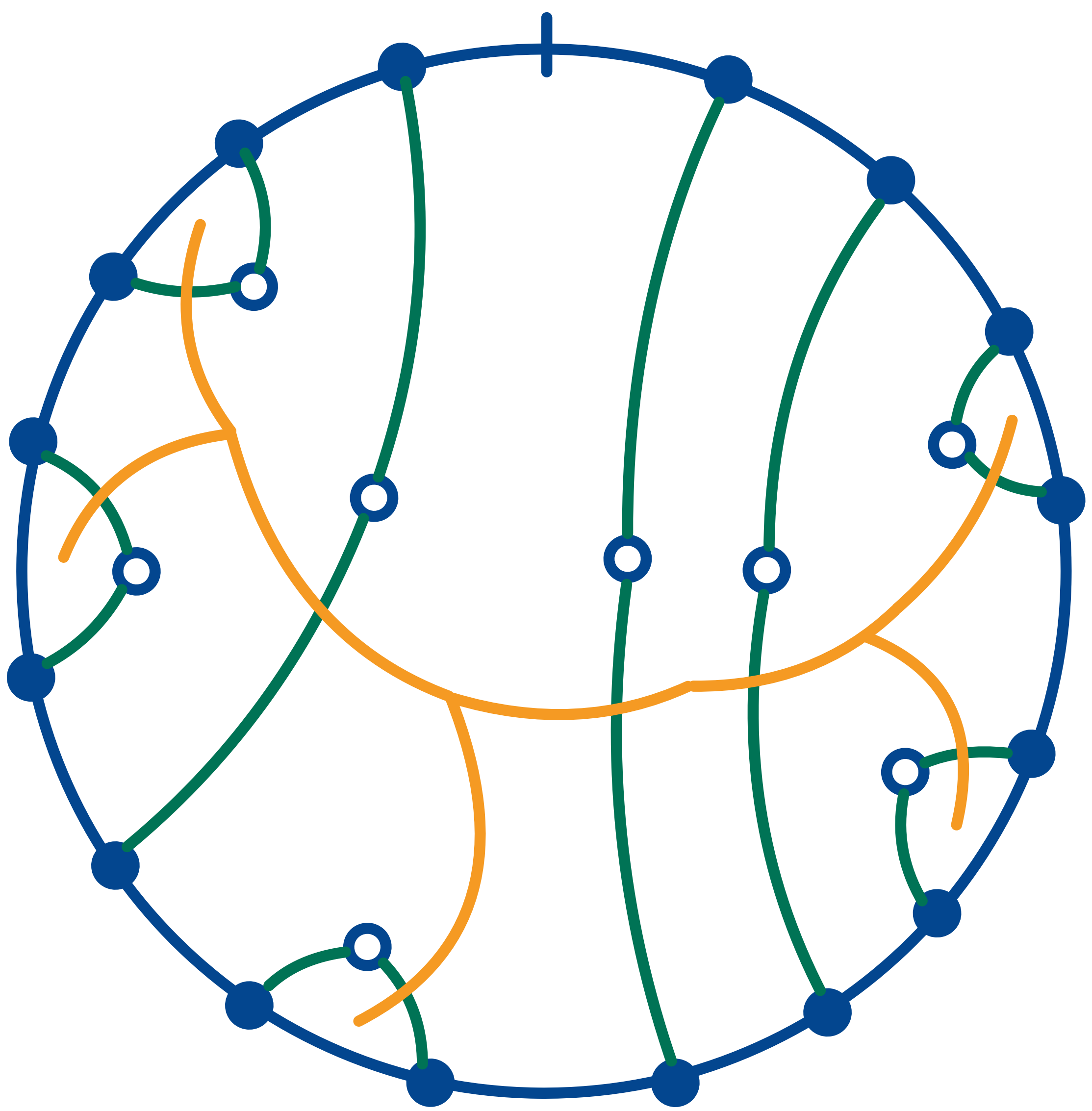
$\longmapsto$



Q: What did the labels mean?

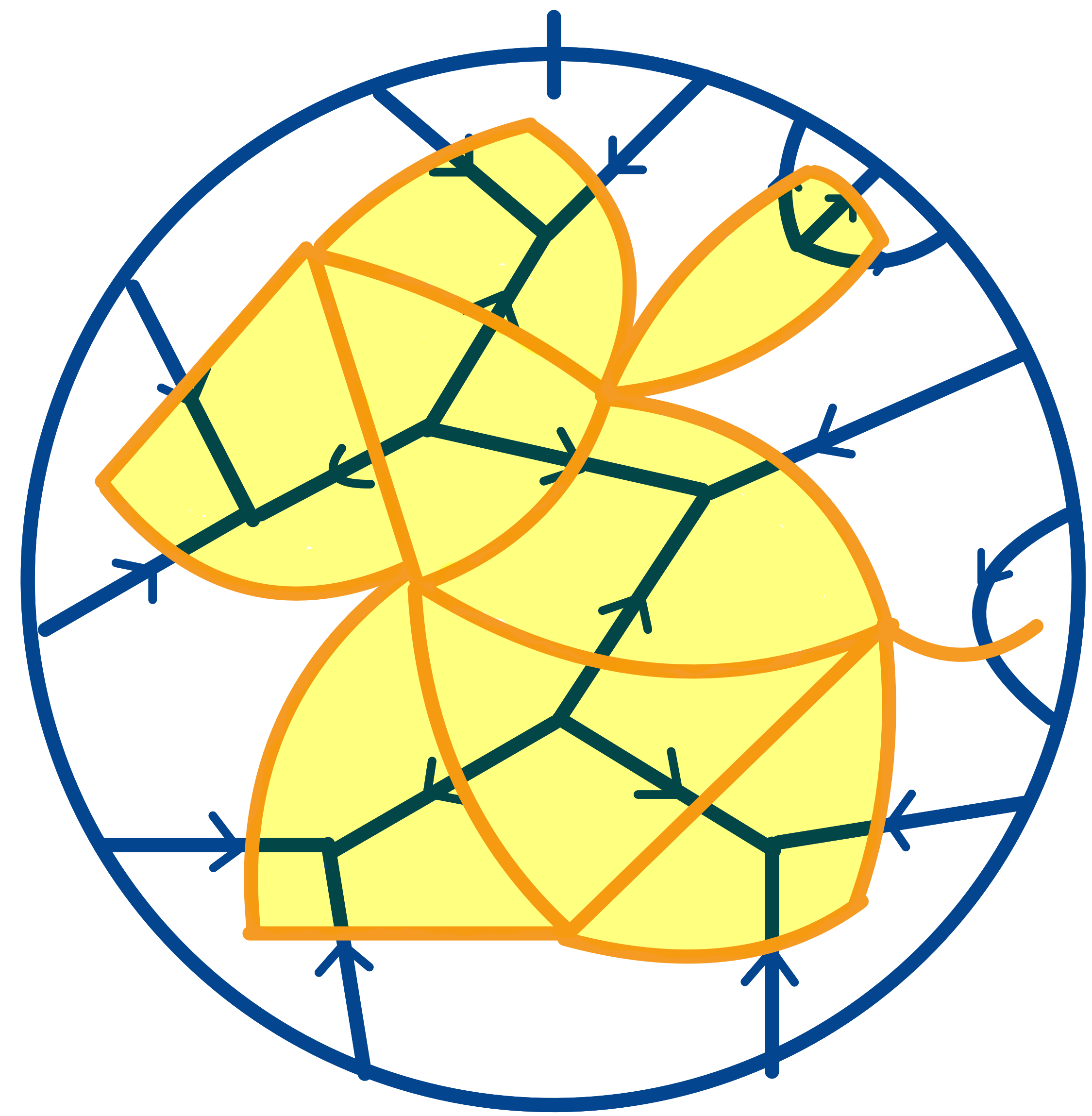
SL_3-web duals

Obs D(W) for SL(3) is a triangulation:



D(W) is 1D for SL(2)

vs.



D(W) is "2D" for SL(3)

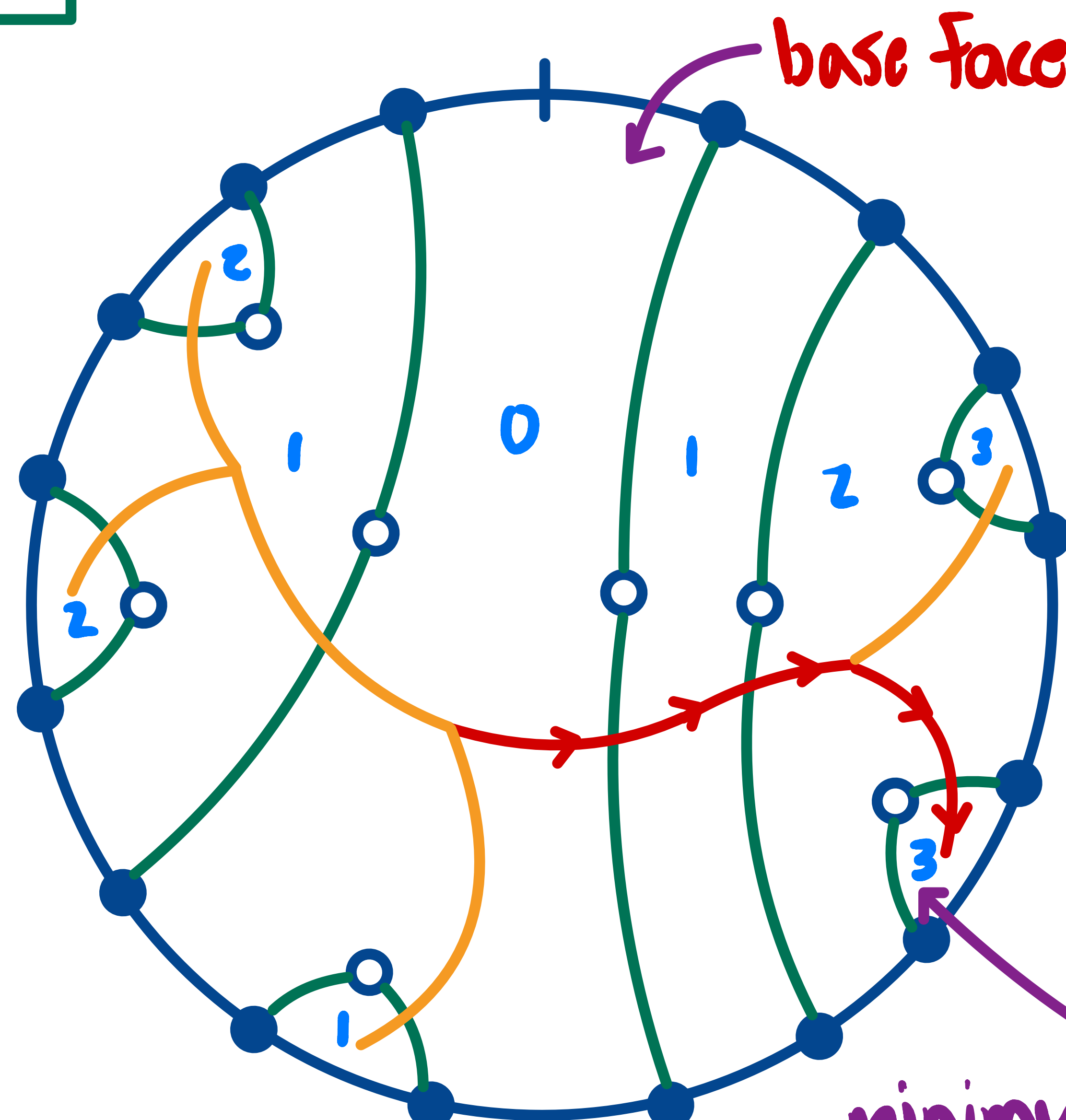
SL_3-web duals

Next • Fontaine-Kamnitzer-Kuperberg '13 showed duals of SL_3 basis webs live inside the affine building $\Delta(SL_3^\vee)$

- Growth rules build one triangle at a time
- Labels encode distance information
- Non-elliptic condition $\Leftrightarrow$ CAT(0)

