Webs, pockets, and buildings

Joshua P. Swanson
University of Southern California (USC), Los Angeles

Based on joint work with subsets of *Christian Gaetz, Oliver Pechenik, Stephan Pfannerer, Jessica Striker, and Haihan Wu*

arXiv:2306.12501 (4-row)
arXiv:2402.13978 (2-column)
arXiv:2306.12506 (promotion permutations)

Slides: https://www.jpswanson.org/talks/2024_UCLA_pockets.pdf

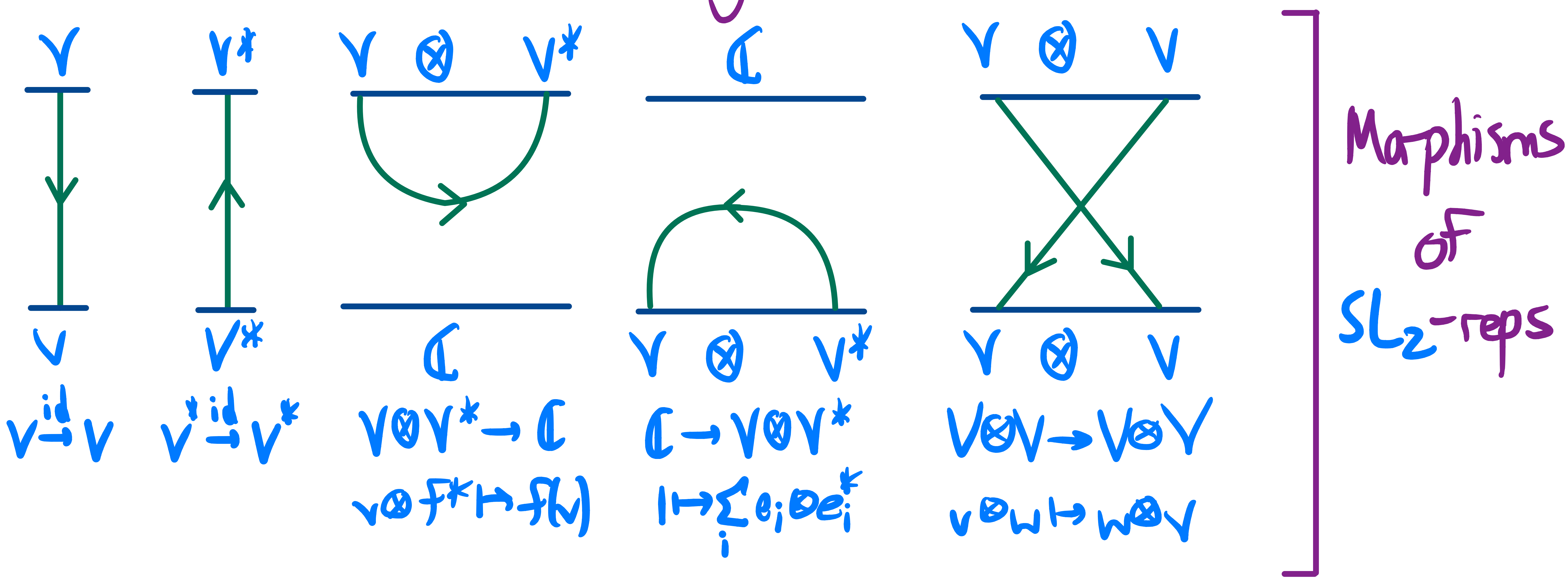
Presented at
UCLA Combinatorics Forum
November 7th, 2024

Outline

- SL_2-webs and SL_3-webs
  - Temperly-Lieb and non-elliptic bases
- Hourglass plabic graphs
  - (New!) SL_4 web basis
- Pockets and buildings

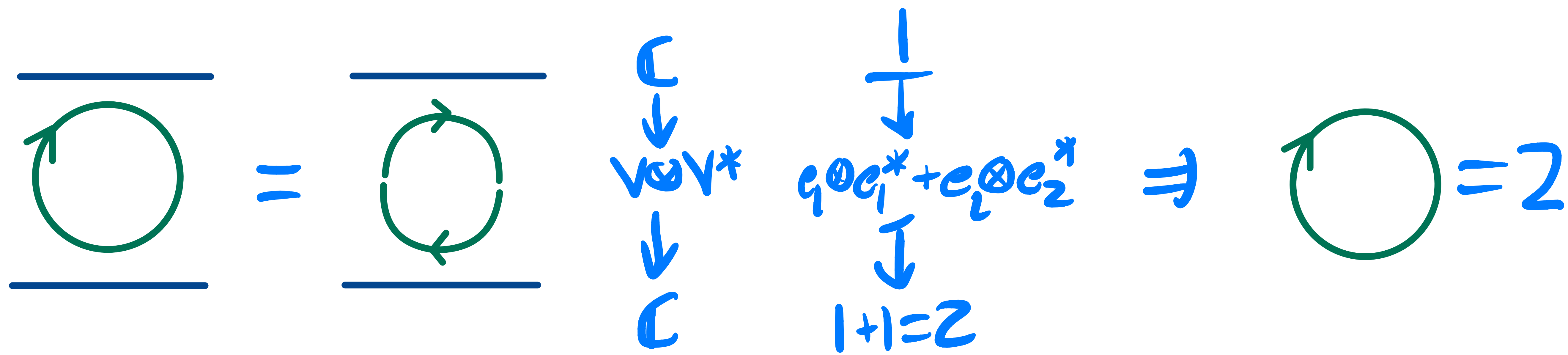
SL_2-Webs

- **Webs** are a graphical calculus for representations.
- Let $V = \mathbb{C}^2$. Some building blocks:

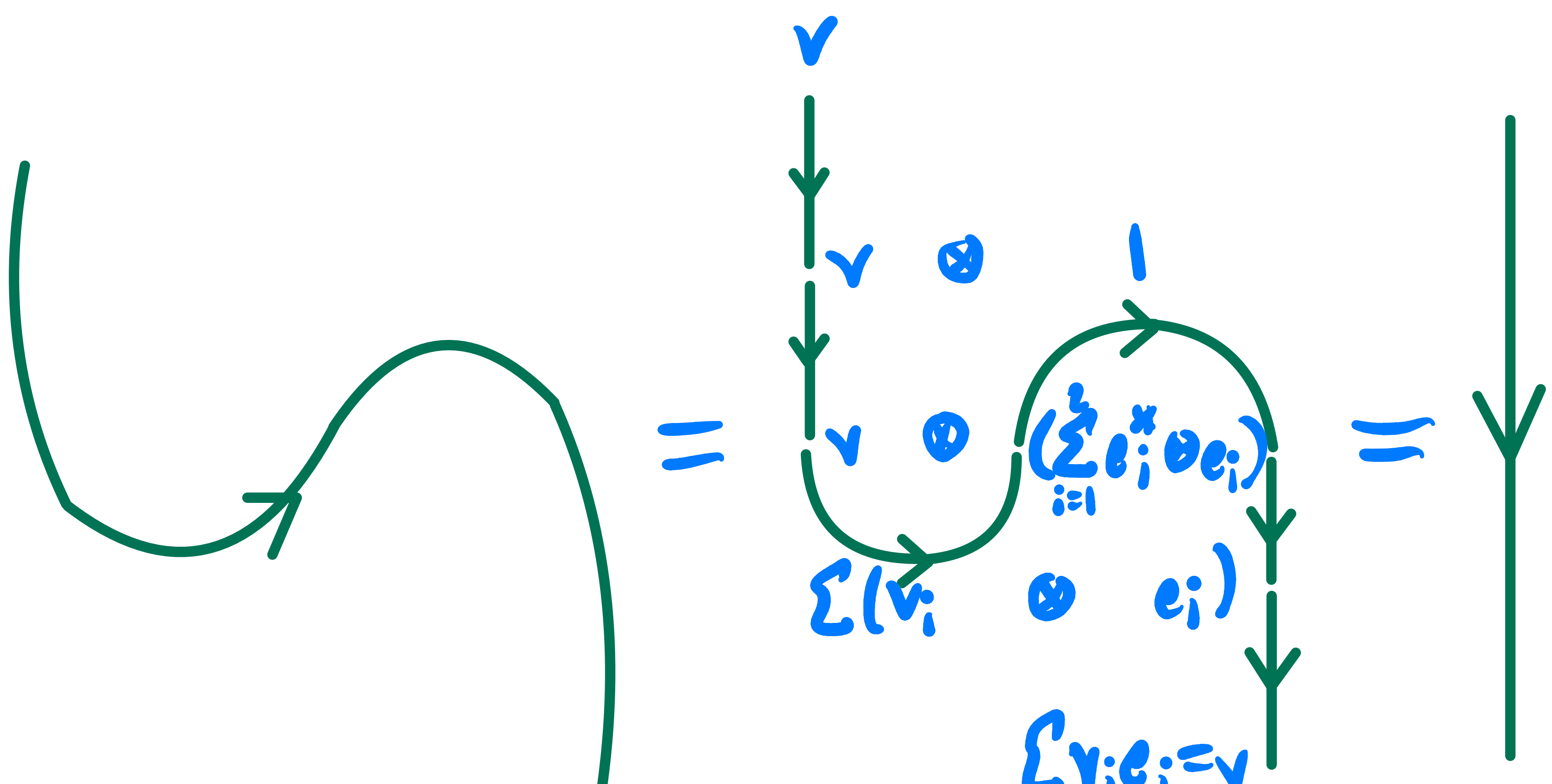


SL₂-Webs

Ex



Ex



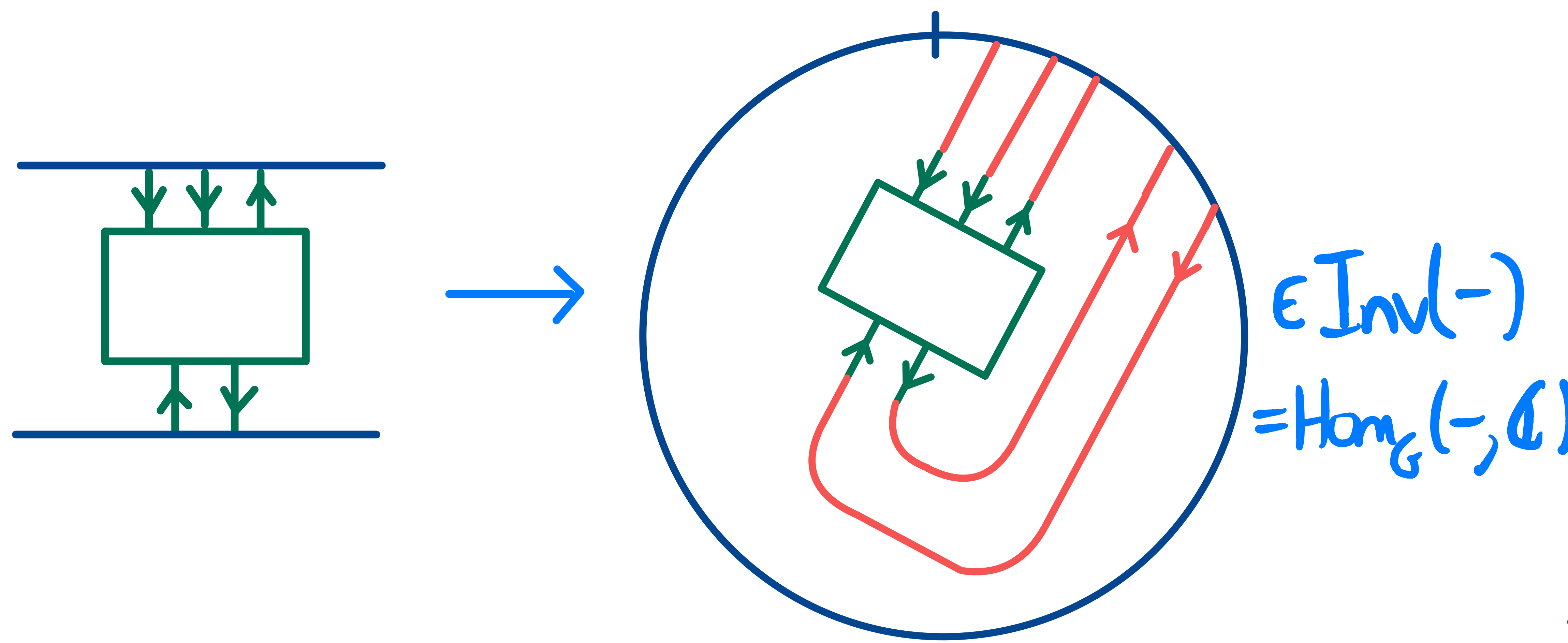
SL_2-Webs

- Can "rotate" factors from codomain to domain with duals:

$$\mathrm{Hom}_G(A, B\otimes C) \cong \mathrm{Hom}_G(A\otimes C^*, B)$$

(Tensor-hom adjunction and $\mathrm{Hom}_{\mathbb{C}}(X,Y) \cong Y^*\otimes X$.)