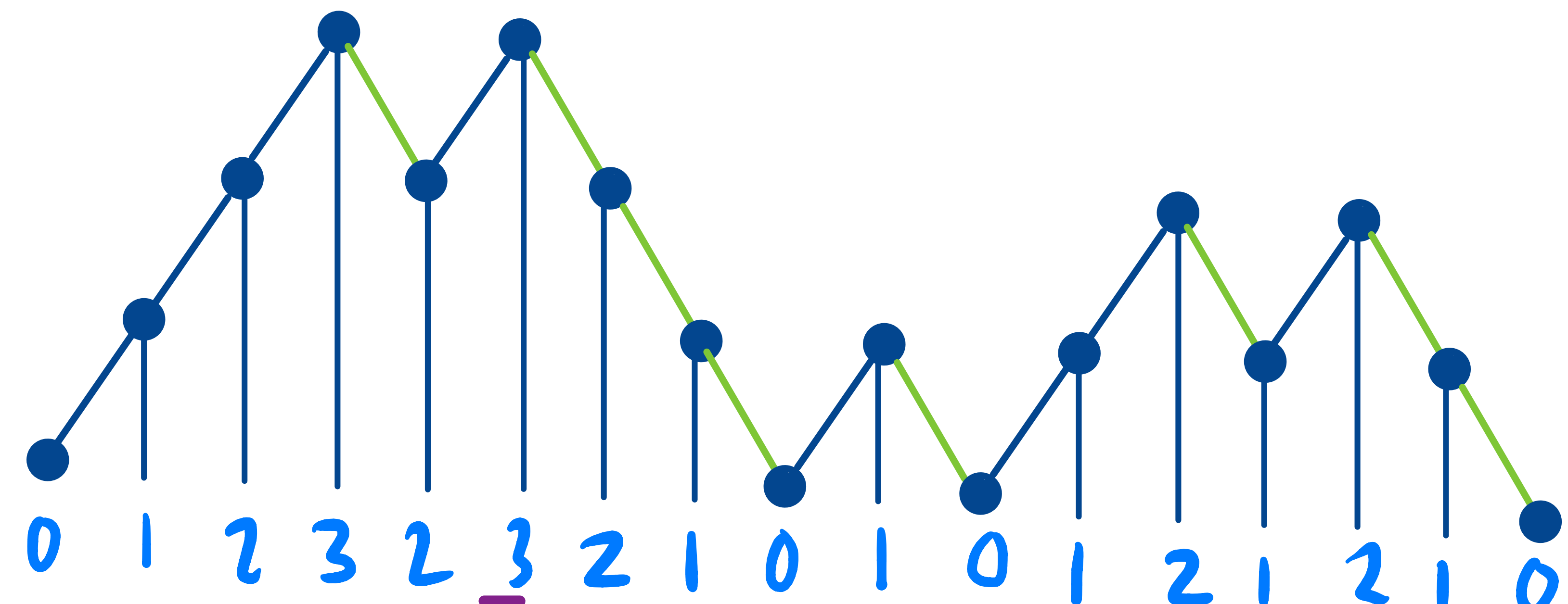
Distances

Ex



minimum depth from base face

1	2	3	5	9	11	12	14
4	6	7	8	10	13	15	16



- Hence depths around the boundary encode the tableau

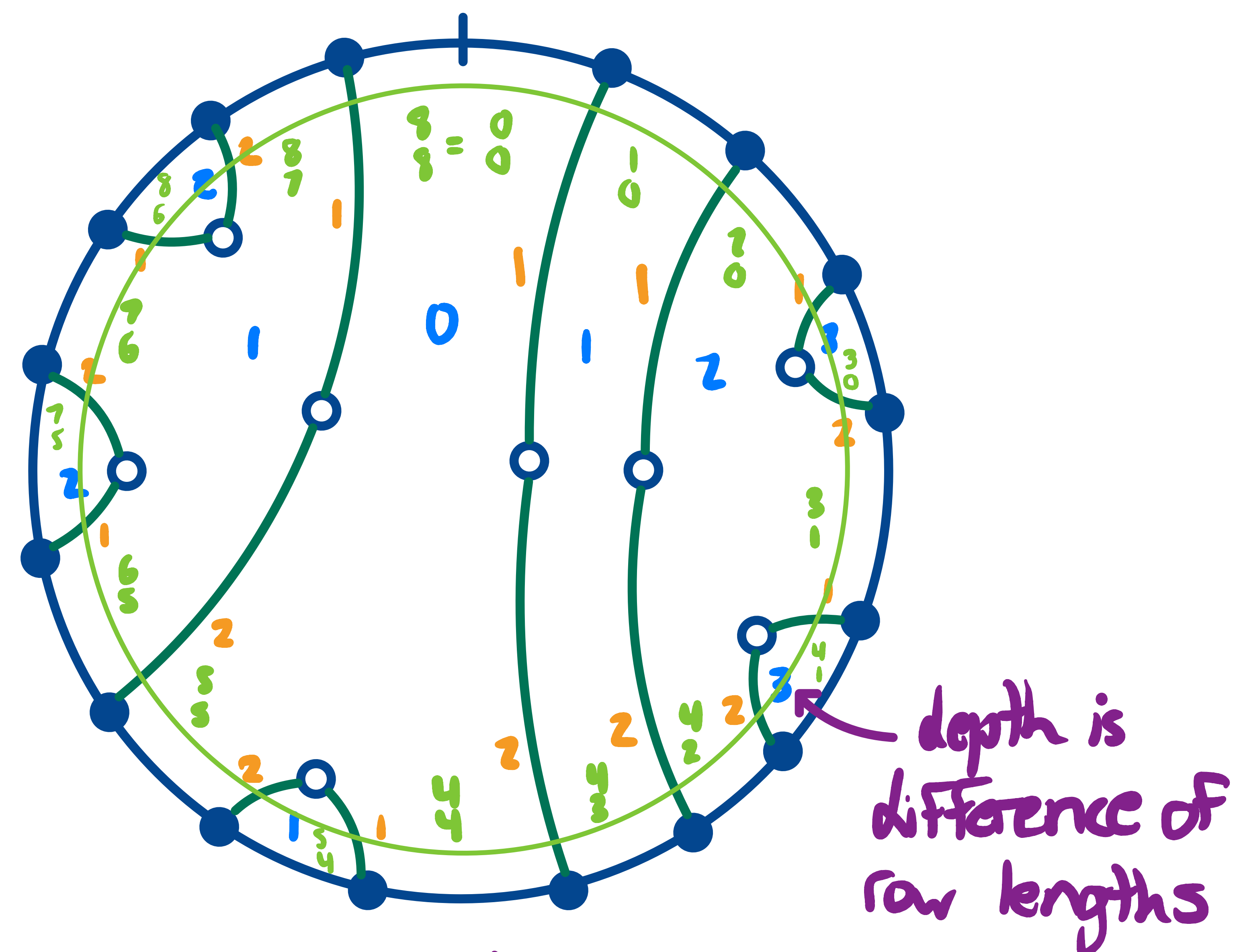
Distances

Ex

1	2	3	5	9	11	12	14
4	6	7	8	10	13	15	16

↕

$$\emptyset \subset (1,0) \subset (2,0) \subset (3,0) \subset (3,1) \subset \cdots \subset (8,8)$$



Label so that boundary is
1112122212112122

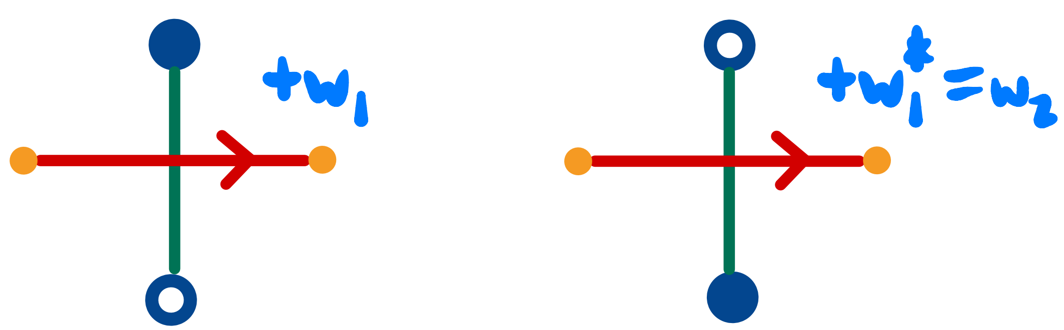
Distances

Recall

- The $SL(r)$ dominant weights are
$$\lambda = (\lambda_1 \geq \cdots \geq \lambda_r) \in \mathbb{Z}^r / \langle (1, \dots, 1) \rangle.$$
- The fundamental weights of $SL(r)$ are
$$\omega_i = (1^i, 0^{r-i})$$
$$\omega_i^* = (0^{r-i}, -1^i) \equiv \omega_{r-i}.$$
- If $\lambda, \mu \in \mathbb{Z}^r_{dec}$ where $|\lambda| = |\mu| + rk$ for some $k \in \mathbb{Z}$, say
$$\lambda \geq \mu \iff \forall m, \sum_{j=1}^{m} \lambda_j \geq \sum_{j=1}^{m} (\mu_j + k).$$

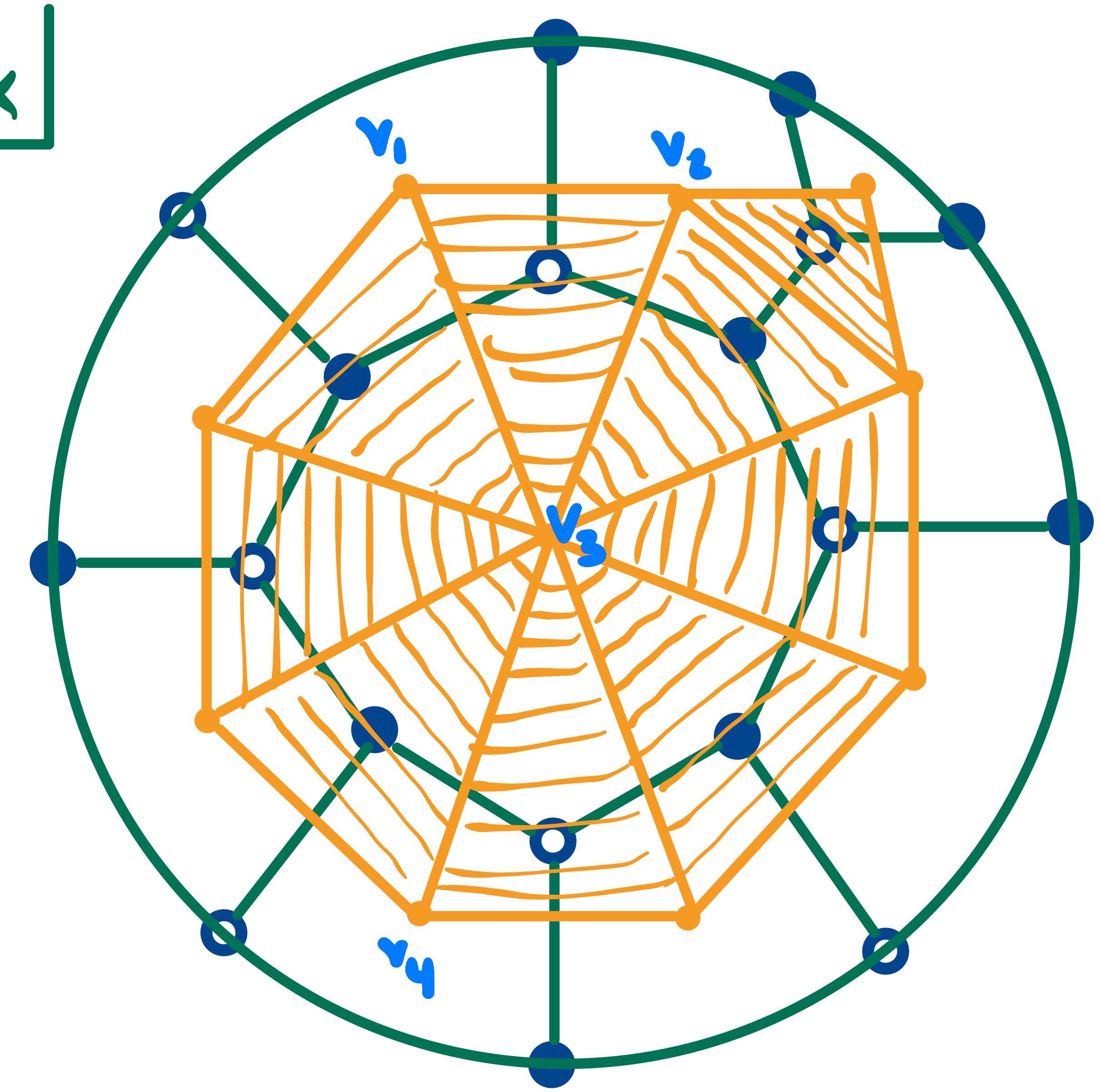
Distances

Idea: For an SL(3) web, walk along the dual graph, picking up SL(3) weights according to:



Call this the **distance** of the walk.

Ex



$$d\left(v_1 \xrightarrow{w_1} v_2 \xrightarrow{w_1} v_3 \xrightarrow{w_2} v_4\right) = 2w_1 + w_2 = (3,1,0)$$

<u>vs.</u>

$$d\left(v_1 \xrightarrow{w_2} v_3 \xrightarrow{w_2} v_4\right) = 2w_2 = (2,2,0)$$

smaller!

Geodesics

Def • Such a walk on $D(W)$ for an $SL(3)$-web W is a ("combinatorial") geodesic from p to q if it is a minimizer of

$$\{\ell(\gamma) \mid \gamma : p \to \cdots \to q\}.$$

• We have coherent geodesics if

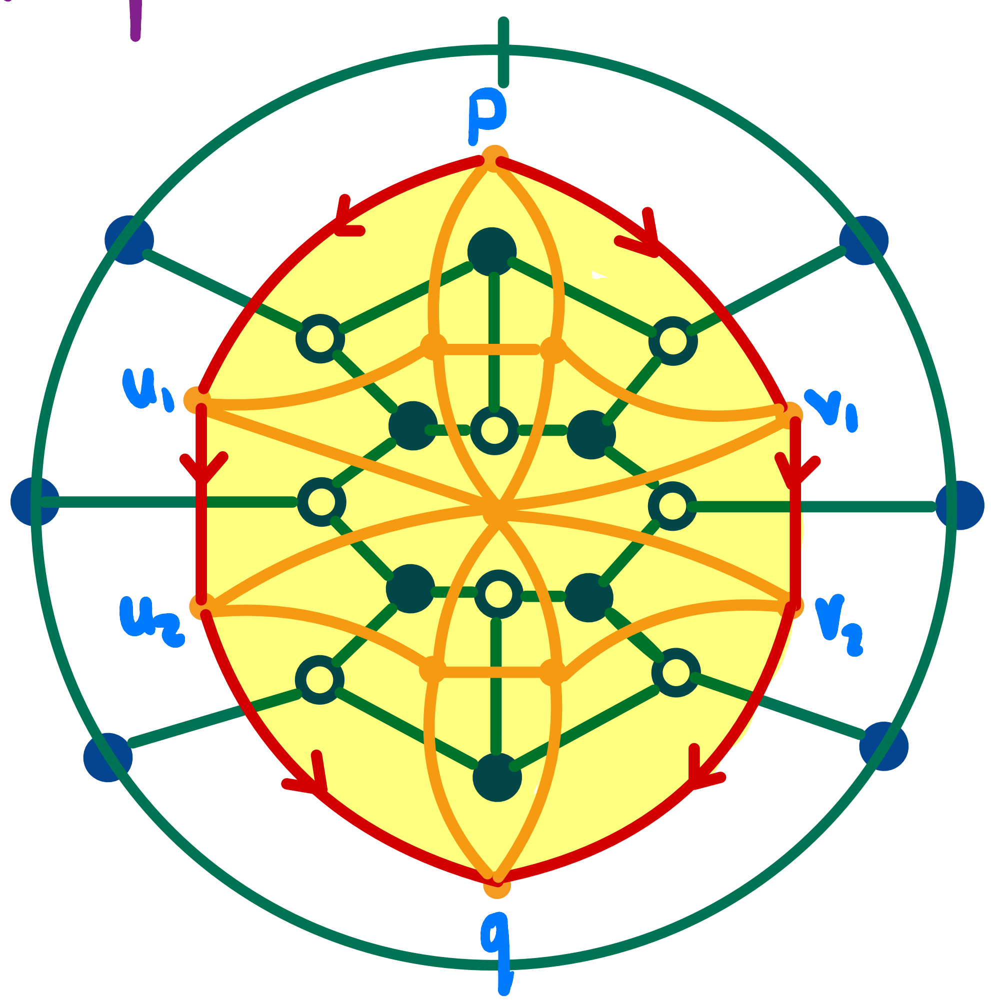
$\forall p, q,\ \exists!$ geodesic from p to q.

unique!

Geodesics

Thm (FKK) If W is a non-elliptic $SL(3)$ web, it has coherent geodesics.

Ex Non-example:



$$d(p \to u_1 \to u_2 \to q) = (3,3,0)$$

$$d(p \to v_1 \to v_2 \to q) = (3,0,0) = (4,1,1)$$

incomparable!

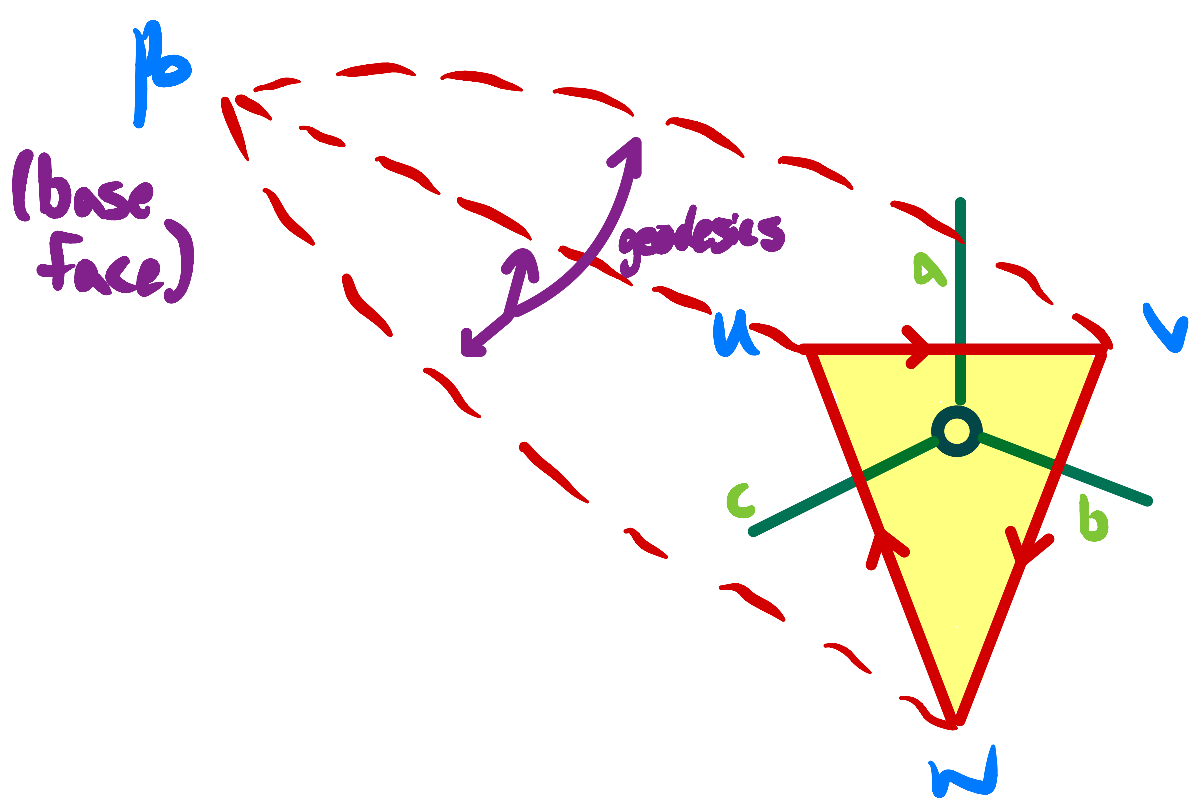
(has squares)

Geodesics

Fact Geodesic distances from base face to faces around the boundary encodes original $T \in SYT(3 \times \frac{n}{3})$!

Q Growth rule label meaning?

A



Fact $d(p_0, w) - d(p_0, v) \in S_{r^{\prime} w_1}$

Hence

$$d(p_0, w) - d(p_0, v) = e_a$$
$$d(p_0, v) - d(p_0, w) = e_b$$
$$d(p_0, w) - d(p_0, u) = e_c$$

$0 = e_a + e_b + e_c$

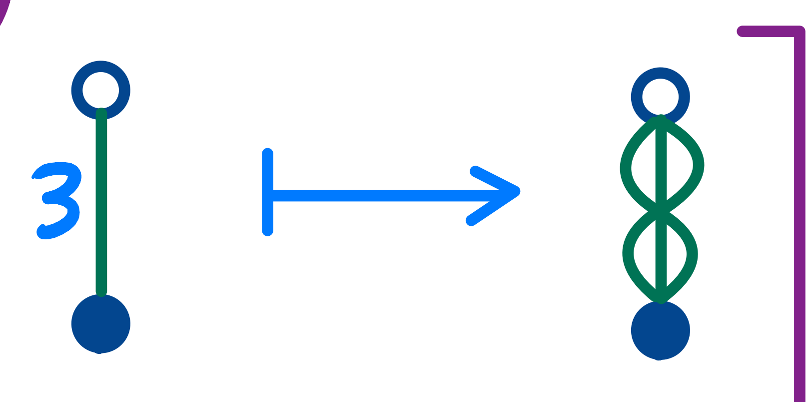
$\Leftrightarrow \{a, b, c\} = \{1, 2, 3\}$

$\Leftrightarrow$ proper labeling!

Hourglass plabic graphs

Def ([Goetz–Pechenik–Pfannerer–Striker–S. '23])
An r-hourglass plabic graph (r-HPG) is a planar bipartite graph embedded in a disk with edge weights in [r] which sum to r around internal vertices, and boundary vertices have degree 1.

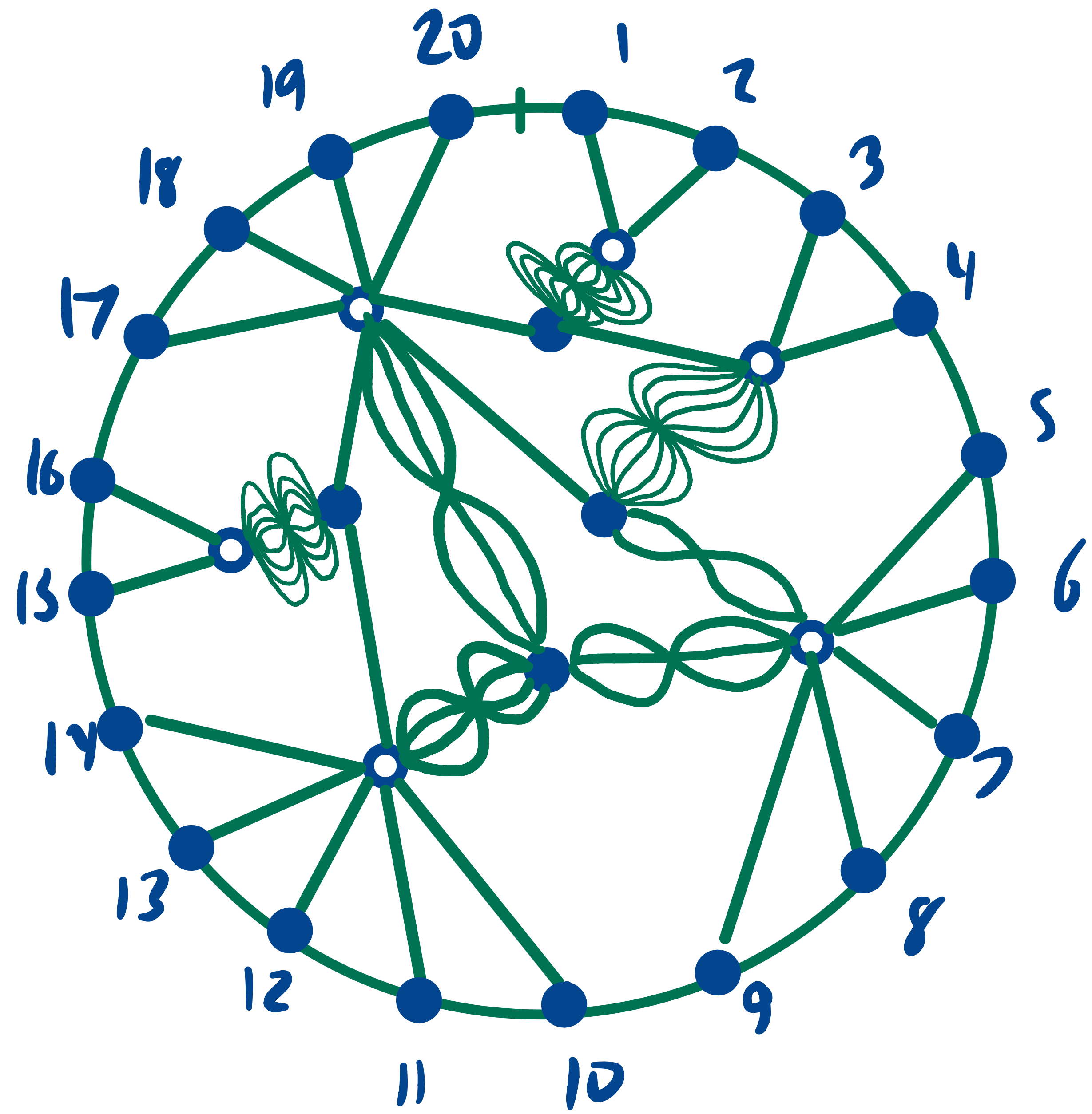
An edge with weight m is drawn as an m-hourglass:



a 3-hourglass

Hourglass plabic graphs

Ex

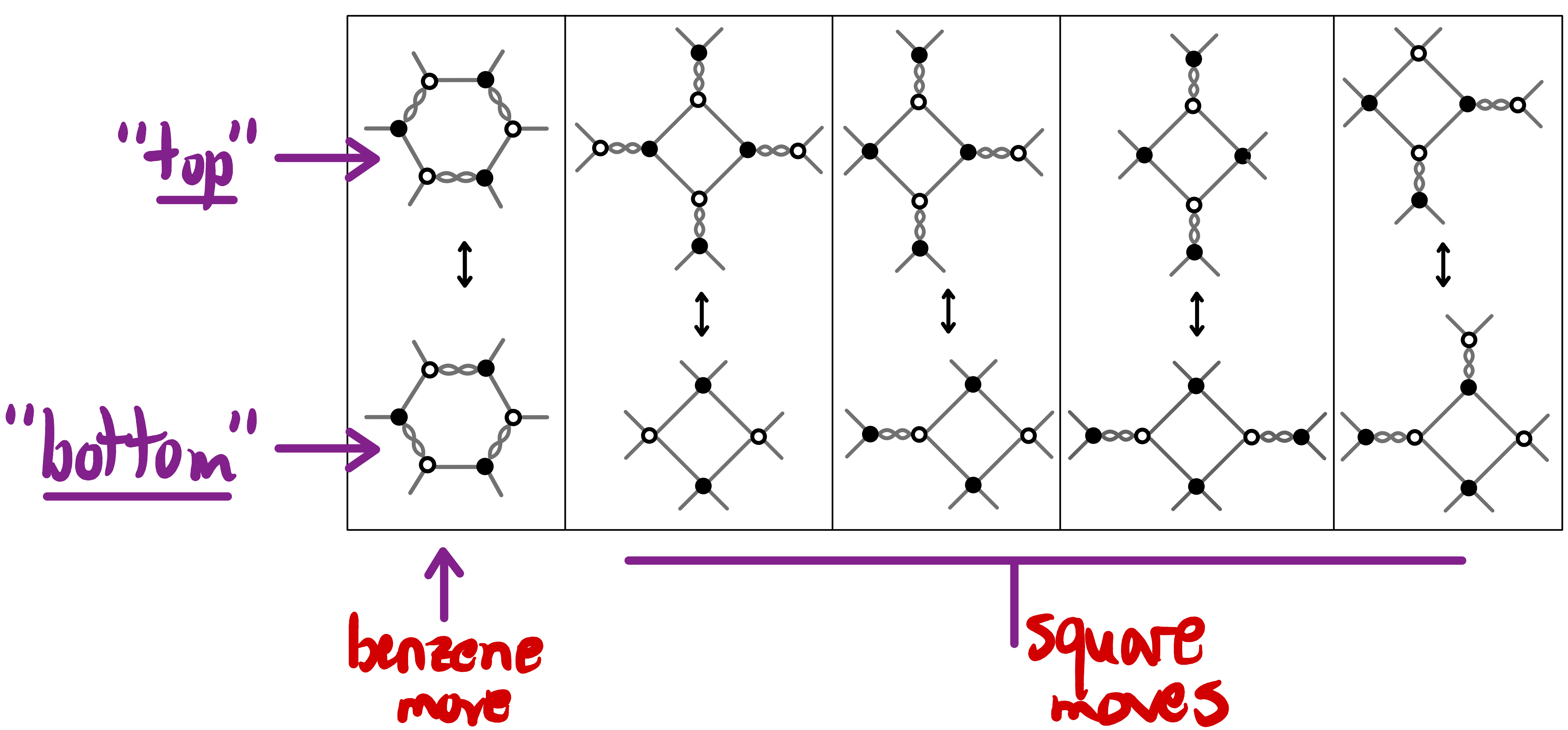


(r = 10)

r=4 moves

Thm (GPPSS '23) Two contracted, fully reduced 4-HPG's have the same trip permutations

$\Leftrightarrow$ they are related by moves:



$r=4$ moves

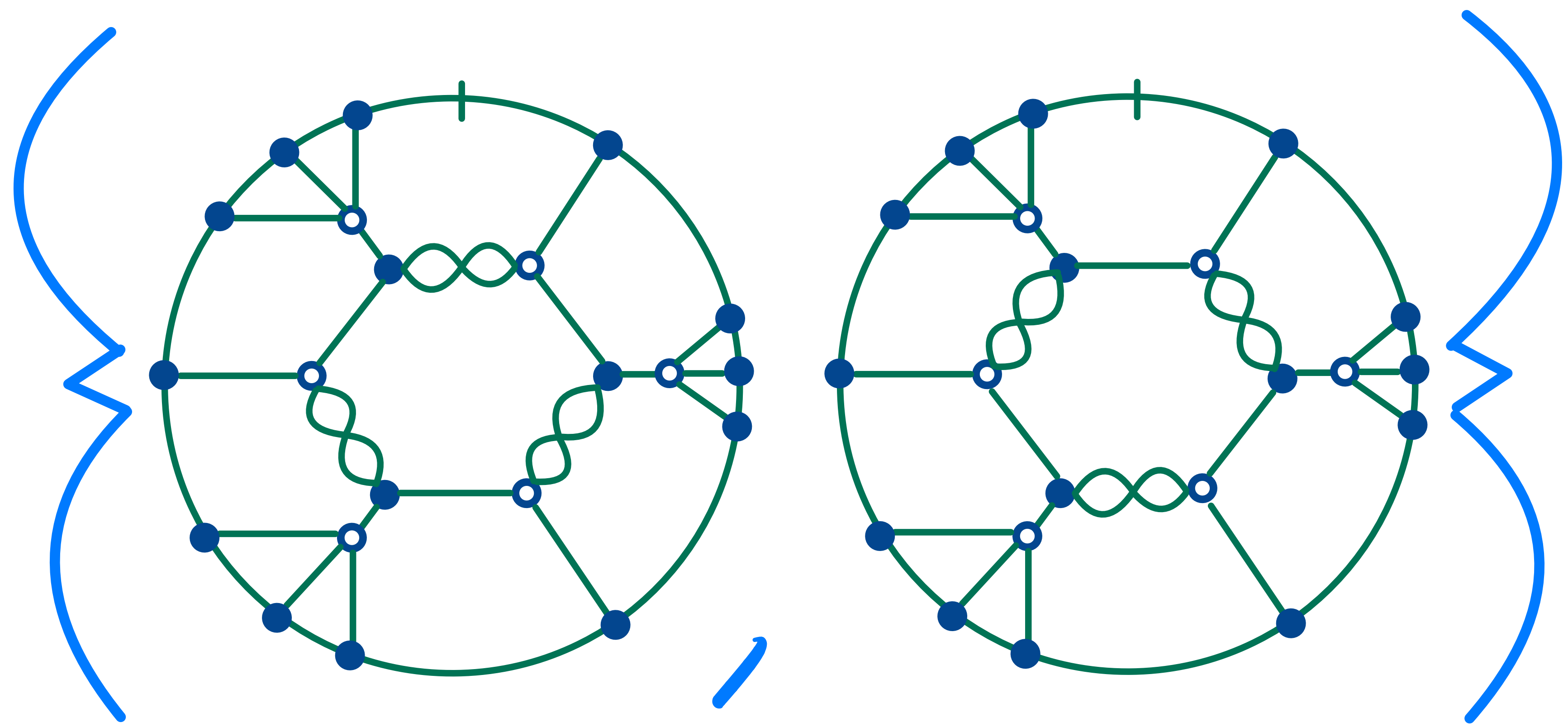
Thm (GPPSS '23) There is a bijection between $SYT(4\times c)$ and such <u>move-classes</u>. It sends $\mathrm{prom}_i(T)$ to $\mathrm{trip}_i(G)$.

Ex

$T=$

1	2	6
3	5	10
4	7	11
8	9	12

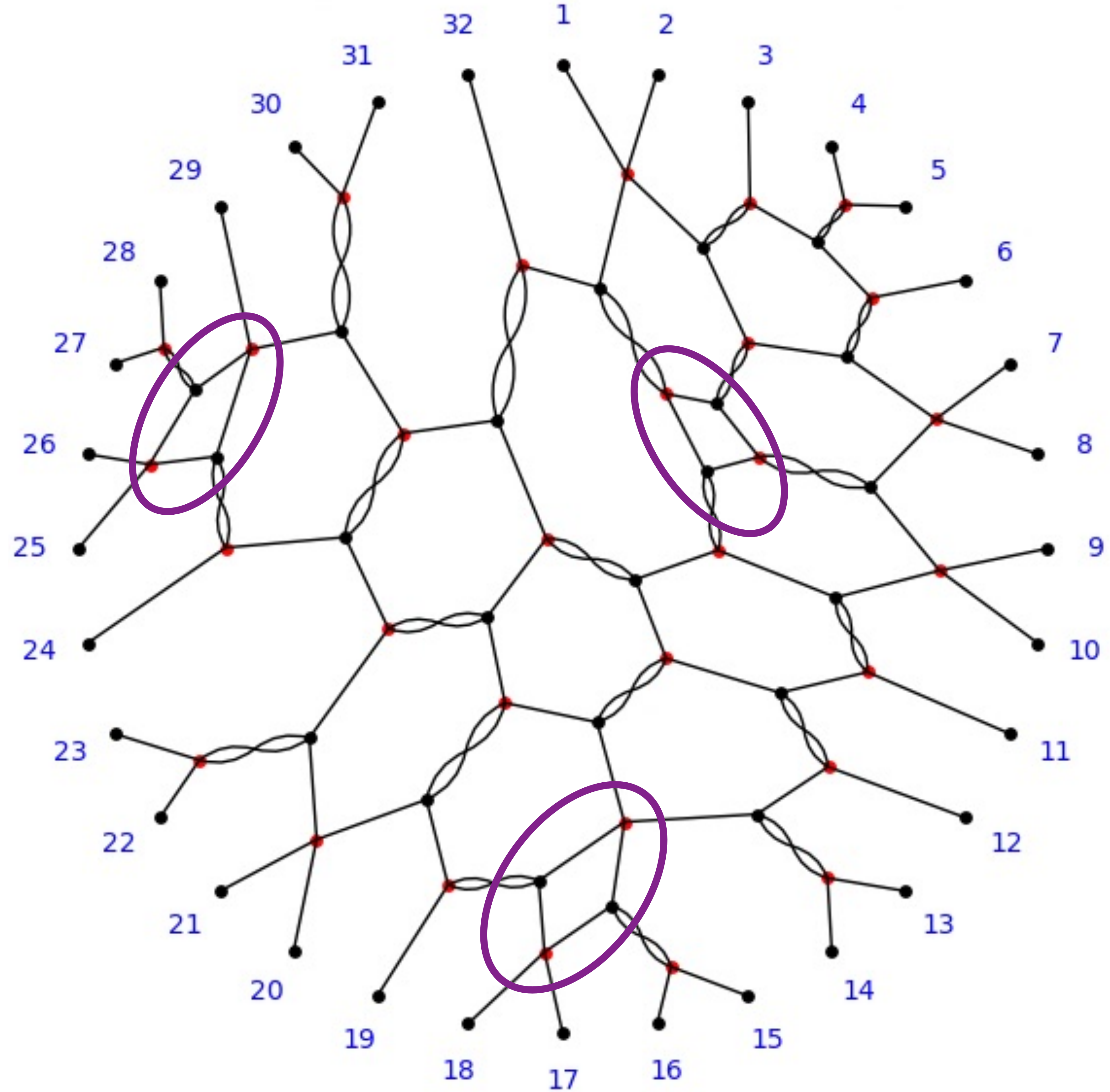
$\longmapsto$



r=4 moves

Ex

1	3	4	7	8	17	19	23
2	5	6	9	14	18	21	24
10	12	13	15	16	25	26	28
11	20	22	27	29	30	31	32