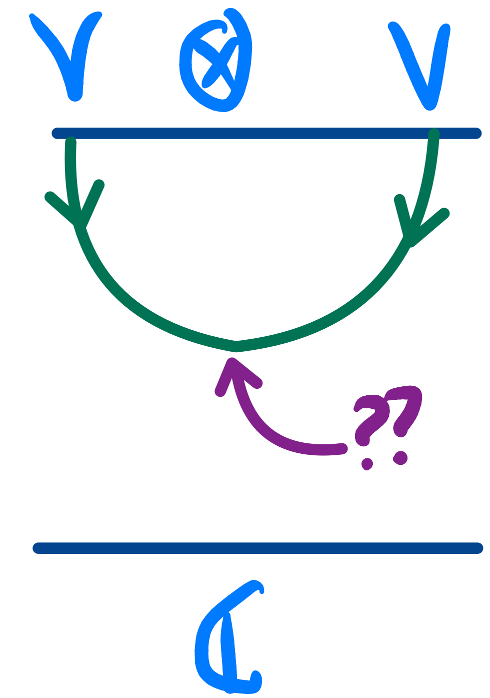
Ex



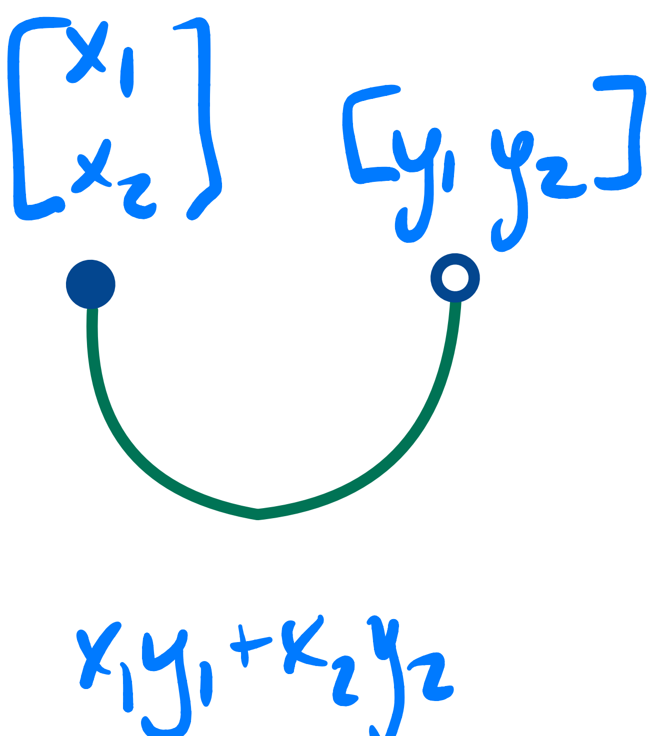
Graphs in disks

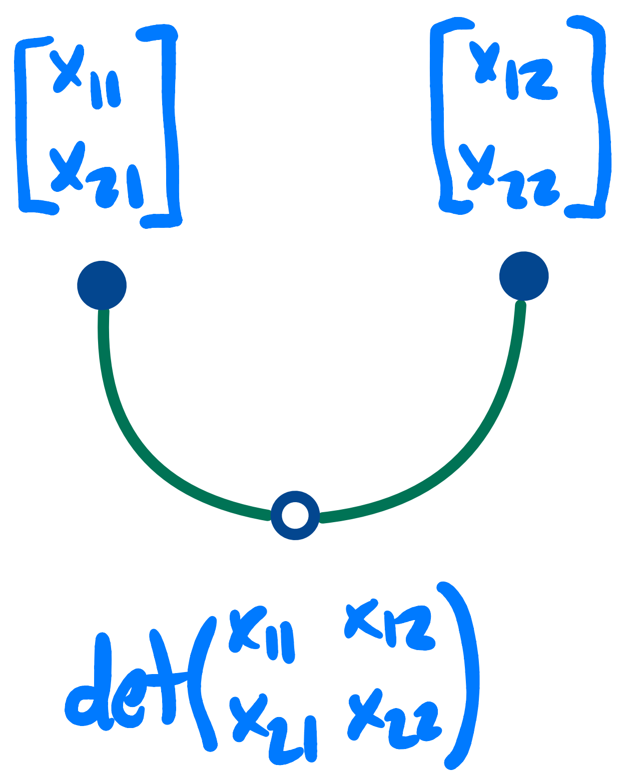
SL_2-Webs

- What about $\det: V \otimes V \to \mathbb{C}$? Use



- Switch to <u>bipartite</u> graphs:

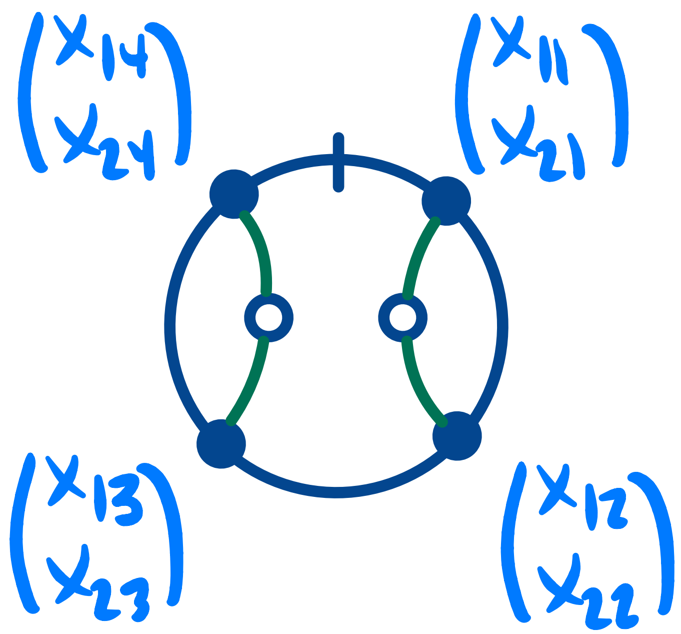




• $= V$ ∘ $= V^*$
in domain
(opposite in codomain...)

SL_2-Webs

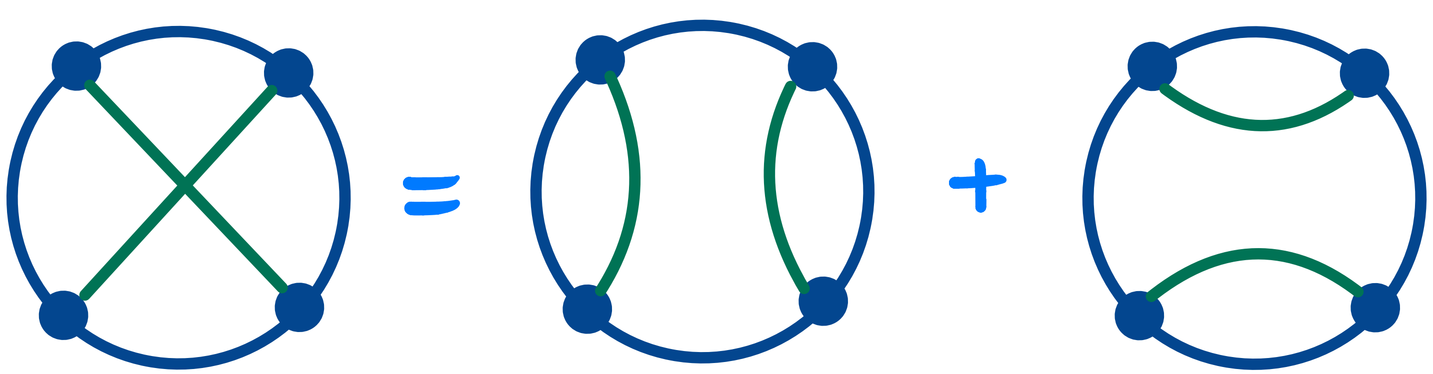
Ex



$$= \det\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \cdot \det\begin{pmatrix} x_{13} & x_{14} \\ x_{23} & x_{24} \end{pmatrix} \in \mathrm{Inv}(V^{\otimes 4})$$

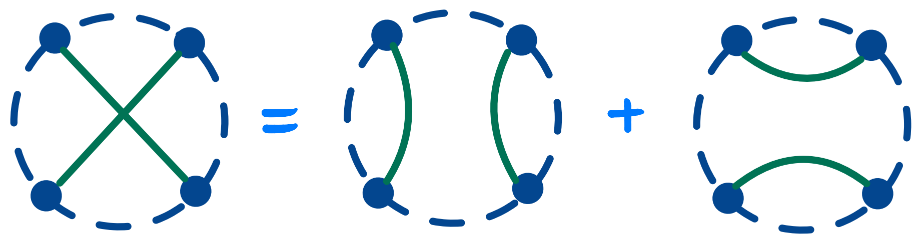
$$= \mathrm{Hom}_{SL_2}(V^{\otimes 4}, \mathbb{C})$$

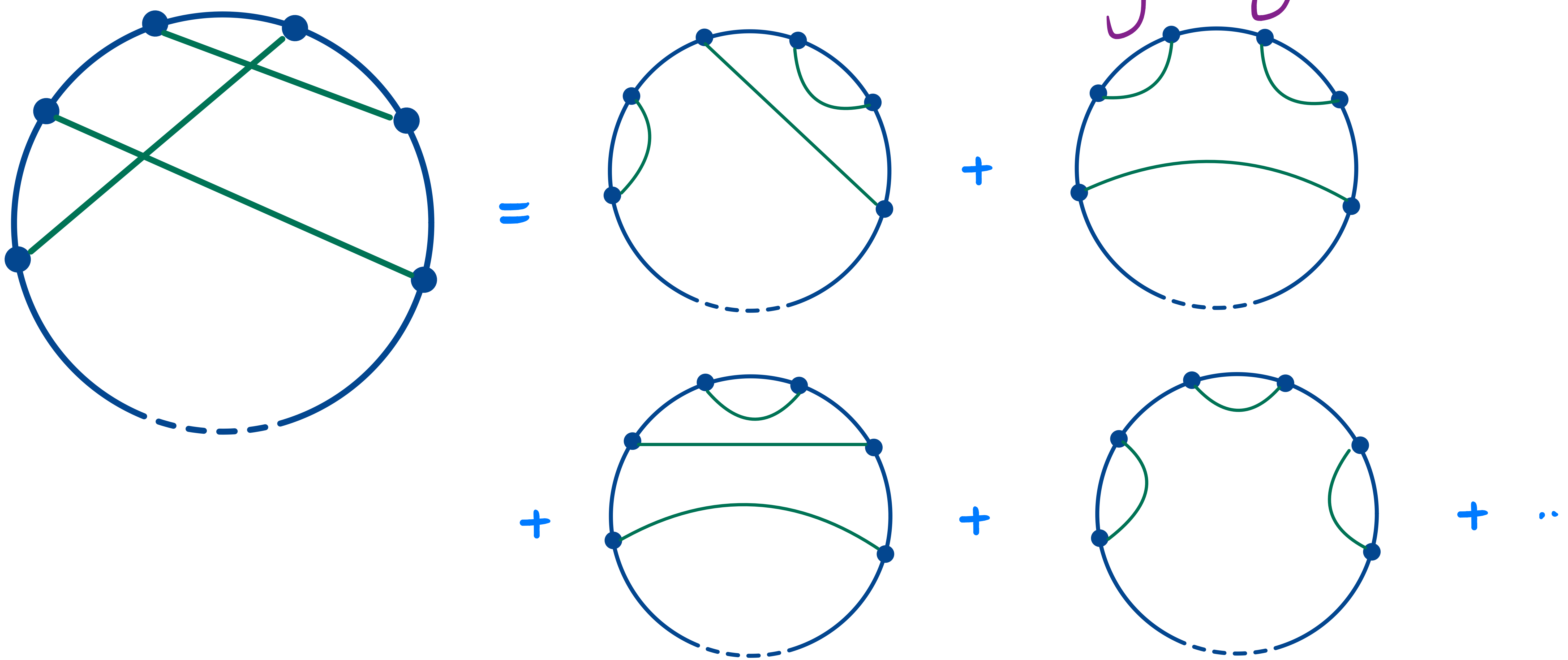
Ex Plücker relations:



$$(x_{11}x_{23} - x_{21}x_{13})(x_{12}x_{24} - x_{22}x_{14})$$
$$=$$
$$(x_{11}x_{22} - x_{21}x_{12})(x_{13}x_{24} - x_{23}x_{14})$$
$$+$$
$$(x_{11}x_{24} - x_{21}x_{14})(x_{12}x_{23} - x_{22}x_{13})$$

Temperly-Lieb basis

- Using  , can reduce any matching diagram to a linear combination of matching diagrams:



Temperly-Lieb basis

Thm The noncrossing 2-row webs are a basis for

$$\mathrm{Inv}_{SL_2}(V_1 \otimes \cdots \otimes V_n) \qquad (V_i \in \{V, V^*\})$$

called the Temperly-Lieb basis.

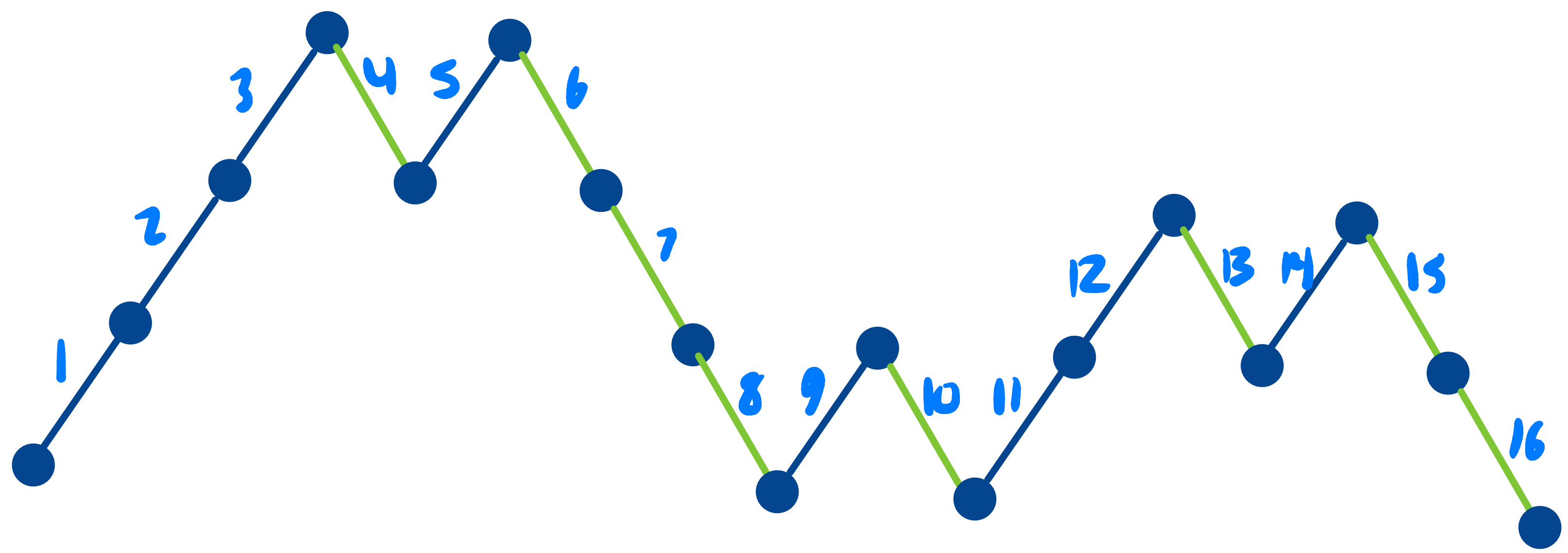
Pf
- Spanning: diagrams span by classical invariant theory, noncrossing by uncrossing rule.
- Independence: by Pieri rule,

$$\dim \mathrm{Inv}_{SL_2}(V^{\otimes n}) = \#SYT(2 \times \tfrac{n}{2})$$

→

Temperly-Lieb basis

Some Catalan bijections:



Dyck path

$\mapsto$ ((()()))()(()()) $\mapsto$ 1112122212112122

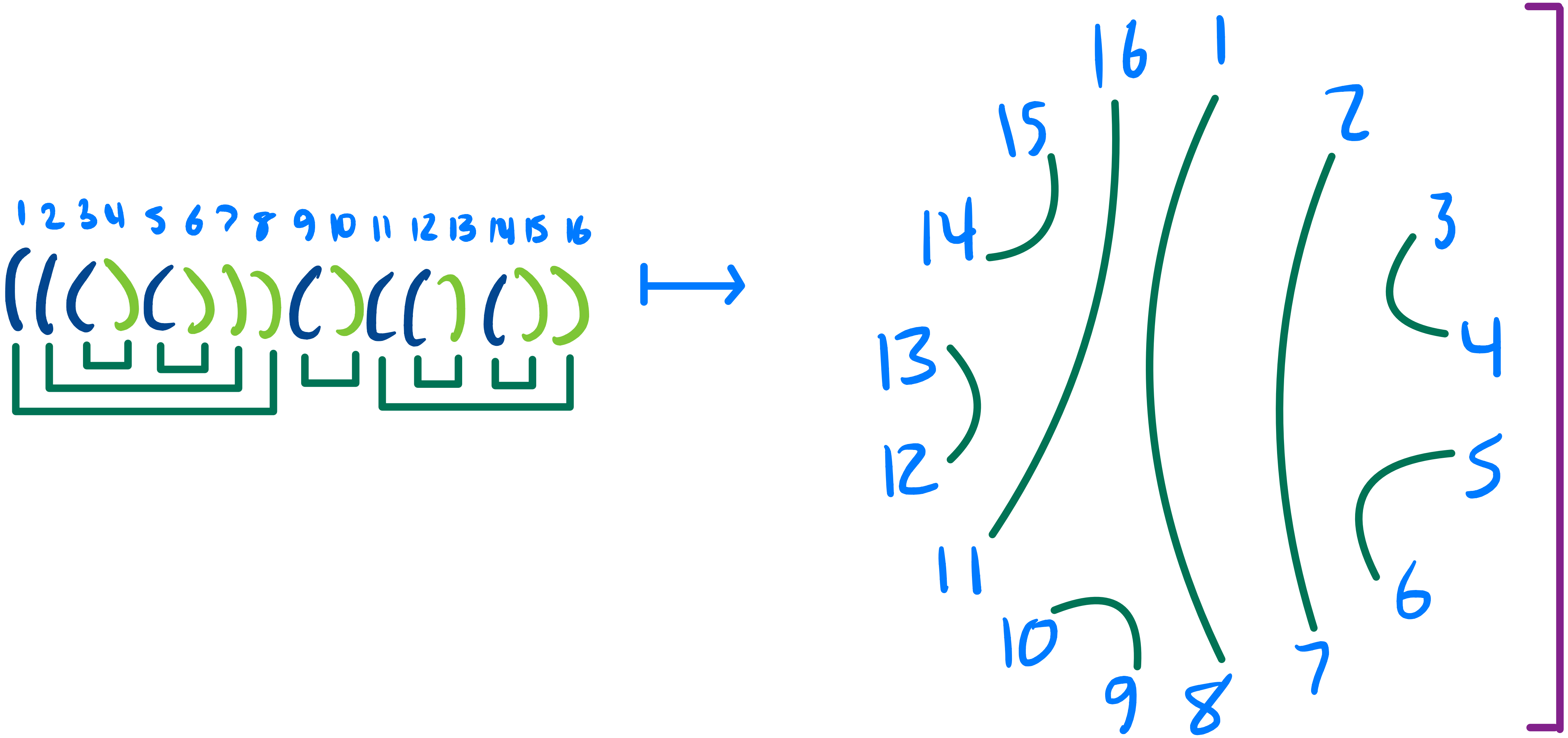
lattice word

$\mapsto$

1	2	3	5	9	11	12	14
4	6	7	8	10	13	15	16

Standard Young tableau

Temperly-Lieb basis



Non-crossing perfect matching ✓

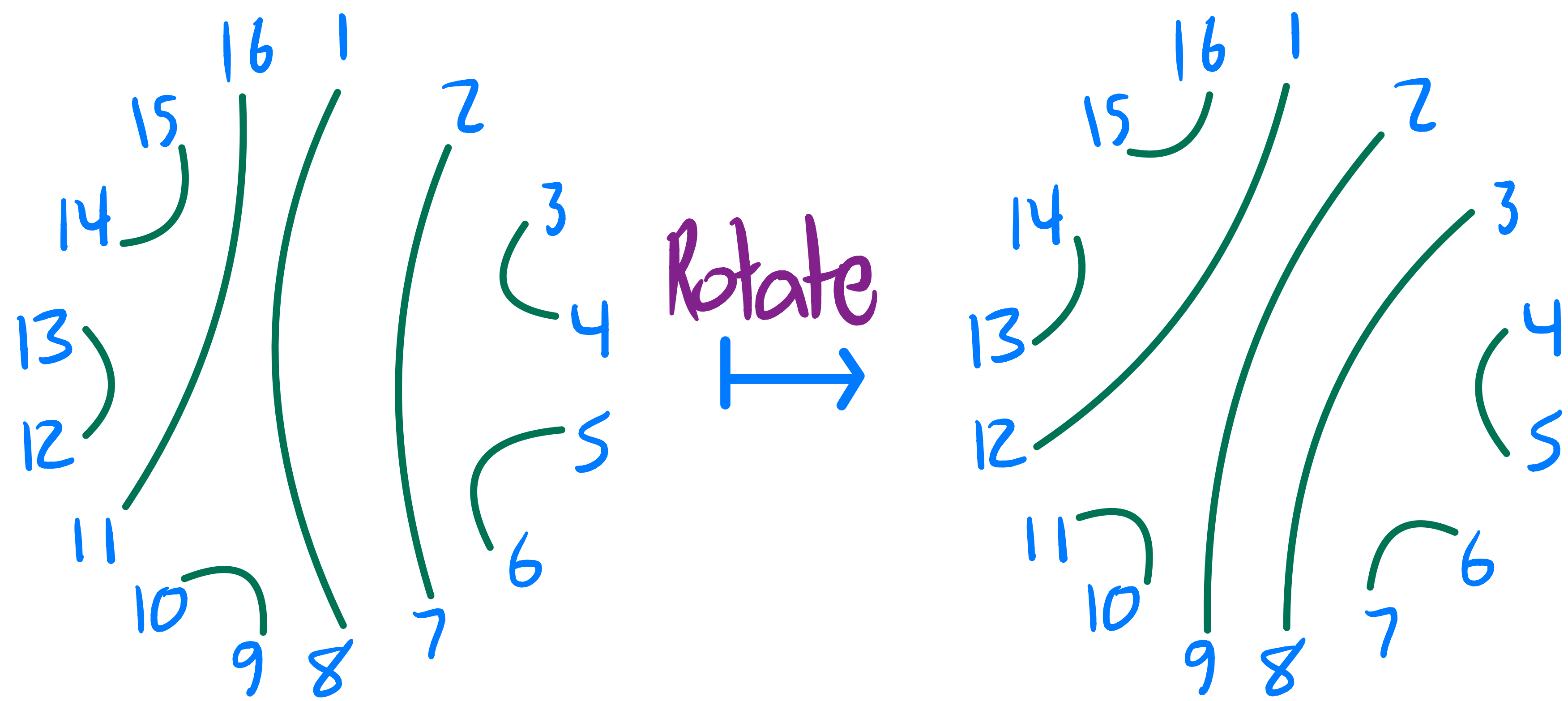
Promotion/evacuation

Thm The bijection $NCM(2n) \xrightarrow{\sim} SYT(2\times n)$ sends rotation to promotion, reflection to evacuation.

- "Hidden" dihedral action on $SYT(2\times n)$!

Promotion/evacuation

Ex



1	2	3	5	9	11	12	14
4	6	7	8	10	13	15	16

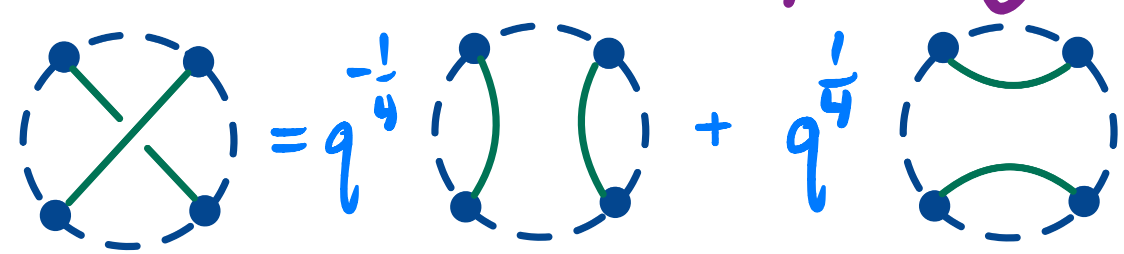
Promote ⟼

1	2	3	4	6	10	13	15
5	7	8	9	11	12	14	16

Quantum Link Invariants

The quantum group $U_q(\mathfrak{sl}_2)$ is an algebra deforming the (universal enveloping algebra of) $\mathfrak{sl}_2$.

- Sending $q \to 1$ recovers the usual case.
- $\mathrm{Inv}(V_1 \otimes \cdots \otimes V_n)$ still has a Temperley-Lieb web basis:



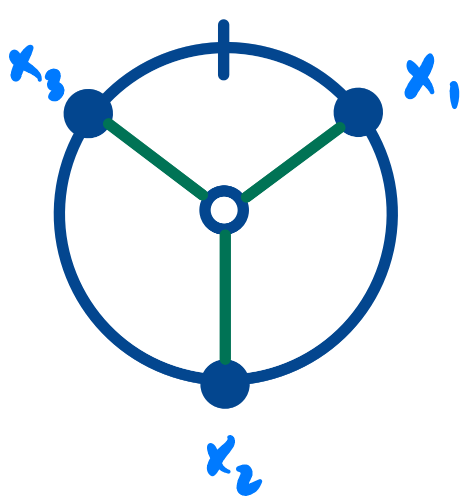
- Projections of knots/links/tangles become polynomials in q; refined version of Jones polynomial

SL_3-webs

- Let $V = \mathbb{C}^3$, $V_i \in \{V, V^*\}$.

  Q: Is there a nice web basis for $\mathrm{Inv}_{SL_3}(V_1 \otimes \cdots \otimes V_n)$?

- Use bipartite planar graphs in a disk built from



$= \det\begin{pmatrix} | & | & | \\ x_1 & x_2 & x_3 \\ | & | & | \end{pmatrix}$ (so trivalent).

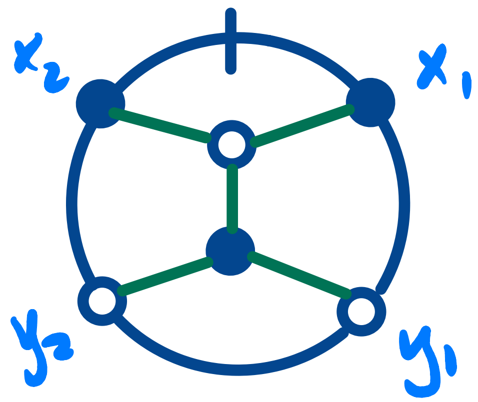
(+ dual)

Have univalent boundary vertices and connected to boundary.

SL_3-webs

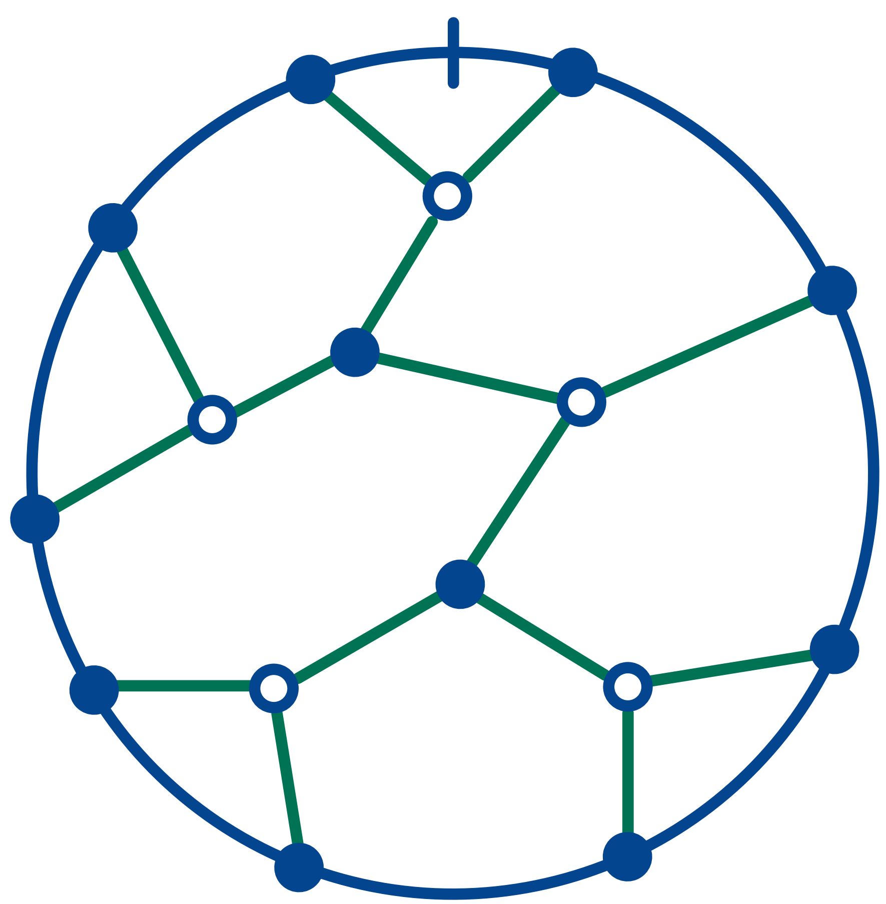
SL_3-webs

Ex



$$= \det\begin{pmatrix} | & | & | \\ x_1 & x_2 & y_1 \wedge y_2 \\ | & | & | \end{pmatrix}$$