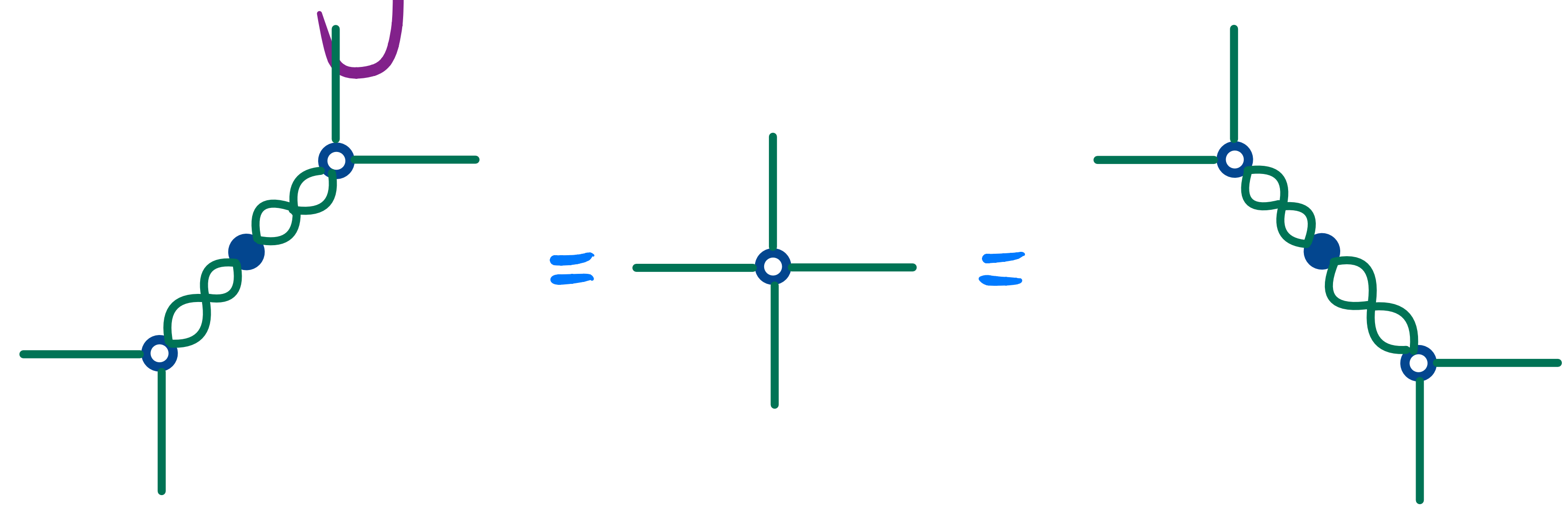


Pockets

Def (Gaetz–Striker–S.–Wu)

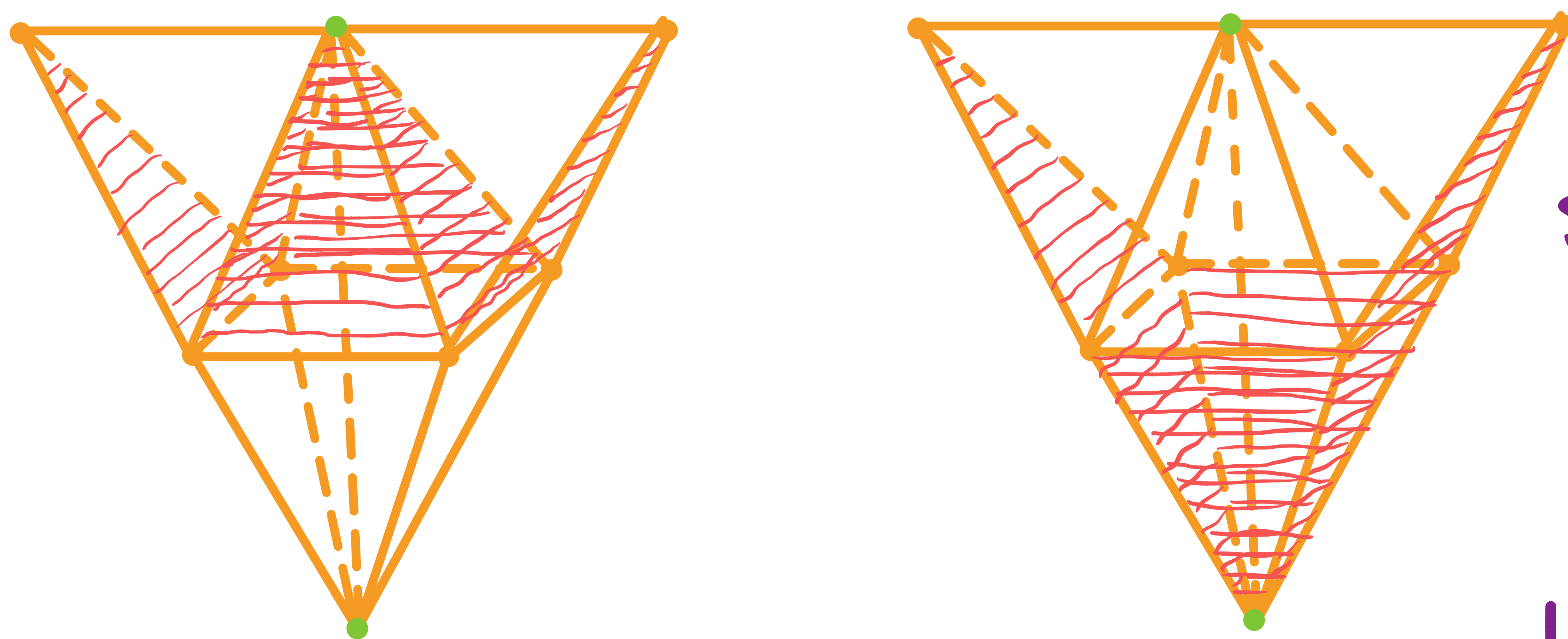
Let $\mathcal{C}$ be the move-class of some fully reduced 4-HPG W. Then there is a 3D flag simplicial complex $P(\mathcal{C})$ called the **pocket** of $\mathcal{C}$, which $D(W)$ embeds into.

- Requires expanding sources/sinks via $I=H$ moves:



Pockets

Ex



Pocket of square + benzene + IH class of

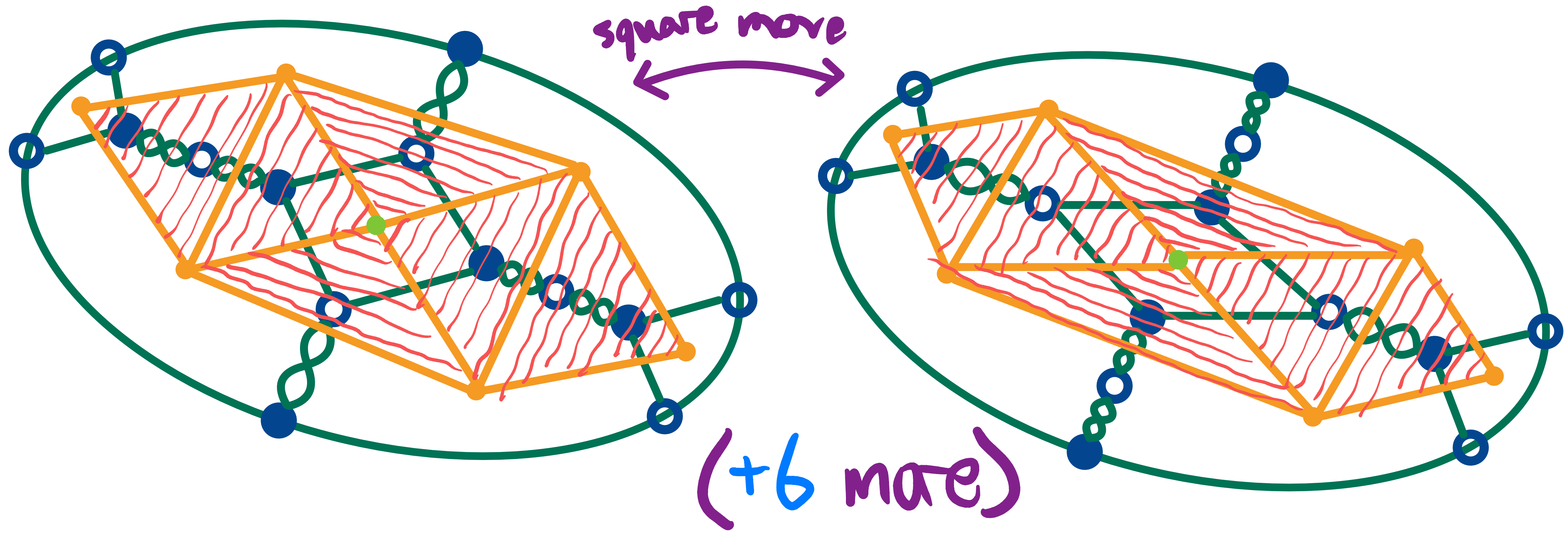
$\bar{4}\{1,2\}\bar{3}\,\bar{2}\{3,4\}\bar{1}$

has 6 tetrahedra

14 triangles

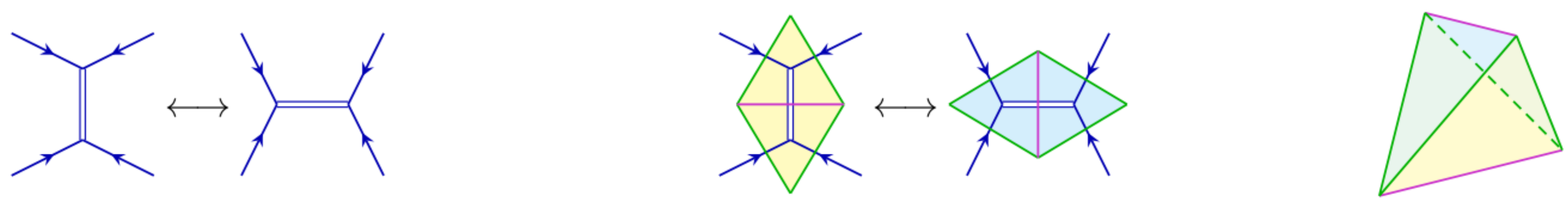
19 edges

8 vertices

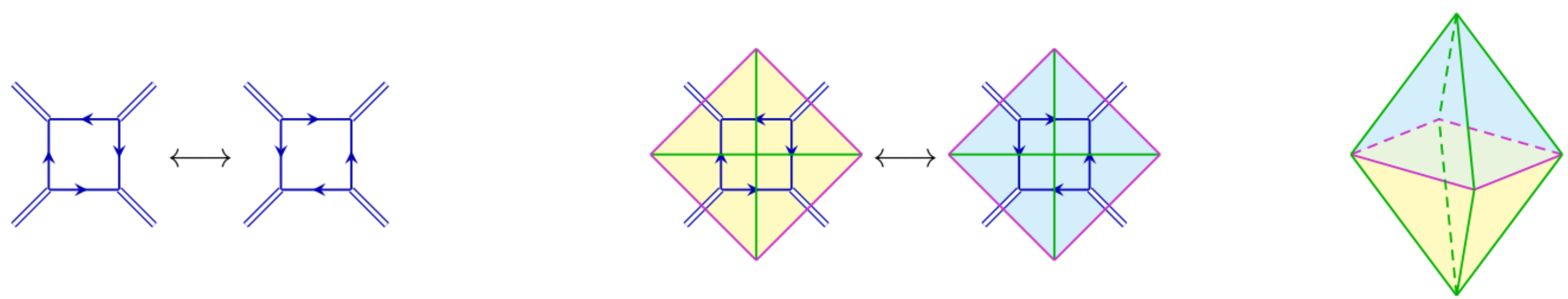


Pockets

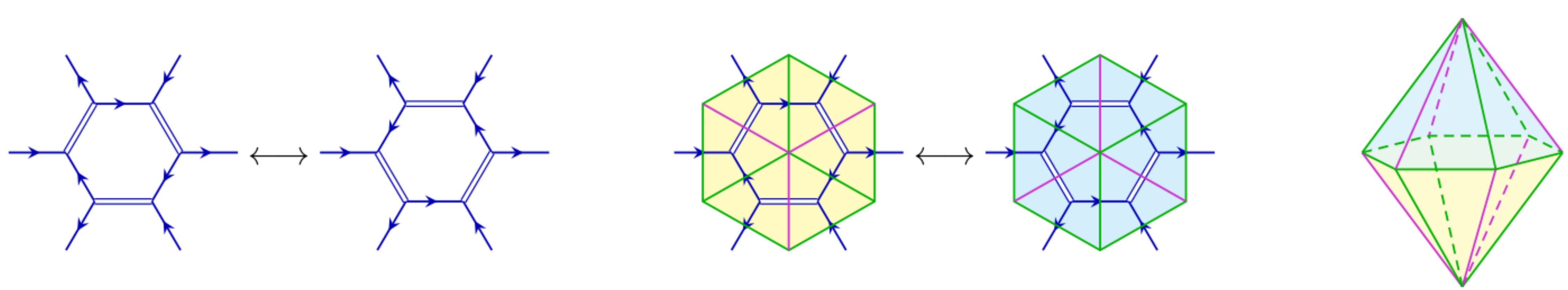
- Build pockets from 4-HPG basis web move classes:



IH move between webs | Dual diskoids | tetrahedron



Square move between webs | Dual diskoids | octahedron



Benzene move between webs | Dual diskoids | dodecahedron

Pockets

Thm (GSSW)

- The pocket $P(\mathcal{C})$ has coherent geodesics.
- It is an interval bundle over the disk.
- The webs $W \in \mathcal{C}$ are its "sections."
- The separation labeling/growth labeling encodes geodesic distances from the base face.

Buildings

Thm (FKK) If W is a non-elliptic $SL(3)$ web, then $D(W)$ embeds in the affine building $\Delta(SL(3)^\vee)$, preserving distances.

next!