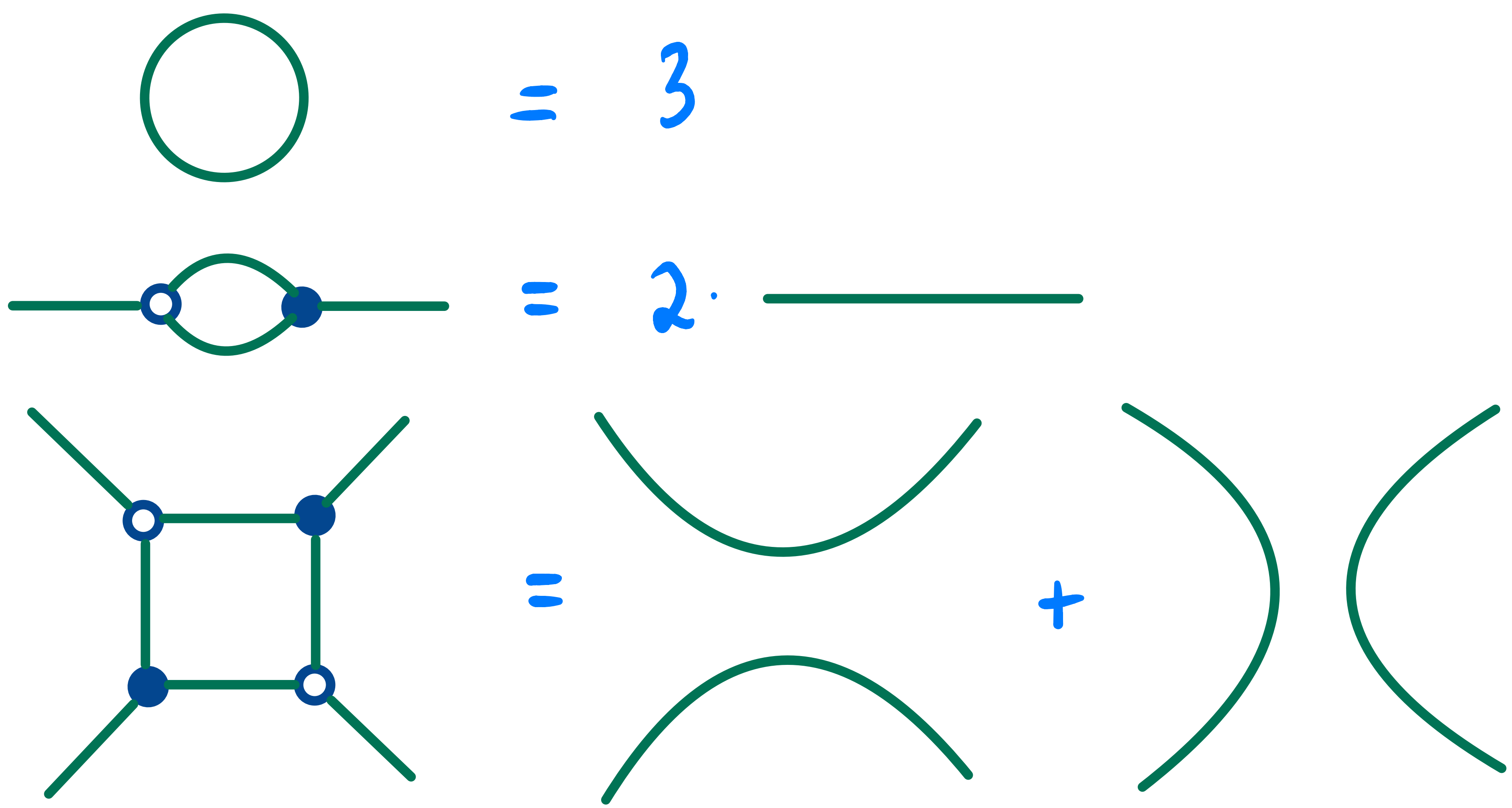
Ex



= … a polynomial obtained by summing over <u>proper edge 3-colorings</u> …

SL_3-webs

Thm (Kuperberg '94) The generating SL_3-web relations are



Non-elliptic web basis

Thm (Kuperberg '94)

Call an SL_3-web non-elliptic if it has no 2-faces or 4-faces.

The non-elliptic webs form a basis of

$$\mathrm{Inv}_{SL_3}(V_1 \otimes \cdots \otimes V_n). \qquad (V_i \in \{V, V^*\})$$

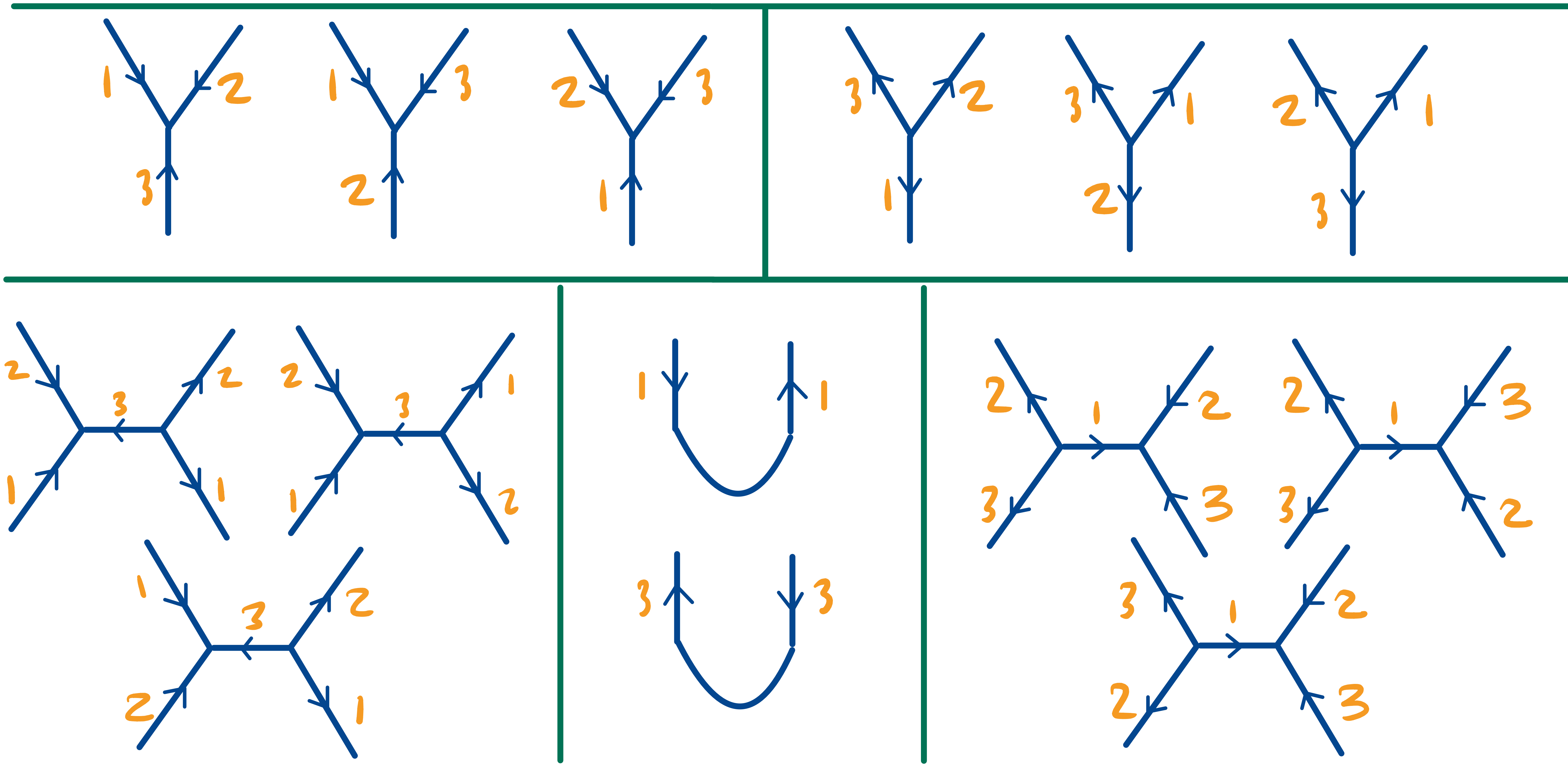
Non-elliptic web basis

Pf | • Spanning: similar to SL_2 case.

• Independence: bijection to $SYT(3 \times \frac{n}{3})$ using growth rules. (Other descriptions of this bijection have since been found.)

SL_3-growth rules

Kuperberg-Khovanov growth rules:



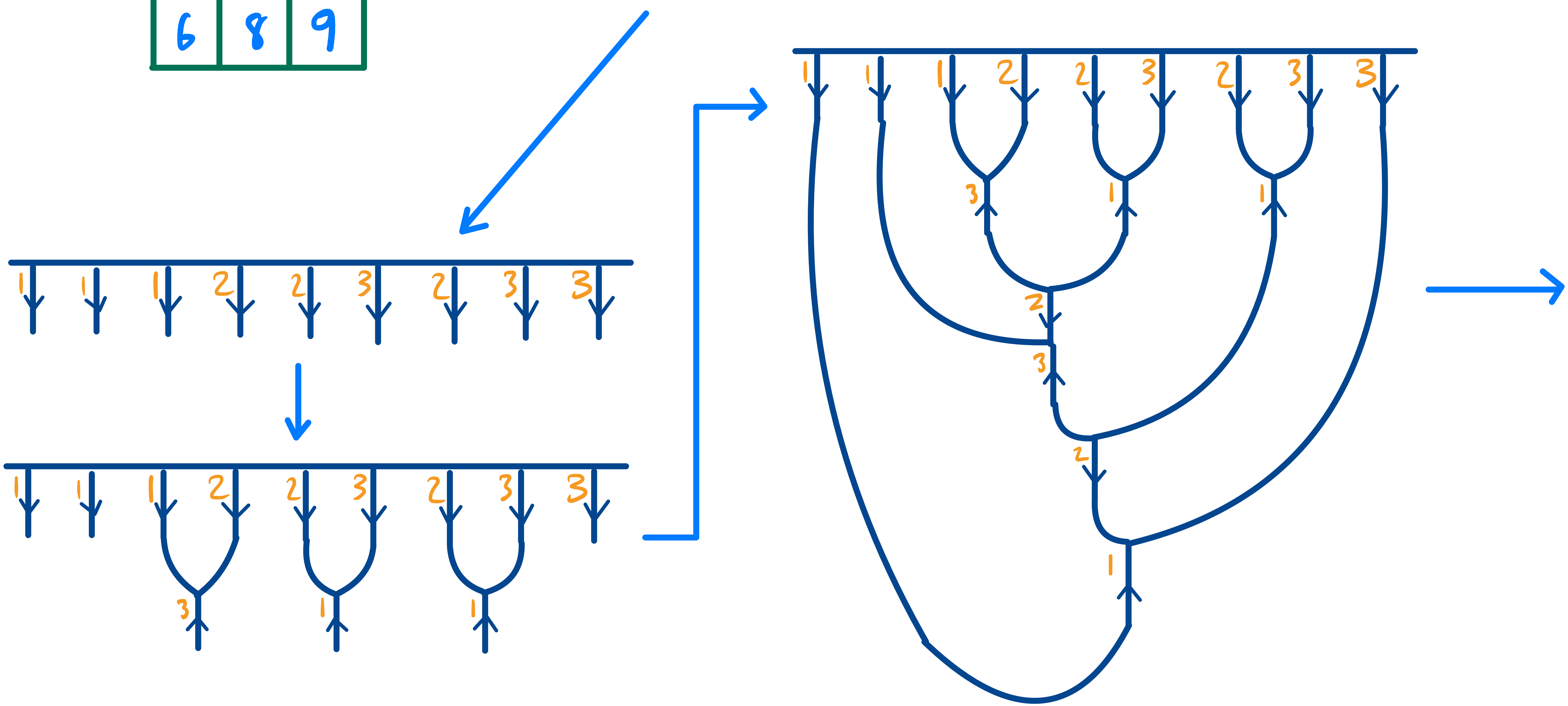
SL_3-growth rules

Ex

$T =$

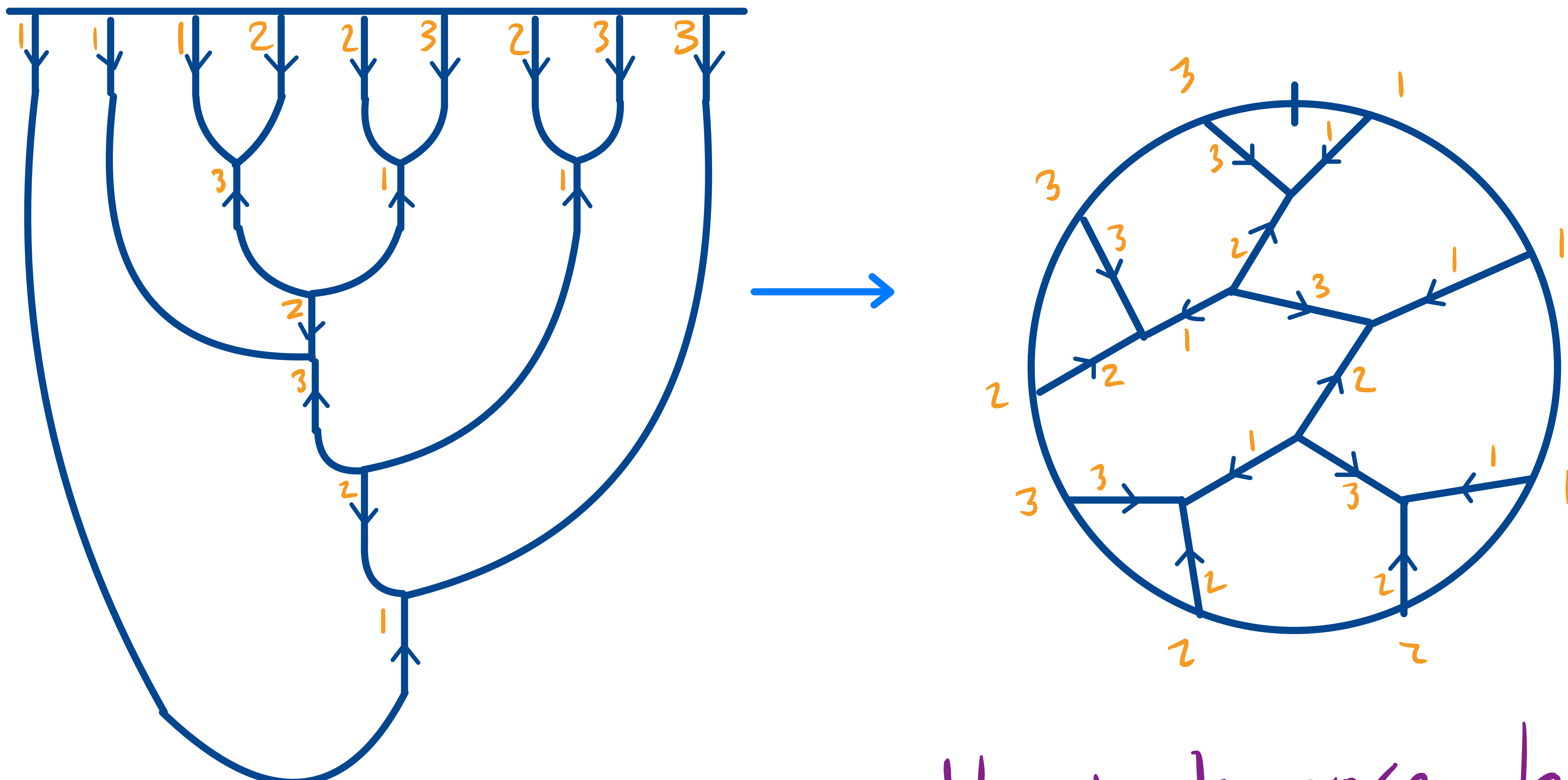
1	2	3
4	5	7
6	8	9

→ 111223233



SL_3-growth rules

$(T \longrightarrow 111223233)$



Now just erase labels!

Promotion/evacuation

Thm (Petersen–Pylyavskyy–Rhoades)
The bijection from non-elliptic webs to $SYT(3\times\frac{n}{3})$
sends rotation to promotion
reflection to evacuation.

- "Hidden" dihedral action on $SYT(3\times m)$!