– In fact, $D(W)$ is CAT(0).

Double cosets

Obs Given groups $H \leq G$, have a "distance" on G: (or G/H)

$$d : G \times G \to H\backslash G/H$$

$$d(p,q) = \underbrace{Hp^{-1}qH}_{\text{a double coset}}$$

Ex For $\mathbb{R}_{>0} \leq \mathbb{C}^\times$, (double) cosets are represented by polar angles. Then $d(re^{i\theta}, se^{i\phi}) = e^{i(\phi-\theta)}\mathbb{R}_{>0}$.

Basically $(x,y) \mapsto \arg\left(\frac{y}{x}\right) = \mathrm{atan2}(y,x)$!

Double cosets

Ex For $\mathbb{C}[[t]]^\times < \mathbb{C}((t))$, coset reps are $\{t^n \mid n \in \mathbb{Z}\}$

Ex For $GL_r(\mathbb{C}[[t]]) < GL_r(\mathbb{C}((t)))$, double coset reps are

$$t^\lambda = \begin{pmatrix} t^{\lambda_1} & & & \\ & t^{\lambda_2} & & \\ & & \ddots & \\ & & & t^{\lambda_r} \end{pmatrix}$$

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$ integers.

$GL(r)$ dominant weights

Ex For $PGL_r(\mathbb{C}[[t]]) < PGL_r(\mathbb{C}((t)))$, now have

$$t^\lambda = t^{\lambda + (1^r)}$$

$SL(r)$ dominant weights

Affine Grassmannians

Def The affine Grassmannian of $SL(r)^\vee$ is

$$\mathrm{AffGr}_r = PGL_r(\mathbb{C}((t)))/PGL_r(\mathbb{C}[[t]]).$$

- Have "distance"

$$d : \mathrm{AffGr}_r \times \mathrm{AffGr}_r \to \mathbb{Z}^r_{dec}/\langle 1,\dots,1\rangle$$] $SL(r)$ dominant weights

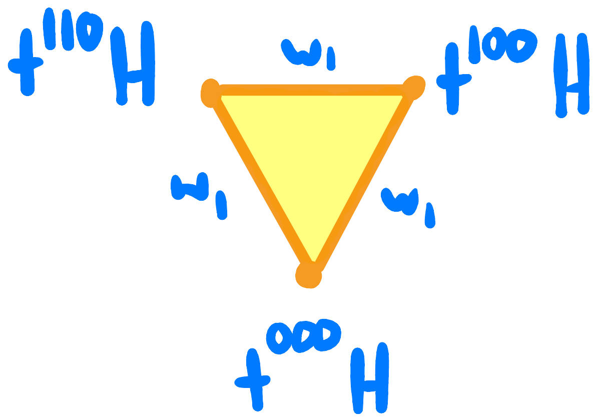
where $d(pH, qH) = d(p,q) = \lambda \iff p^{-1}q \in H t^\lambda H$

- Have $d(p,p) = 0$ $\quad d(p,q) = d(gp, gq)$ for $g \in GL_r(\mathbb{C}((t)))$ $\quad d(q,p) = \underline{-\mathrm{rev}(d(p,q))}$

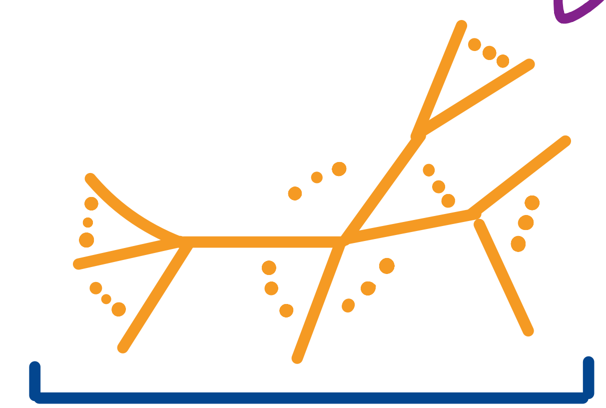
not quite (anti)symmetric

Affine Buildings

Def The affine building on $AffGr_r$ is the simplicial complex Δ_r whose vertices are the points of $AffGr_r$ and whose simplices are collections of points all of whose distances are fundamental weights.

Ex 

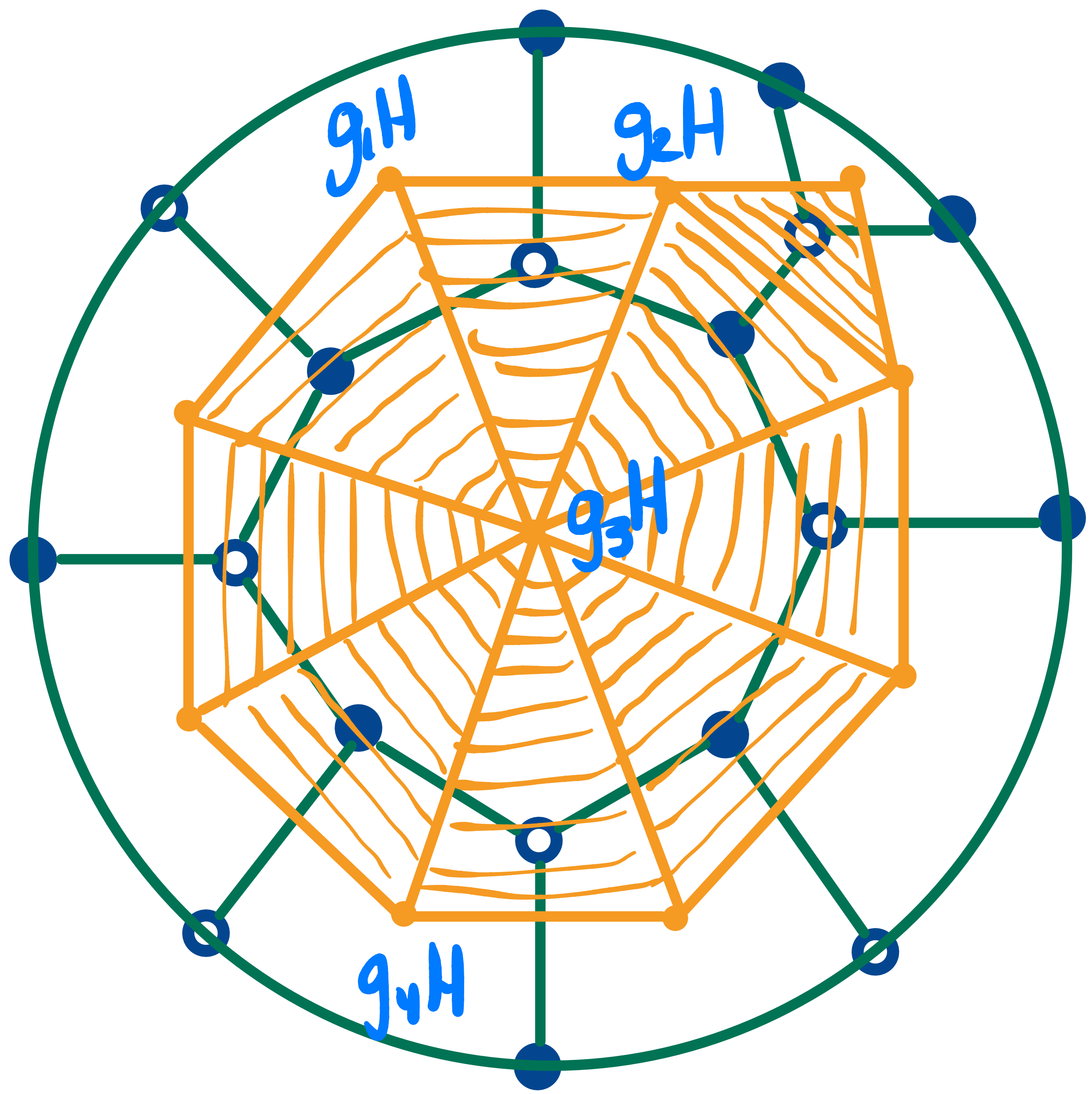
$\in \Delta_3$

Ex $\Delta_2 =$ 

tree where every vertex has uncountable degree

Affine Buildings

Ex There exist $g_1H, g_2H, g_3H, g_4H, \ldots \in \mathrm{AffGr}_3$ s.t.



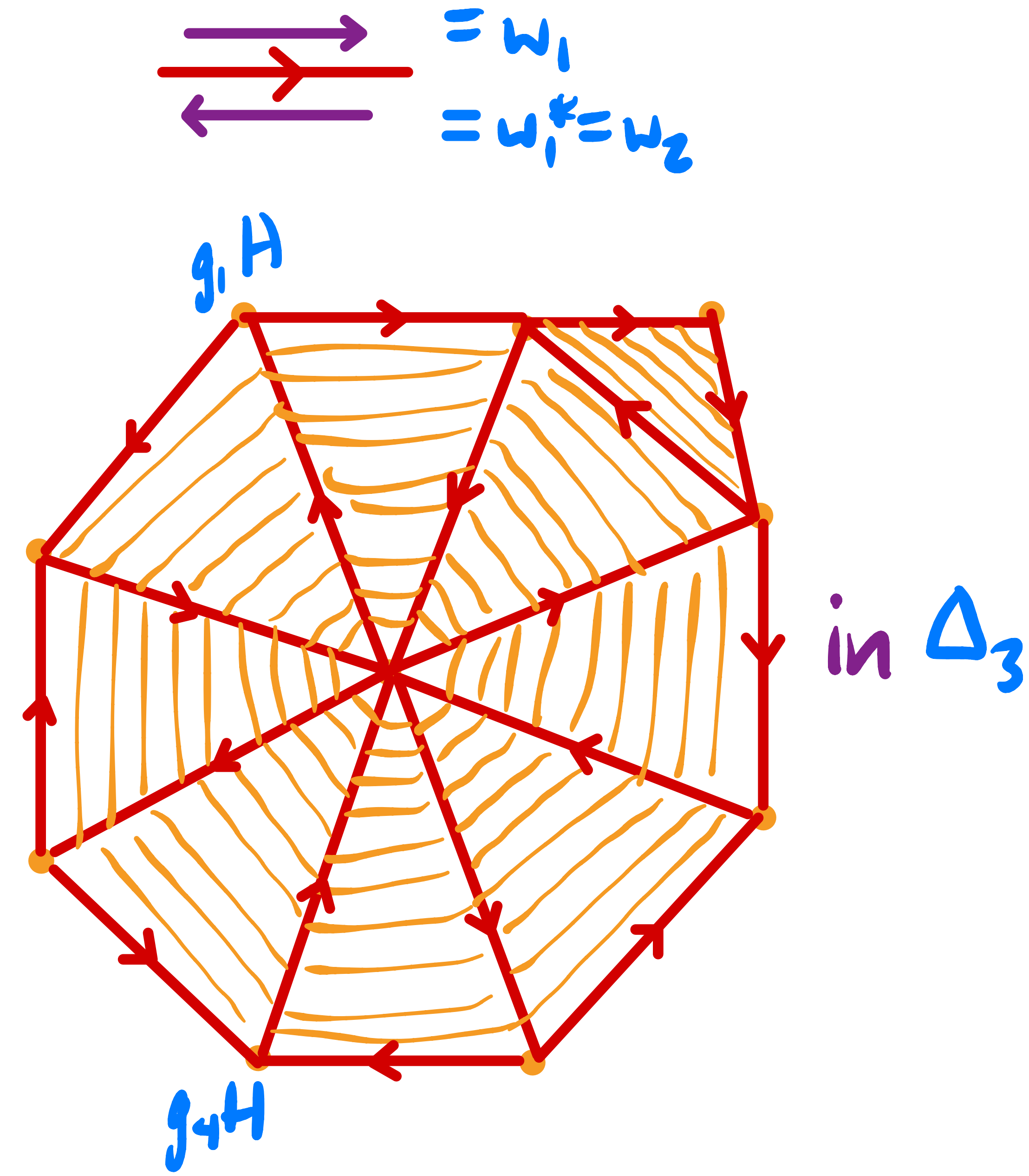
$d(g_1, g_2) = w_1$

$d(g_2, g_3) = w_1$

$\vdots$

$d(g_4, g_1) = 2w_1$ (geodesic distance)

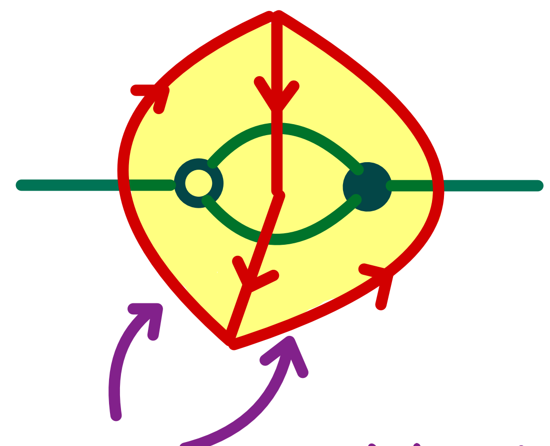
$\vdots$



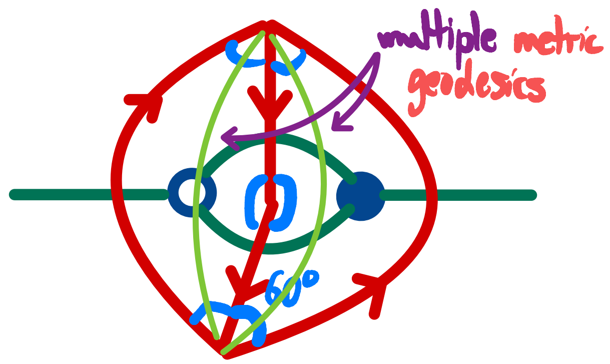
Affine Buildings

Q Non-elliptic condition meaning?

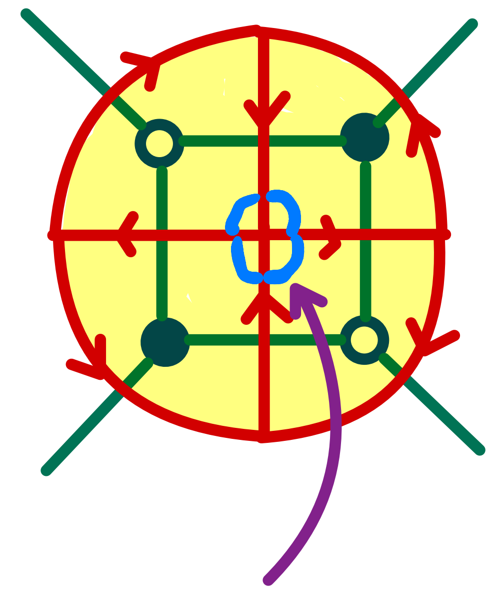
A



must be equilateral



Not CAT(0)!



Angle $4 \cdot 60° = 240° < 360°$!

Also not CAT(0)!

Pockets

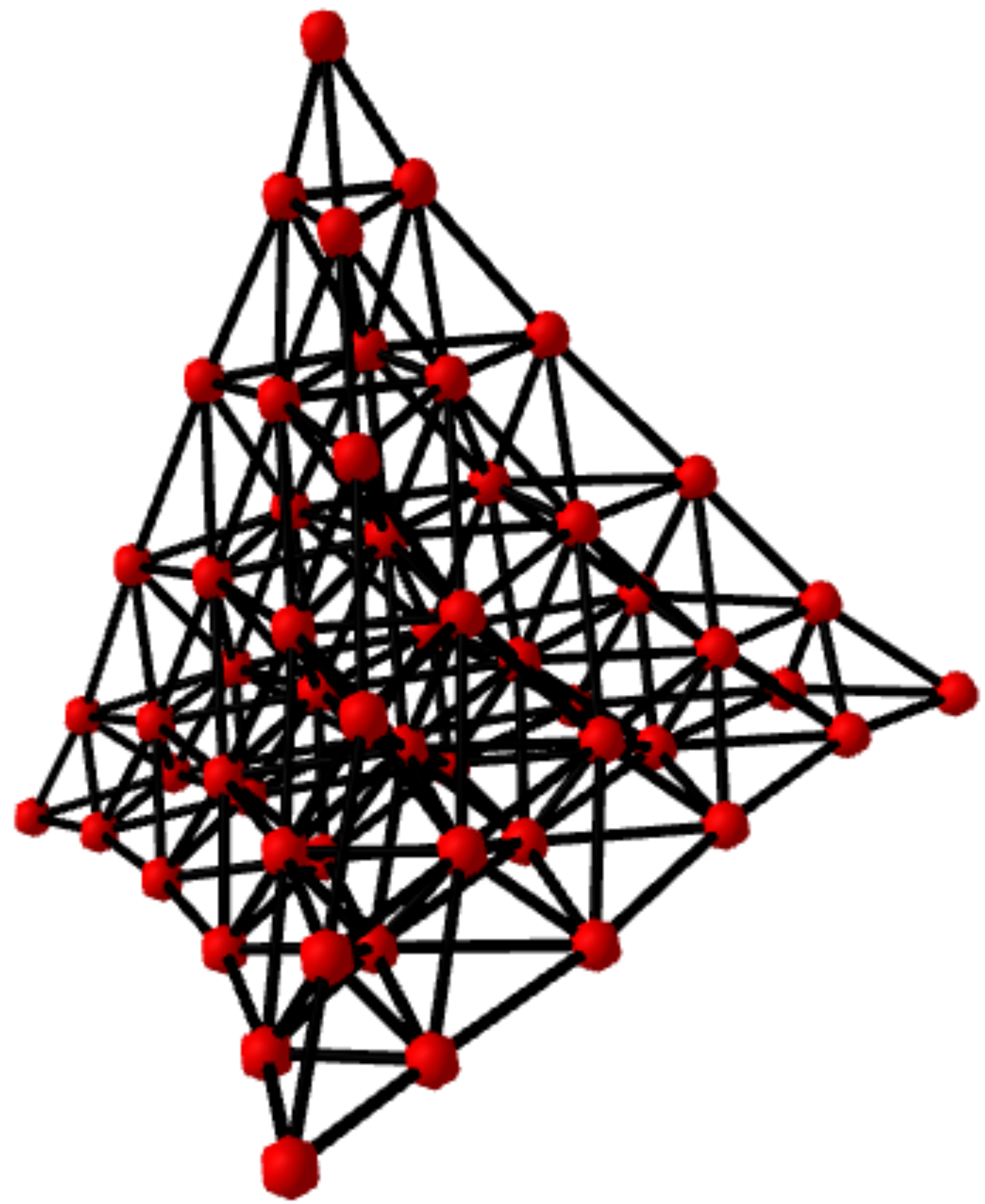
Thm (GSSW)

- Pockets $P(\ell)$ embed in Δ_4.
- Indeed, $P(\ell)$ is CAT(0).
- Given an irreducible component of a Satake fiber of Δ_4 indexed by T, there is a dense open subset U s.t. every point of U extends uniquely to a configuration $P(\ell) \hookrightarrow \Delta_4$ which preserves distances.

Pockets

Ex Pocket of 5×5 ASM:

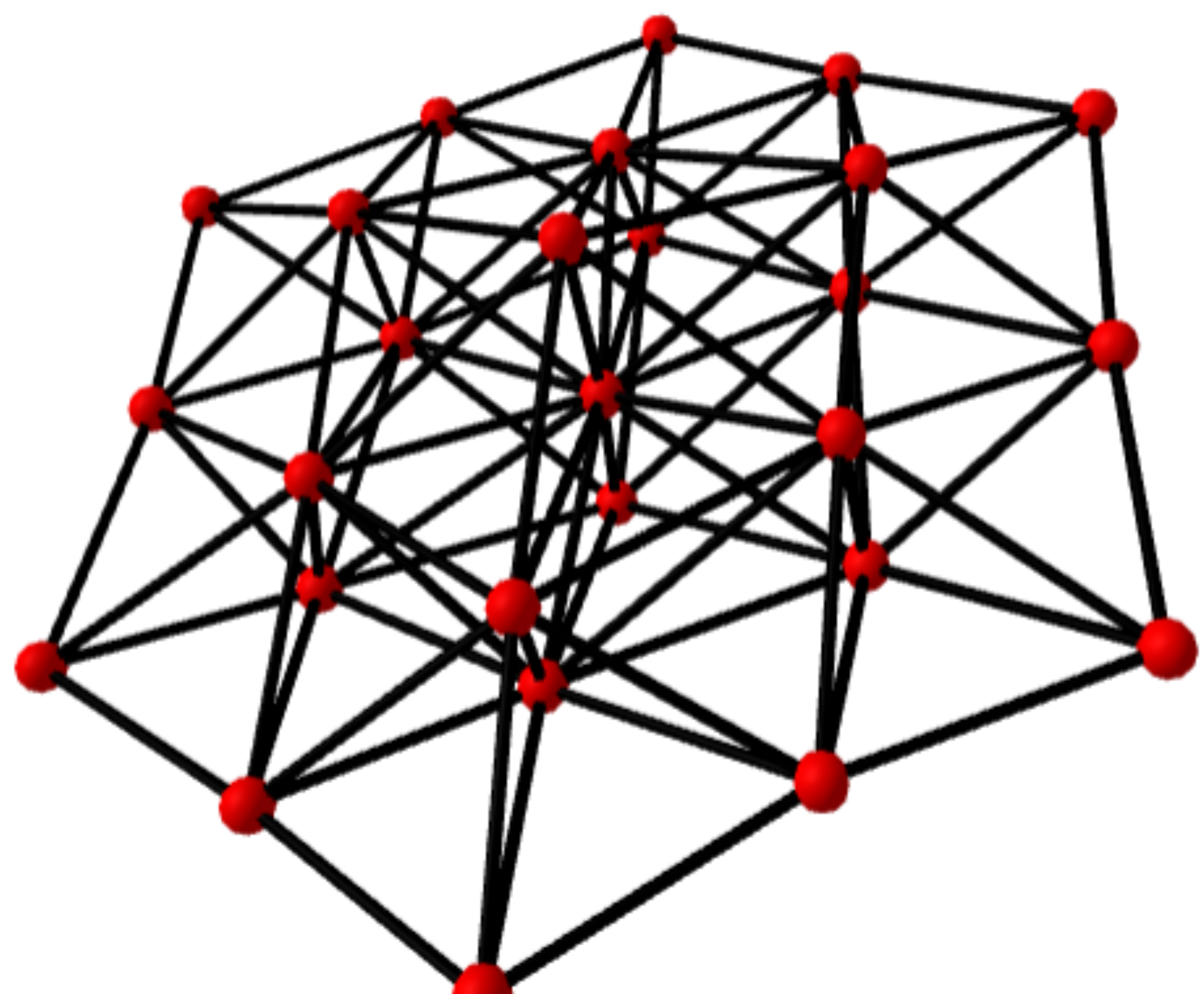
$L = 1^5 2^5 3^5 4^5$



Note Related to height functions, octahedral recurrence, tilings of the Aztec diamond, distributive lattices

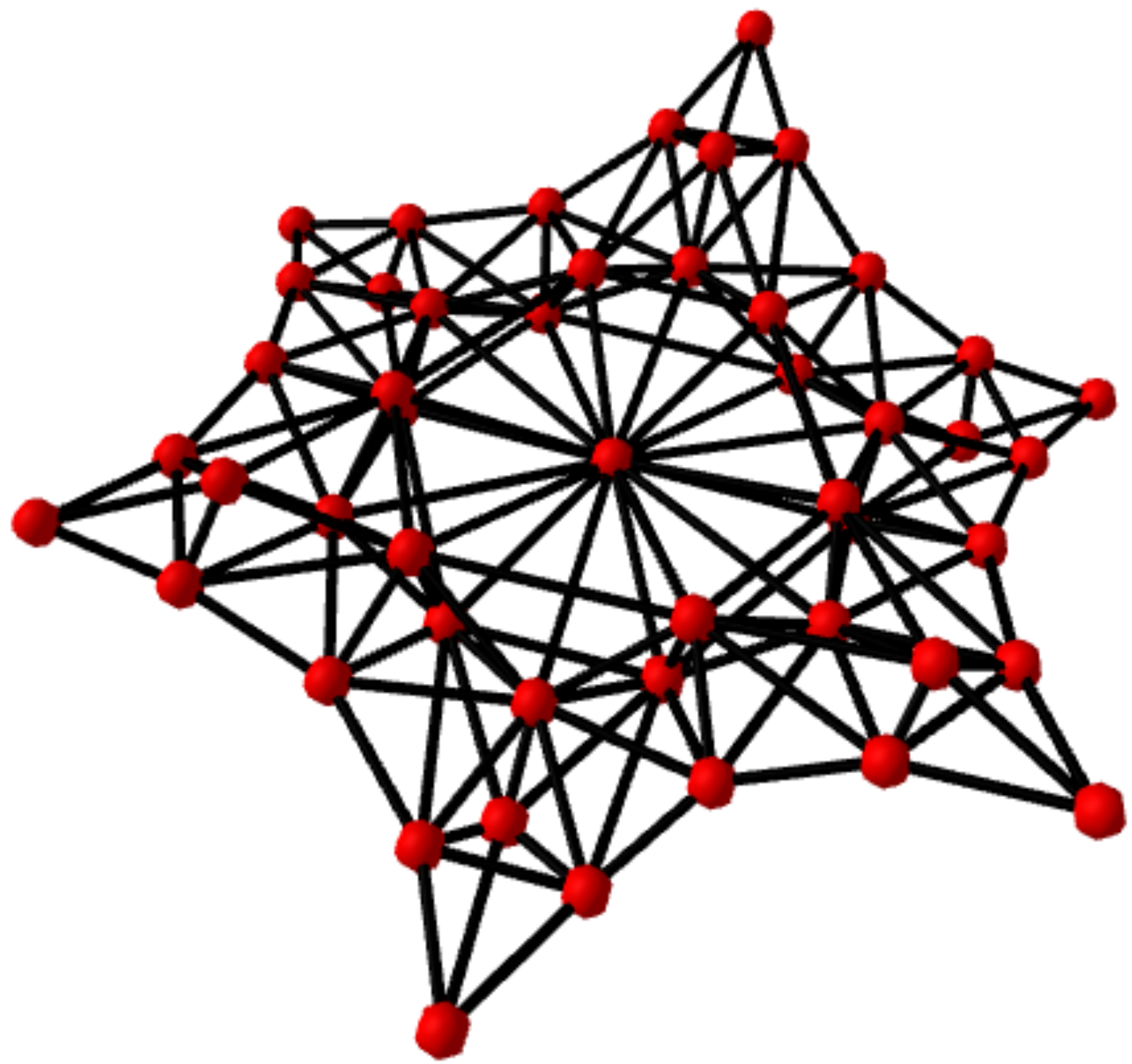
Pockets

Ex | Pocket of $2\times 2\times 2$ PP:



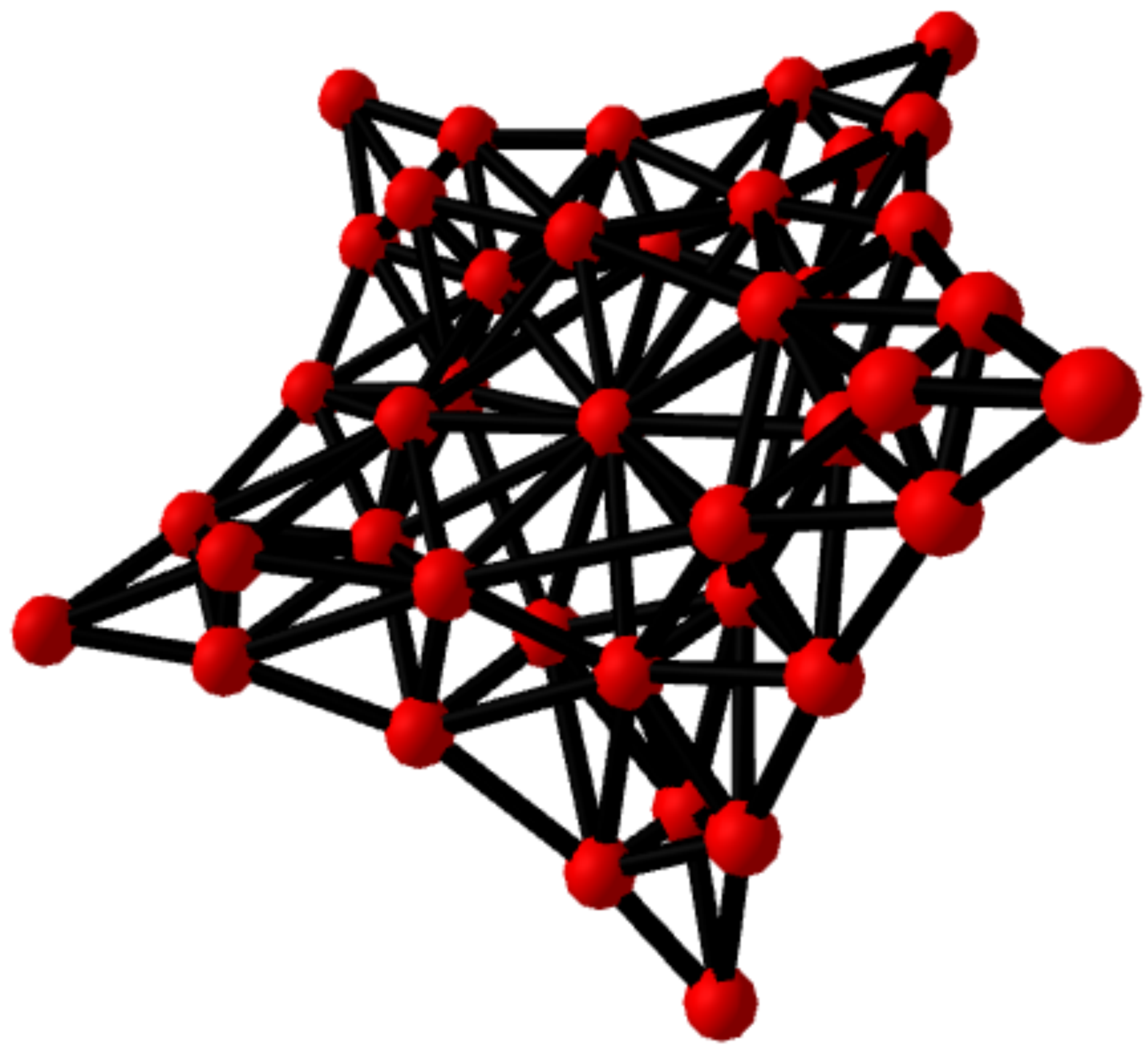
Pockets

Ex Pocket of a "chained hexagon":



Pockets

Ex Pocket of a "chained pentagon":



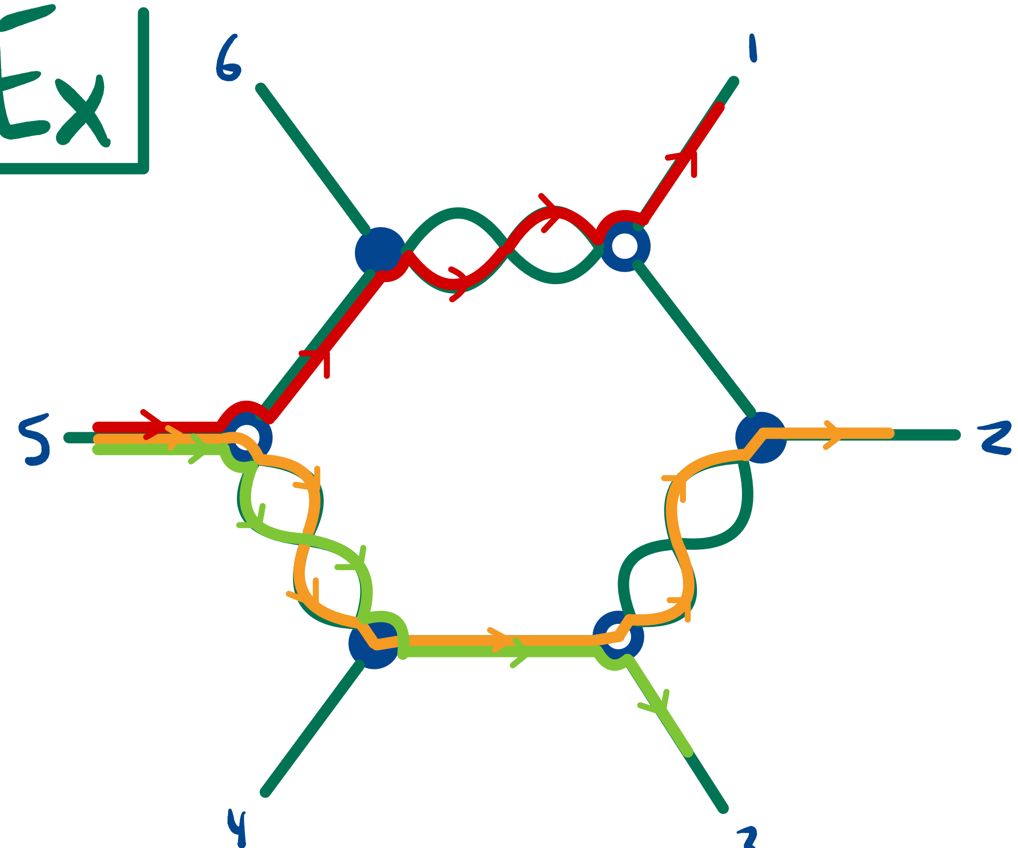
Note Not realizable in $\mathbb{R}^3$

THANKS!

Trip permutations

Def (GPPSS '23) An r-hourglass plabic graph has **trip permutations** $\mathrm{trip}_1, \ldots, \mathrm{trip}_{r-1}$ where trip_i takes the <u>ith left at white</u> and <u>ith right at black</u>:

Ex



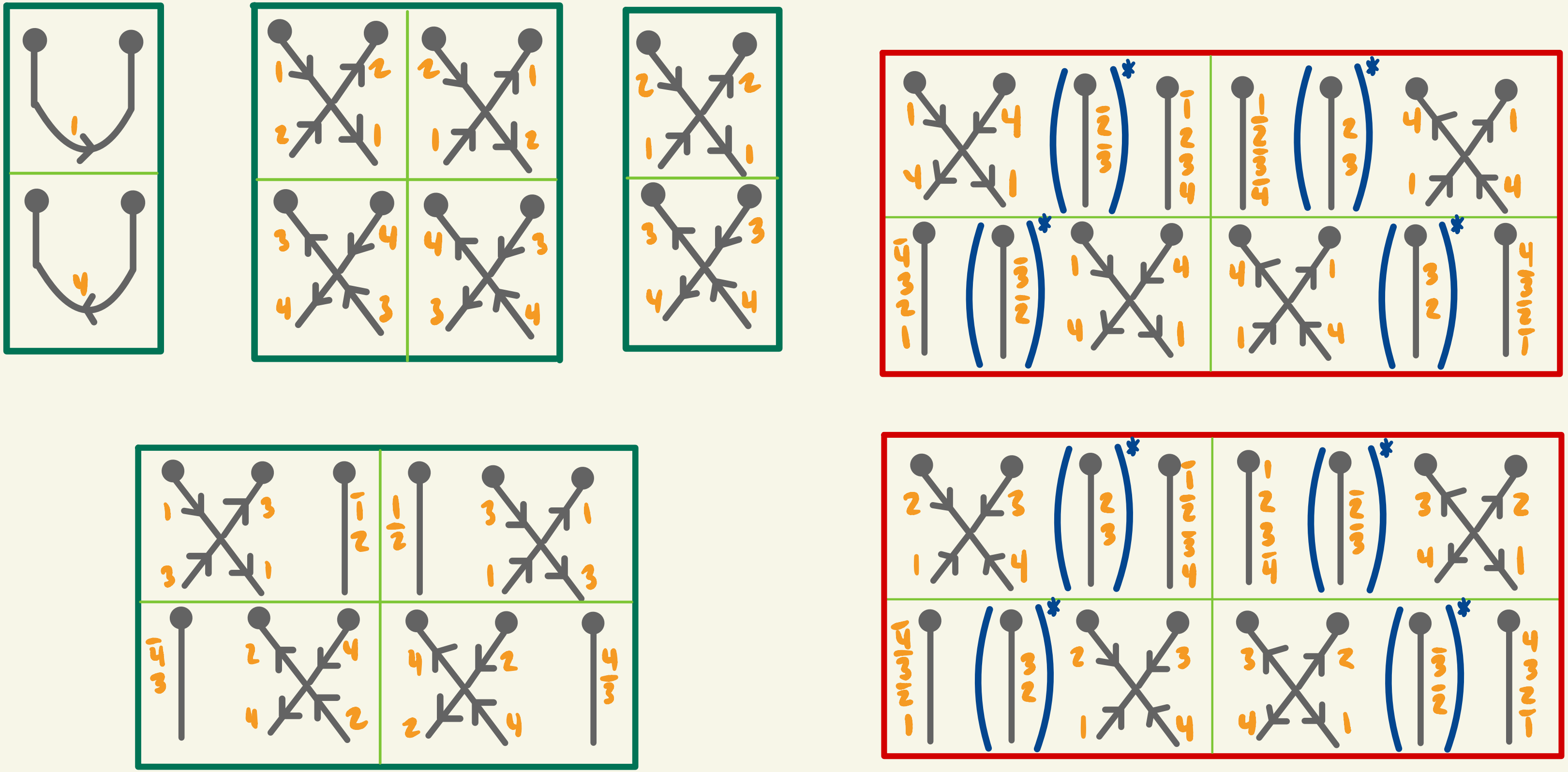
$\longrightarrow$ (red) $= \mathrm{trip}_1 = (1\,3\,5)(6\,4\,2)$

$\longrightarrow$ (orange) $= \mathrm{trip}_2 = (1\,4)(2\,5)(3\,6)$

$\longrightarrow$ (green) $= \mathrm{trip}_3 = (5\,3\,1)(2\,4\,6)$

Note

$\mathrm{trip}_i = \mathrm{trip}_{r-i}^{-1}$!



SL(4) growth rules