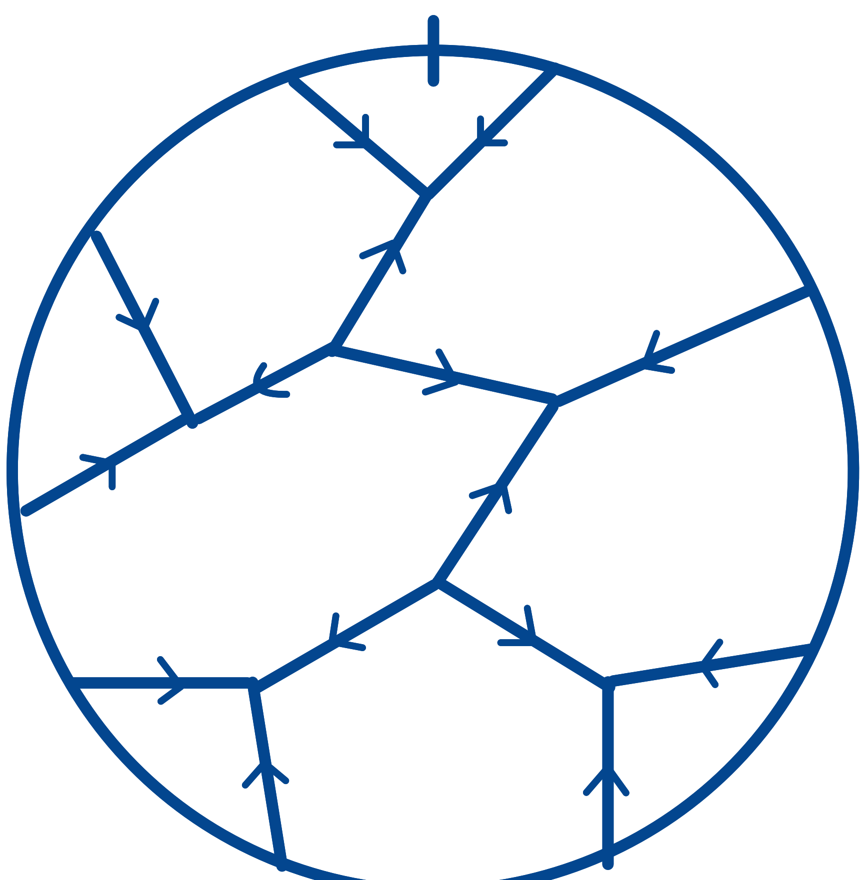
Promotion/evacuation

Ex

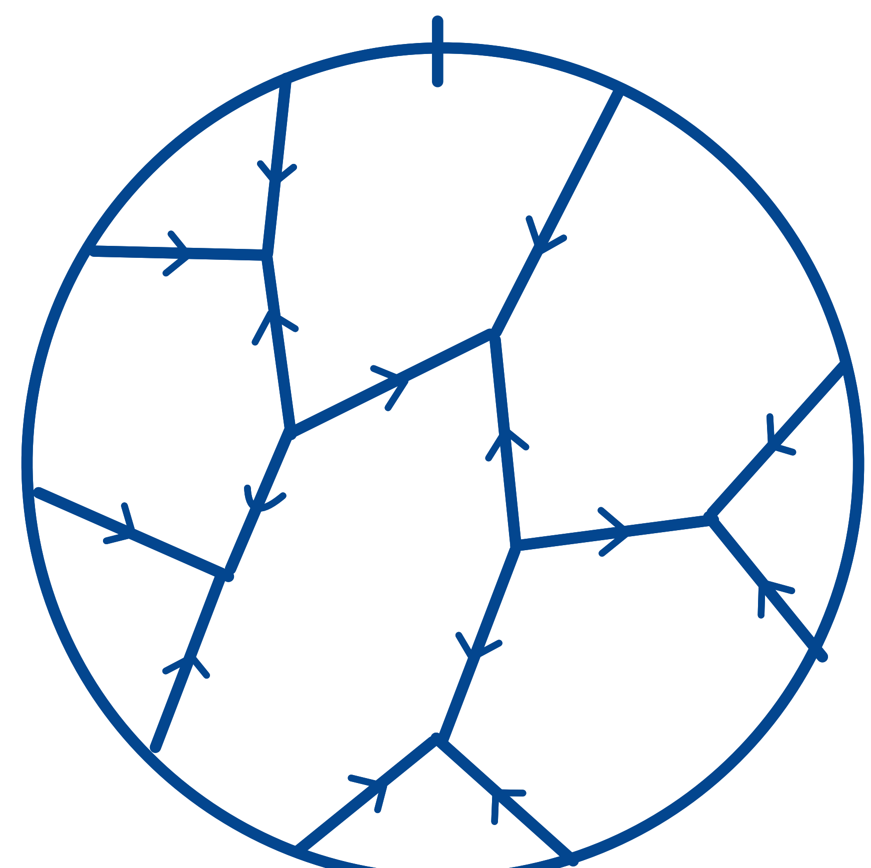
1	2	3
4	5	7
6	8	9

Prom $\mapsto$

1	2	6
3	4	8
5	7	9



Rotation $\mapsto$



SL_3-Web basis

Applications

- Quantum link invariants
  - SL_3 link polynomials, foams, ...
- Cluster algebras
  - Cluster structures on $\mathbb{C}[x_{ij}; y_{kl}]^{SL_3}$
- Enumerative combinatorics
  - promotion, evacuation, cyclic sieving
- Dimer models
- Representation theory

The web basis problem

Problem (Khovanov-Kuperberg '96)

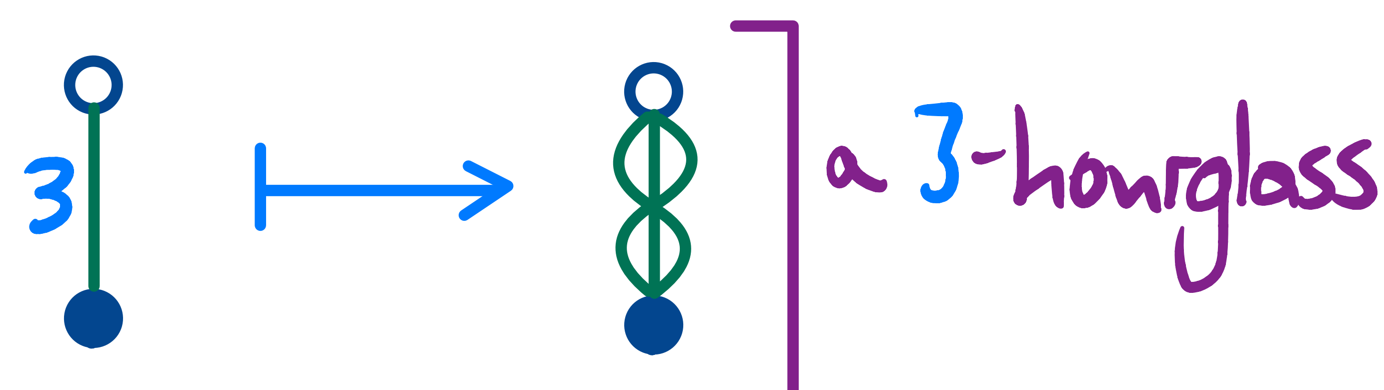
Give a web basis* for $\mathrm{Inv}_{SL_r}(V_1 \otimes \cdots \otimes V_n)$ for $r \geq 4$.

*with desirable properties for use in applications:

- testability
- reduction rules
- rotation invariance

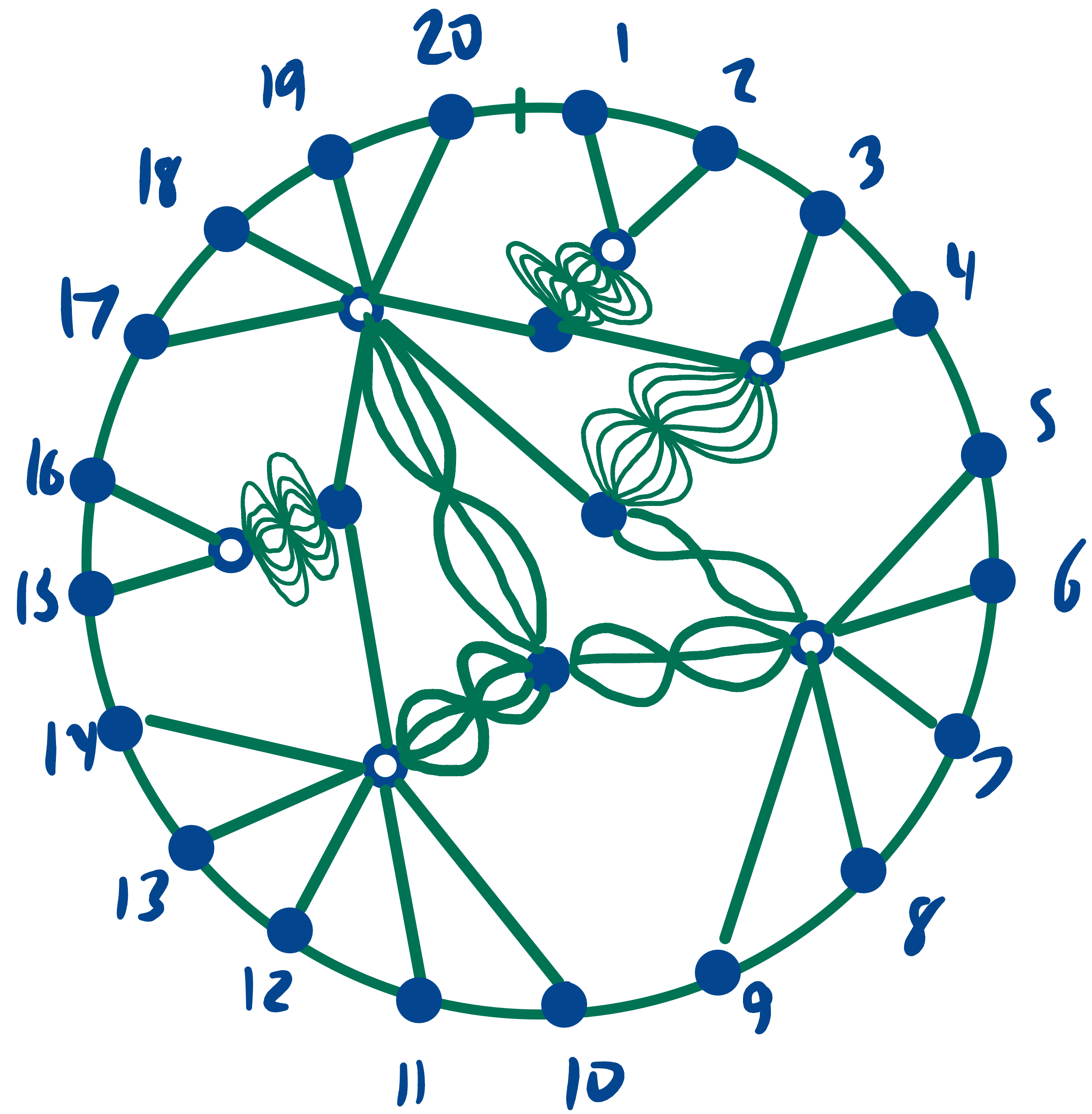
Hourglass plabic graphs

Def ([Goetz–Pechenik–Pfannerer–Striker–S. '23])
An r-hourglass plabic graph (r-HPG) is a planar bipartite graph embedded in a disk with edge weights in [r] which sum to r around internal vertices, and boundary vertices have degree 1.
An edge with weight m is drawn as an m-hourglass:



Hourglass plabic graphs

Ex

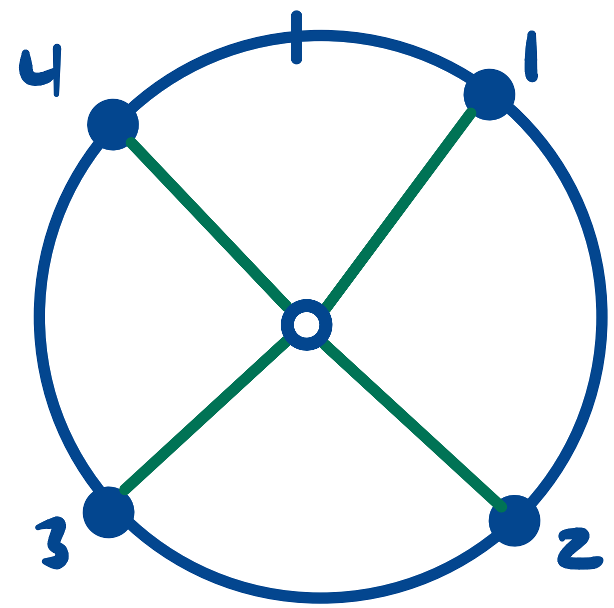


(r = 10)

Hourglass plabic graphs

- Encodes morphisms in $\mathrm{Hom}_{U_q(\mathfrak{sl}_r)}(V_q^{\otimes n}, \mathbb{C}(q))$

e.g.



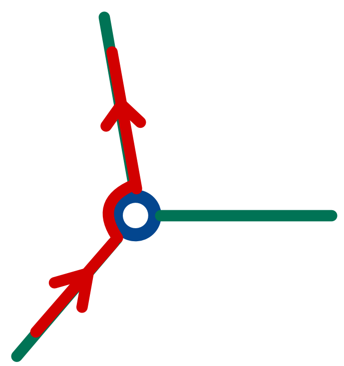
$$= \det \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{pmatrix} \qquad (V_{q=1} = \mathbb{C}^4)$$

- Here  $\longleftrightarrow$ $\Lambda_q^3 V_q$ in domain.

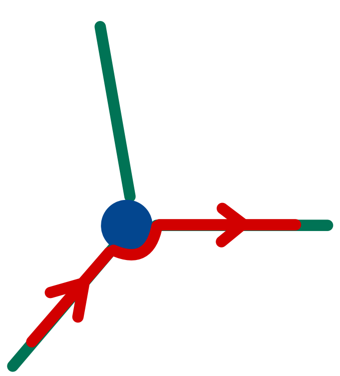
- See [Cautis-Kamnitzer-Morrison] for a trivalent version.

Trip permutations

Recall Postnikov '06 introduced plabic graphs and their trip permutation:

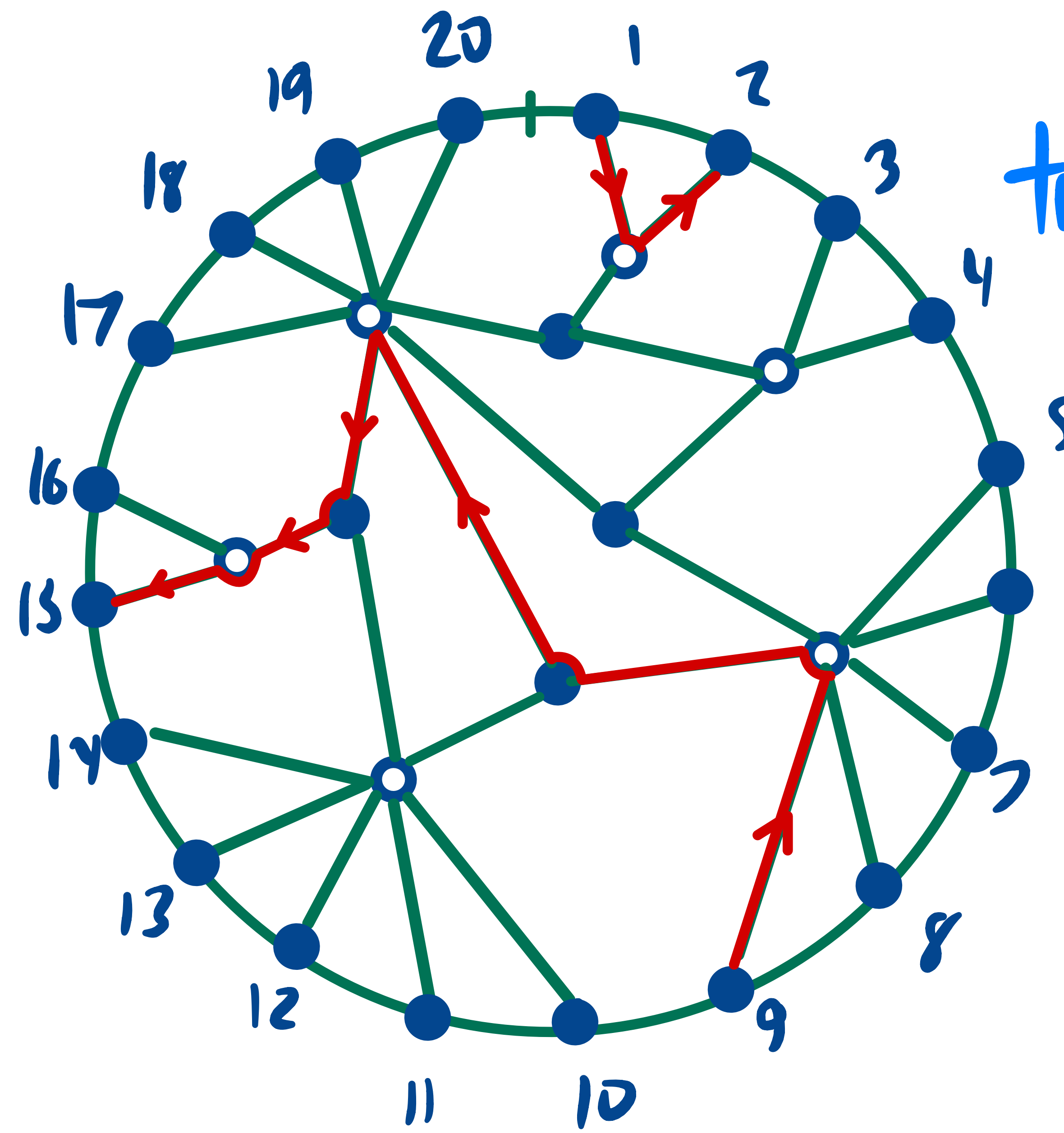


left at white



right at black

Ex

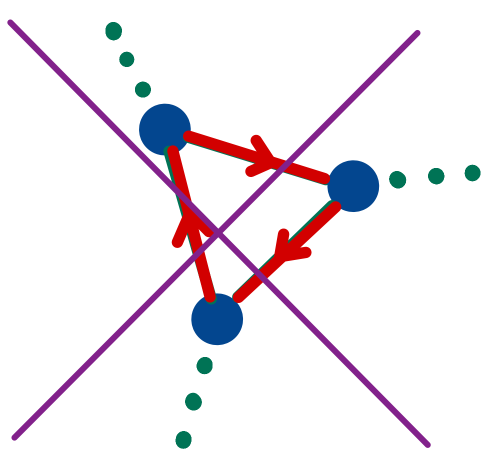


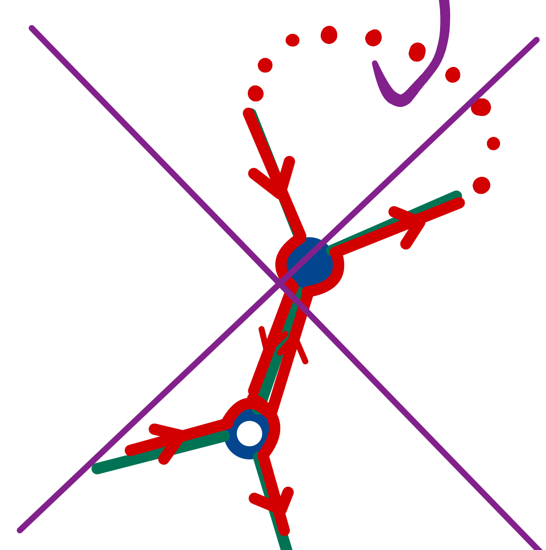
trip = (1 2 5 6 7 8 9 15 16)
(3 4 10 11 12 13
14 17 18 19 20)

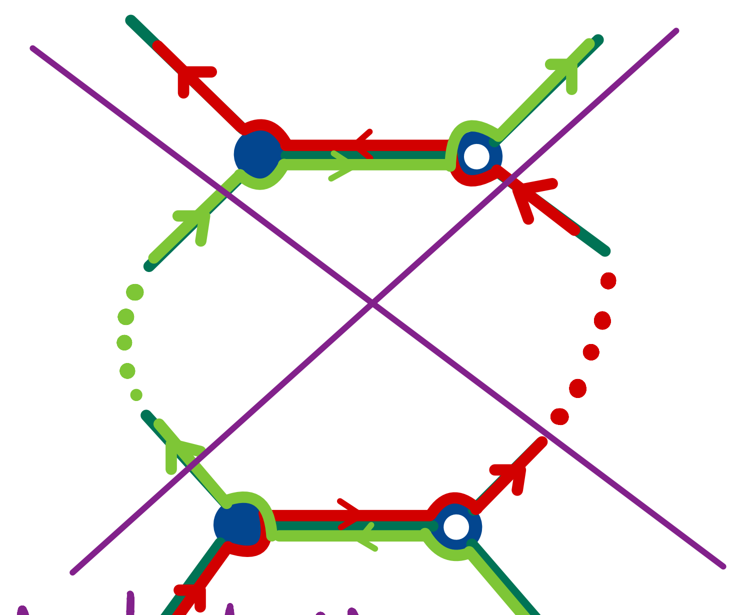
Trip permutations

Thm/Def (Postnikov '06) A (leafless, connected, fixed-point free) plabic graph is **reduced** if it has

1. no round trips
2. no essential self-intersections
3. no bad double crossings



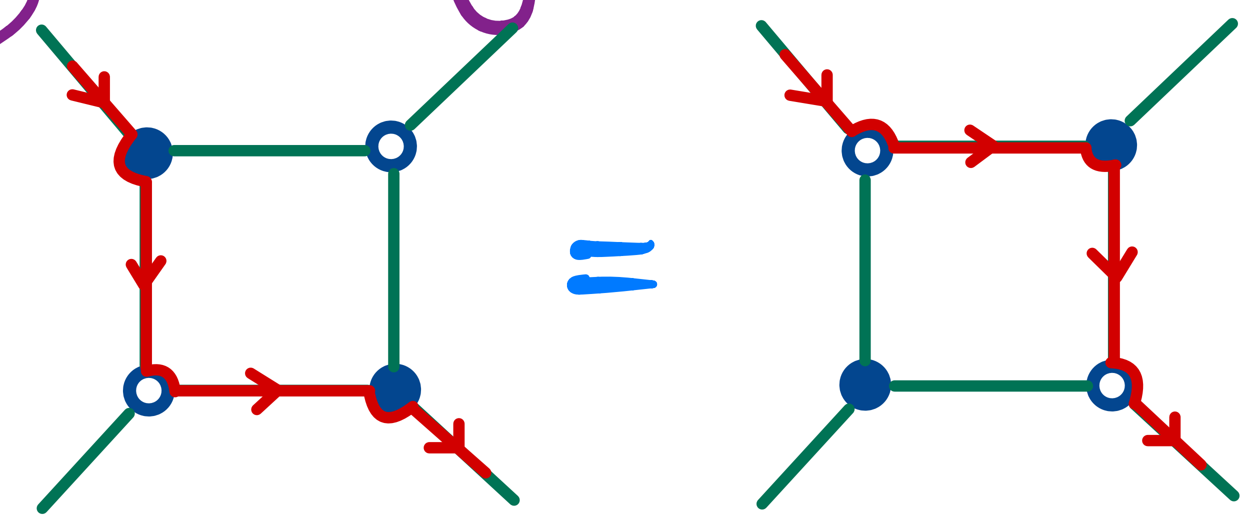




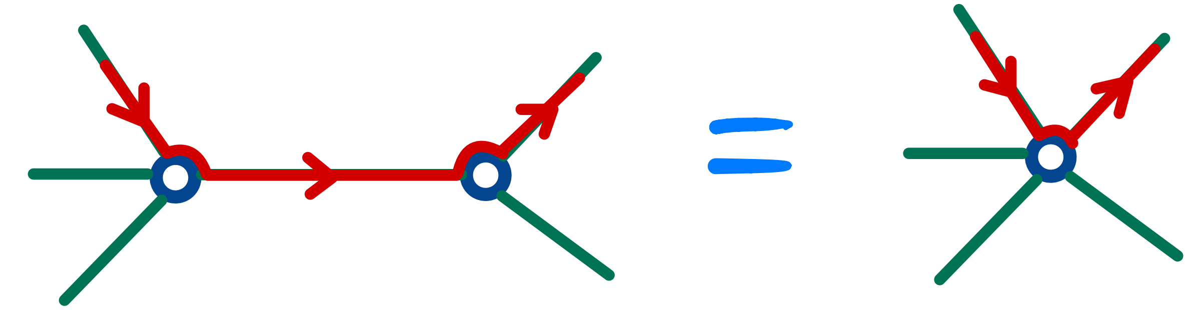
Trip permutations

Thm (Postnikov '06) Two such plabic graphs have the same trip if and only if they are related by moves:

M1 Square move:



M2 Edge contraction:

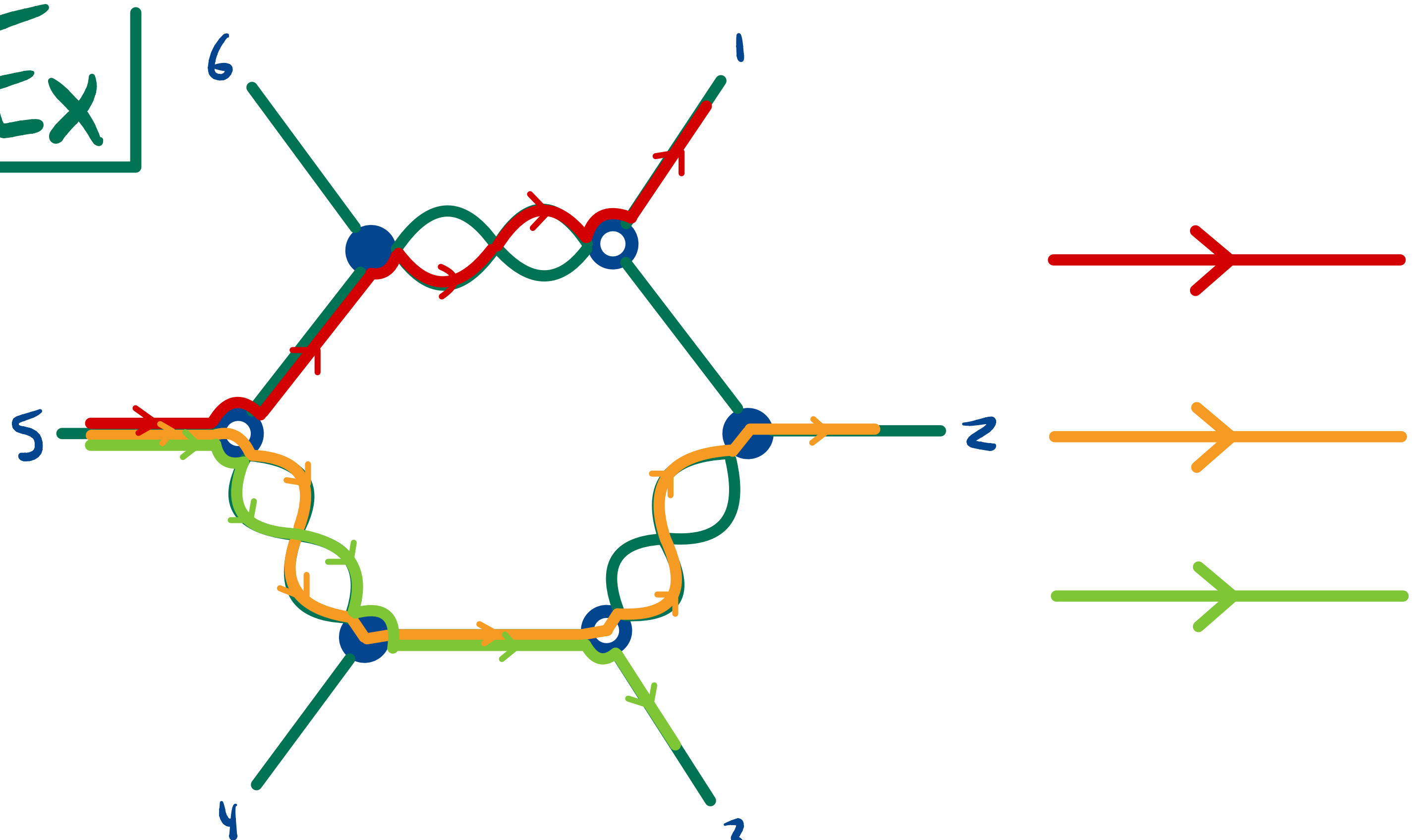


M3 Vertex removal:

Trip permutations

Def (GPPSS '23) An r-hourglass plabic graph has **trip permutations** $\text{trip}_1, \dots, \text{trip}_{r-1}$ where trip_i takes the <u>ith left at white</u> and <u>ith right at black</u>:

Ex



($r=4$)

$= \text{trip}_1 = (135)(642)$

$= \text{trip}_2 = (14)(25)(36)$

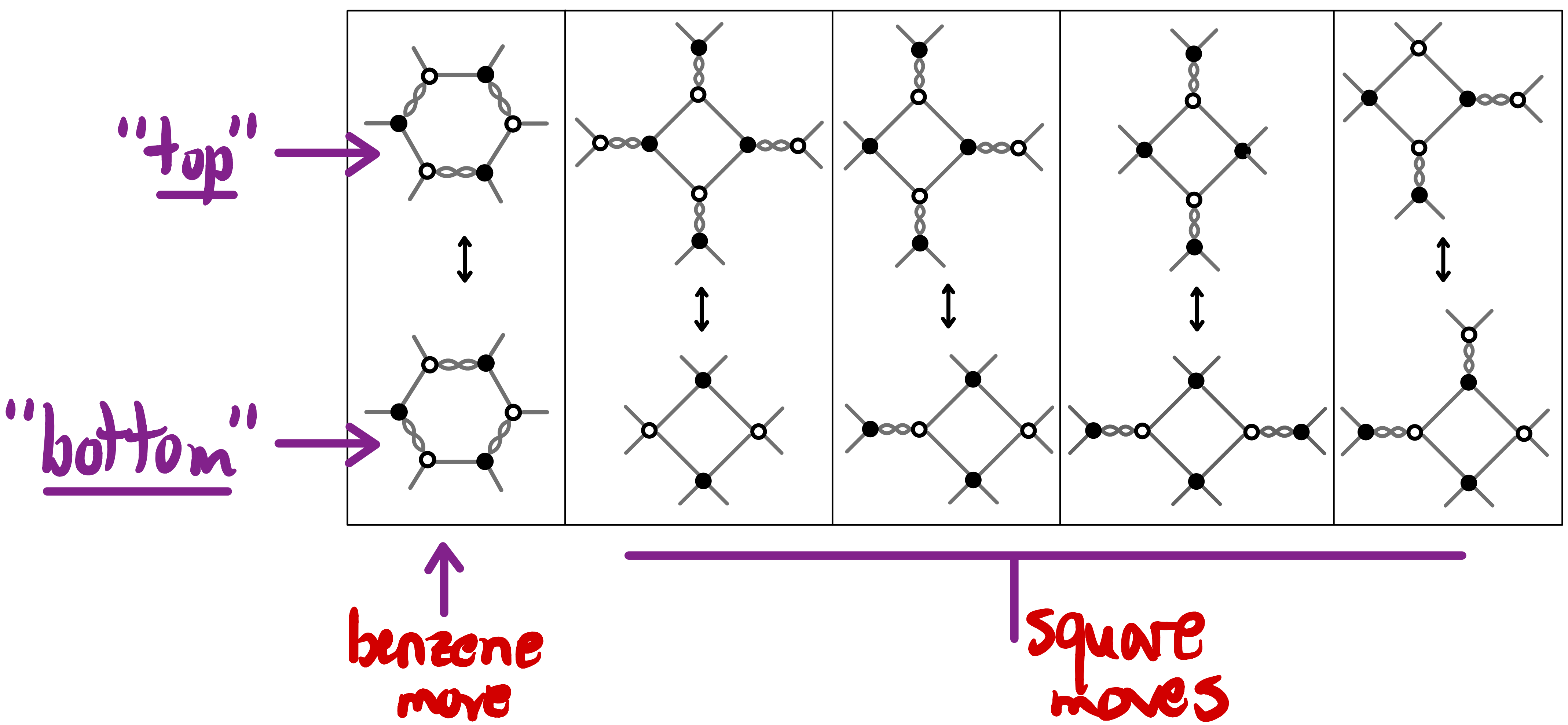
$= \text{trip}_3 = (531)(246)$

Note

$\text{trip}_i = \text{trip}_{r-i}^{-1}$!

$r=4$ moves

Thm (GPPSS '23) Two contracted, fully reduced 4-HPG's have the same $\text{trip}_1, \text{trip}_2, \text{trip}_3$ $\Leftrightarrow$ they are related by moves:



r=4 moves

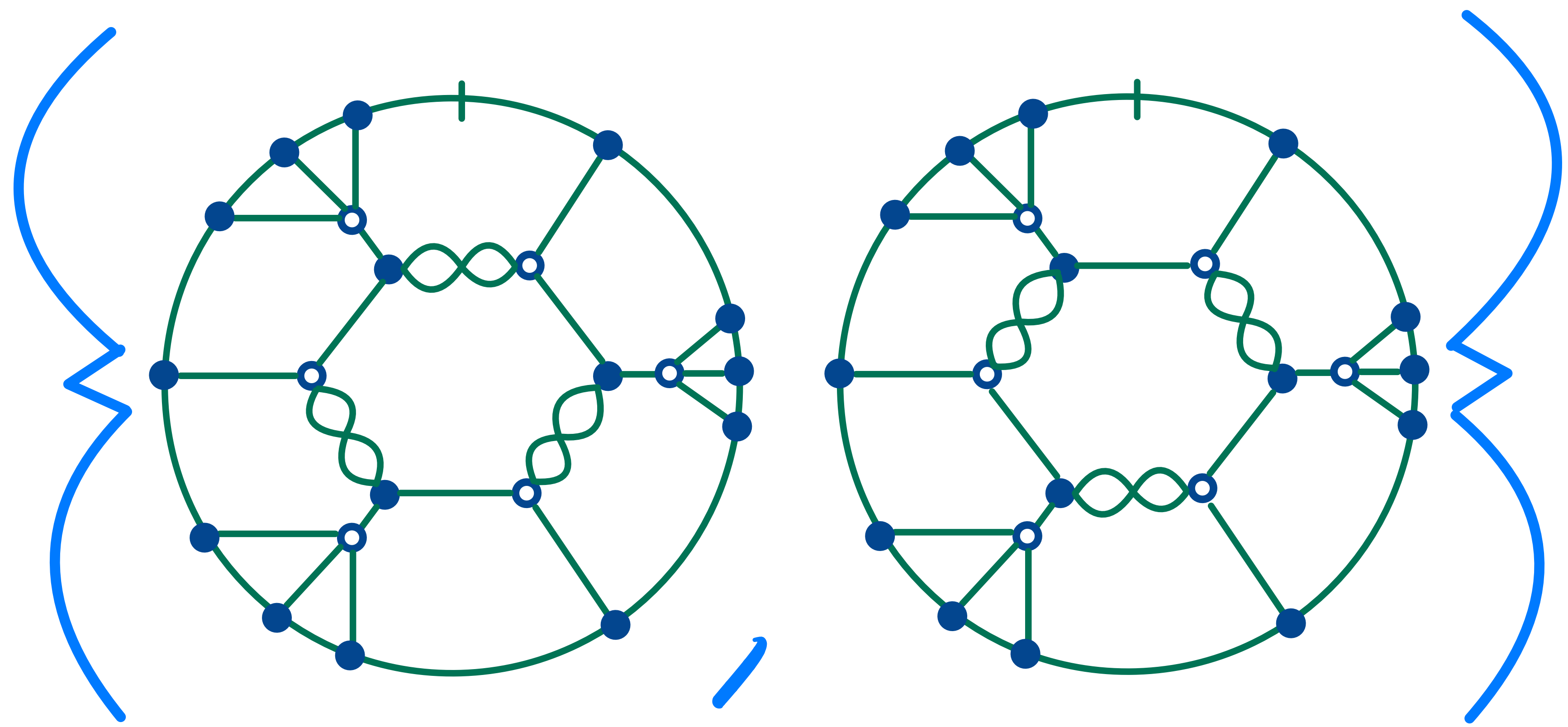
Thm (GPPSS '23) There is a bijection between $SYT(4\times c)$ and such move-classes. It sends $\mathrm{prom}_i(T)$ to $\mathrm{trip}_i(G)$.

Ex

$T=$

1	2	6
3	5	10
4	7	11
8	9	12

$\longmapsto$

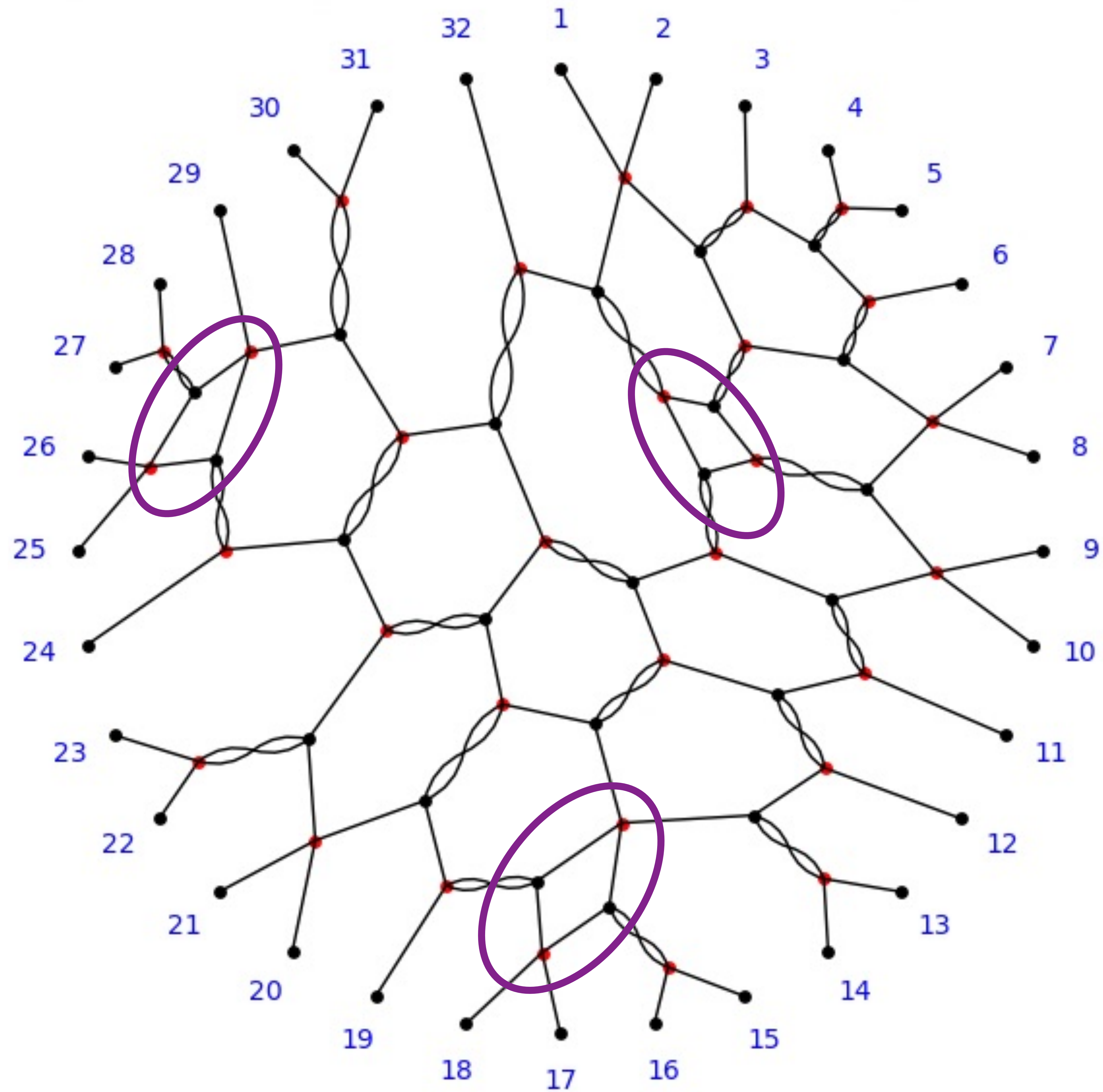


r=4 moves

Ex

1	3	4	7	8	17	19	23
2	5	6	9	14	18	21	24
10	12	13	15	16	25	26	28
11	20	22	27	29	30	31	32

↦



Promotion permutations

Def (GPPSS '23.a building on Hopkins–Rubey)

The **promotion permutations** of $T \in SYT(r \times c)$ are

$$\mathrm{prom}_\bullet(T) = (\mathrm{prom}_1(T), \ldots, \mathrm{prom}_{r-1}(T))$$

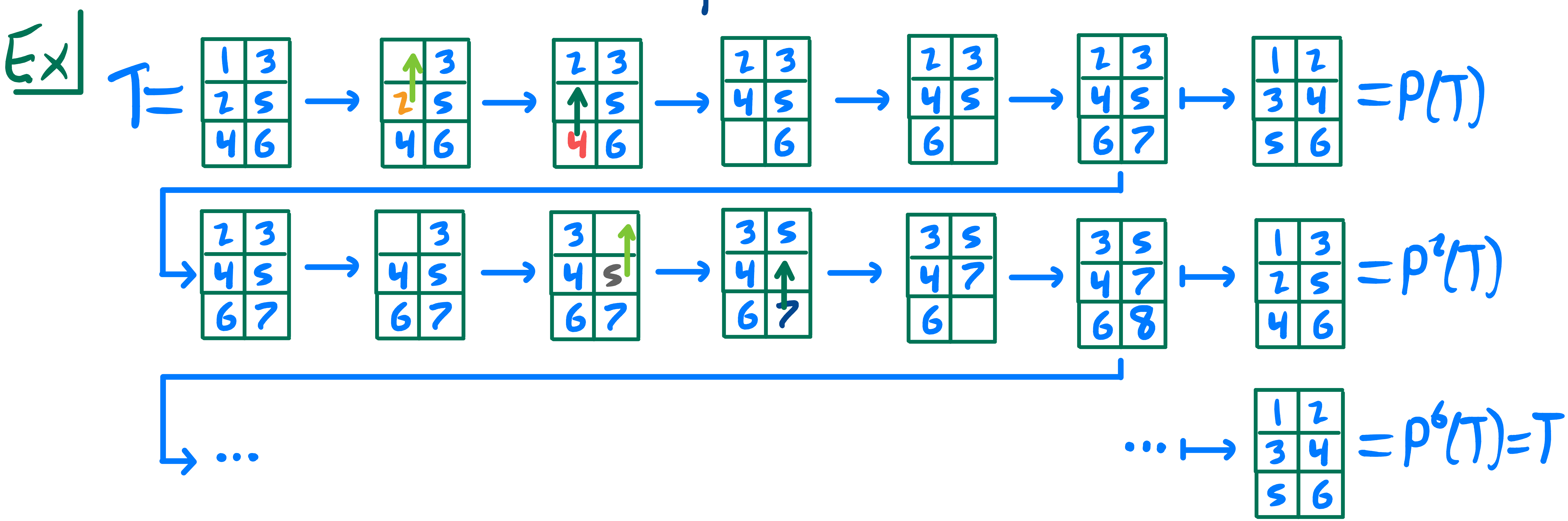
with $\mathrm{prom}_i(T) \in S_n$ defined as follows.

Let $p^{i,j}$ be the unique entry of $P^{j-1}(T)$ which slides from row $i+1$ to row i when computing $P^j(T)$. Set

$$* \quad \mathrm{prom}_i(T) : j \mapsto (p^{i,j} + j - 1) \bmod n. \quad *$$

$(n = rc)$

Promotion permutations



$$\Rightarrow \quad \begin{aligned} \mathrm{prom}_1(T) &= 2\,5\,4\,1\,6\,3 \\ \mathrm{prom}_2(T) &= 4\,1\,6\,3\,2\,5 \end{aligned}$$

Promotion permutations

Thm (GPPSS '23.a) Let $T \in SYT(r \times c)$. Then:

a) $\mathrm{prom}_i(T)$ is a fixed-point free permutation

b) $\mathrm{prom}_i(T)^{-1} = \mathrm{prom}_{r-i}(T)$

c) $c^{-1} \circ \mathrm{prom}_i(T) \circ c = \mathrm{prom}_i(P(T))$ where $c = (1\ 2 \cdots n)$

d) $w_0 \circ \mathrm{prom}_i(T) \circ w_0 = \mathrm{prom}_i(\mathcal{E}(T))$ where $w_0 = n\ n{-}1 \cdots 2\ 1$

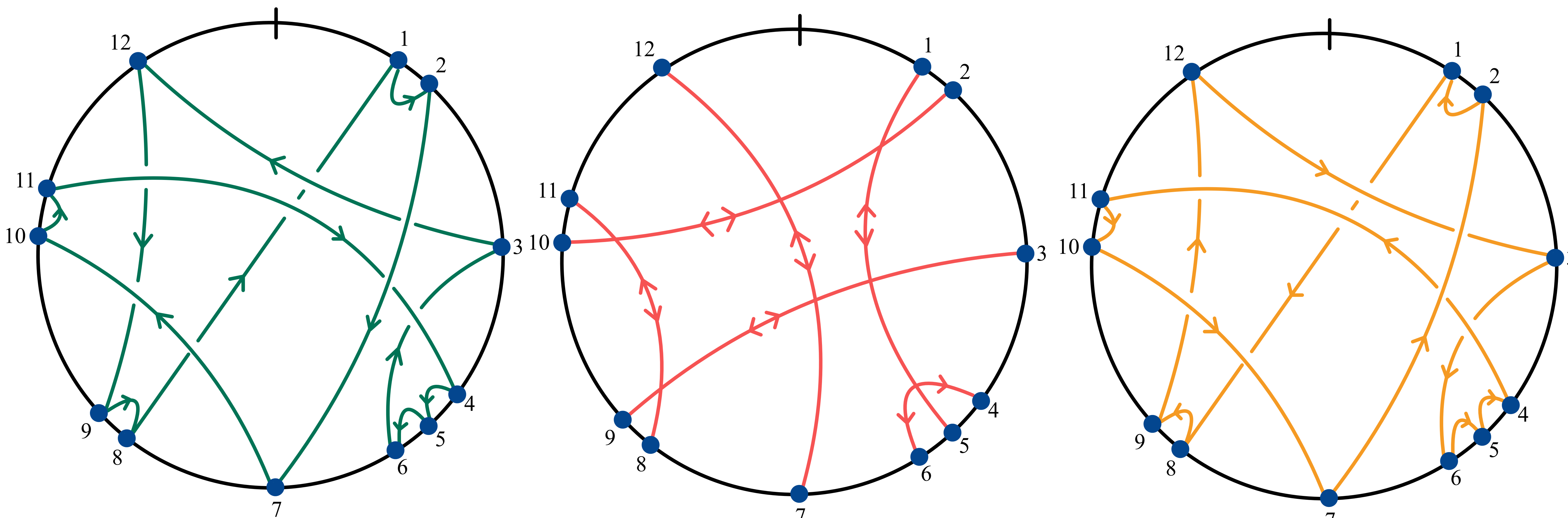
e) $\mathrm{Aexc}(\mathrm{prom}_i(T)) = \{e \mid e \text{ is in the first } i \text{ rows of } T\}$

where $\mathrm{Aexc}(\pi) = \{i : \pi^{-1}(i) > i\}$

Promotion permutations

Cor $\mathrm{prom}_\bullet(T)$ is a combinatorial model manifesting the dihedral structure on $SYT(r \times c)$!

Ex



prom_1 prom_2 prom_3

(Here $T = \{1,2\}\bar{4}\{1,3,4\}2\{\bar{3},\bar{2}\}\{3,4\}T$ is a more general fluctuating tableaux.)

Main theorem

Thm ([GPPSS '23])
$\{[W] \mid W$ is a top fully reduced 4-hourglass plabic graph$\}$

is an SL_4 web basis.

- Solves the SL_4 basis problem from '94!
- Effective graphical calculation with SL_4-webs (Gröbner-like theory)!

→

Pockets

- Fontaine-Kamnitzer-Kuperberg '13 embedded (dual) SL(3) basis webs inside the affine building. Related to geometric Satake correspondence.
- Following a suggestion of Kuperberg, ongoing work of GSSW '24+ associates SL(4) basis move classes with 3D "pockets".
- Beautiful connections to tilings of the Aztec diamond, crystals, and more.

Affine Grassmannians

Def The affine Grassmannian of $SL(r)^\vee$ is

$$AffGr_r = PGL_r(\mathbb{C}((t)))/PGL_r(\mathbb{C}[[t]]).$$

Fact There is a notion of distance d on $AffGr_r$ with values that are dominant $SL(r)$ weights:

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r) \in \mathbb{Z}^r / \langle 1, \ldots, 1 \rangle$$

Affine Grassmannians

- Why? The double cosets

$$GL_r(\mathbb{C}[[t]]) \backslash GL_r(\mathbb{C}((t))) / GL_r(\mathbb{C}[[t]])$$

have canonical representatives

$$t^\lambda = \begin{pmatrix} t^{\lambda_1} & & & \\ & t^{\lambda_2} & & \\ & & \ddots & \\ & & & t^{\lambda_r} \end{pmatrix}.$$

- Similarly for PGL_r.
- Use $d(p,q) = \lambda \Leftrightarrow p^{-1}q \in H t^\lambda H$.

  e.g. $d(p,p) = 0$ $\quad d(p,q) = d(gp,gq)$ $\quad d(q,p) = -\mathrm{rev}(d(p,q))$

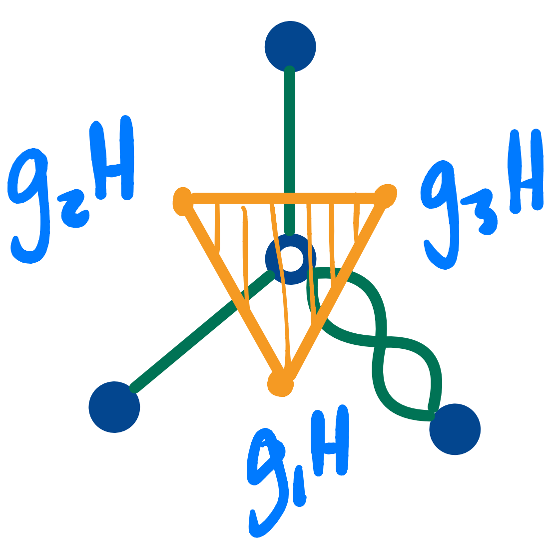
Affine Buildings

- The fundamental weights of $SL(r)$ are $\omega_i = (1^i, 0^{r-i})$
  $\omega_i^* = (0^{r-i}, -1^i) \equiv \omega_{r-i}$.
- Corresponds to $\Lambda^i V$ and $\Lambda^i V^*$.
- Essentially generates $SL(r)$-representation theory
  (e.g. Karoubi envelope...)

Affine Buildings

Def The **affine building** on AffGr_r is the simplicial complex Δ_r whose vertices are the points of AffGr_r and whose simplices are collections of points all of whose distances are fundamental weights.

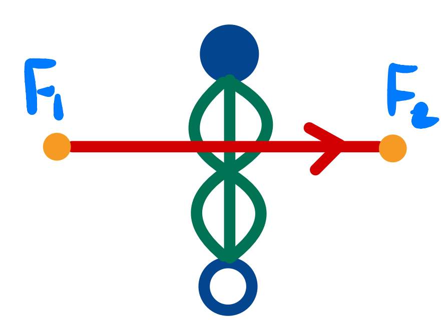
Ex



If $d(g_1, g_2) = \omega_1$, $d(g_2, g_3) = \omega_1$, $d(g_3, g_1) = \omega_2$ then $\{g_1H, g_2H, g_3H\} \in \Delta_4$

Affine Buildings

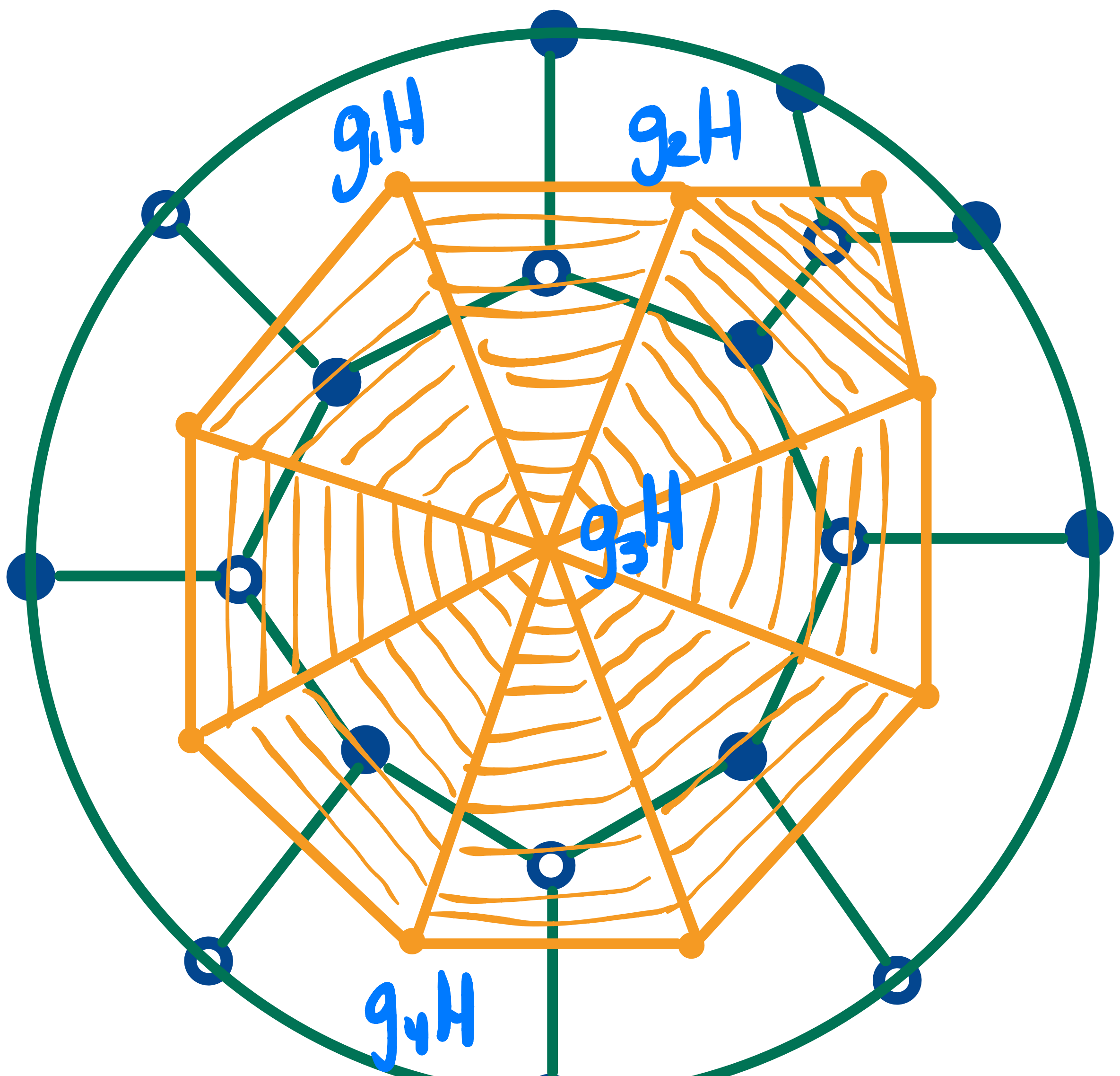
Thm (Fontaine–Kamnitzer–Kuperberg '13) The duals of non-elliptic $SL(3)$ basis webs can be embedded in Δ_3. For faces F_1, F_2, the distance between the corresponding vertices in Δ is the geodesic distance in Δ (or the embedding).



$d(\phi(F_1), \phi(F_2)) = \omega_3$

Affine Buildings

Ex There exist $g_1, g_2, g_3, g_4, \ldots$ s.t.



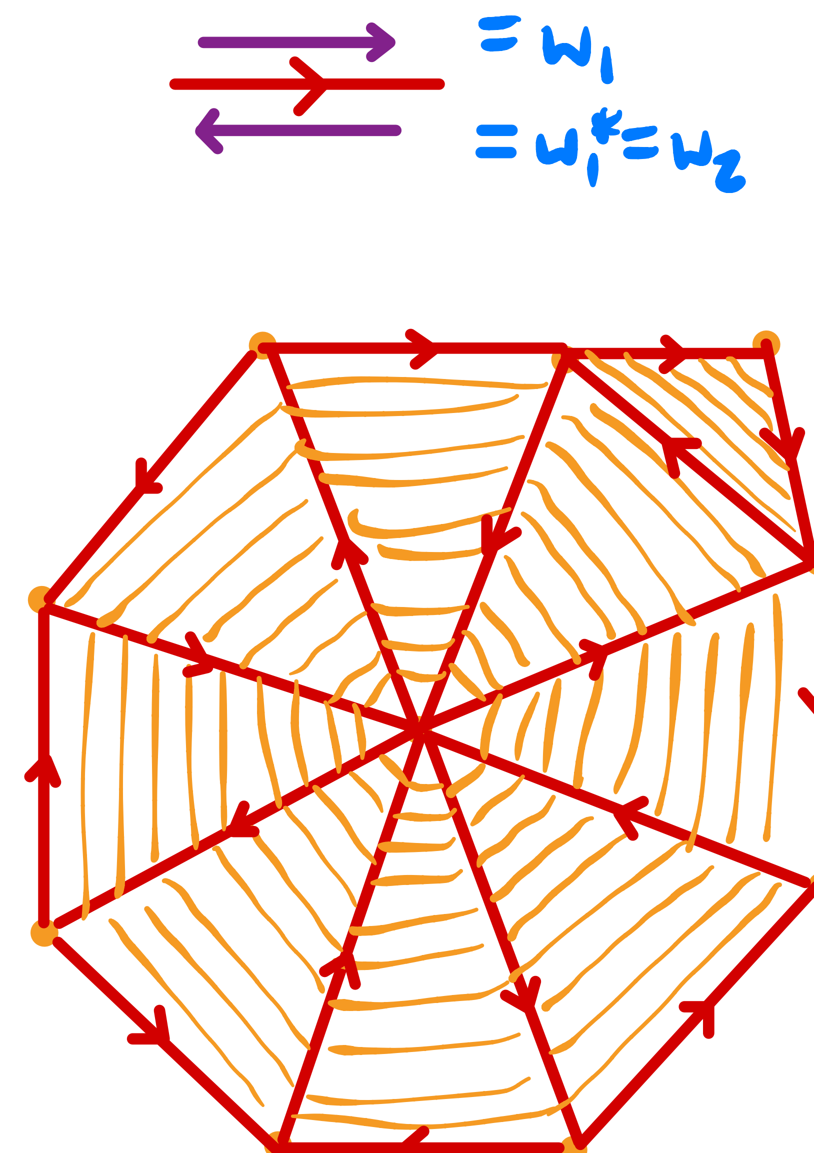
$d(g_1, g_2) = w_1$

$d(g_2, g_3) = w_1$

$\vdots$

$d(g_4, g_1) = 2w_1$

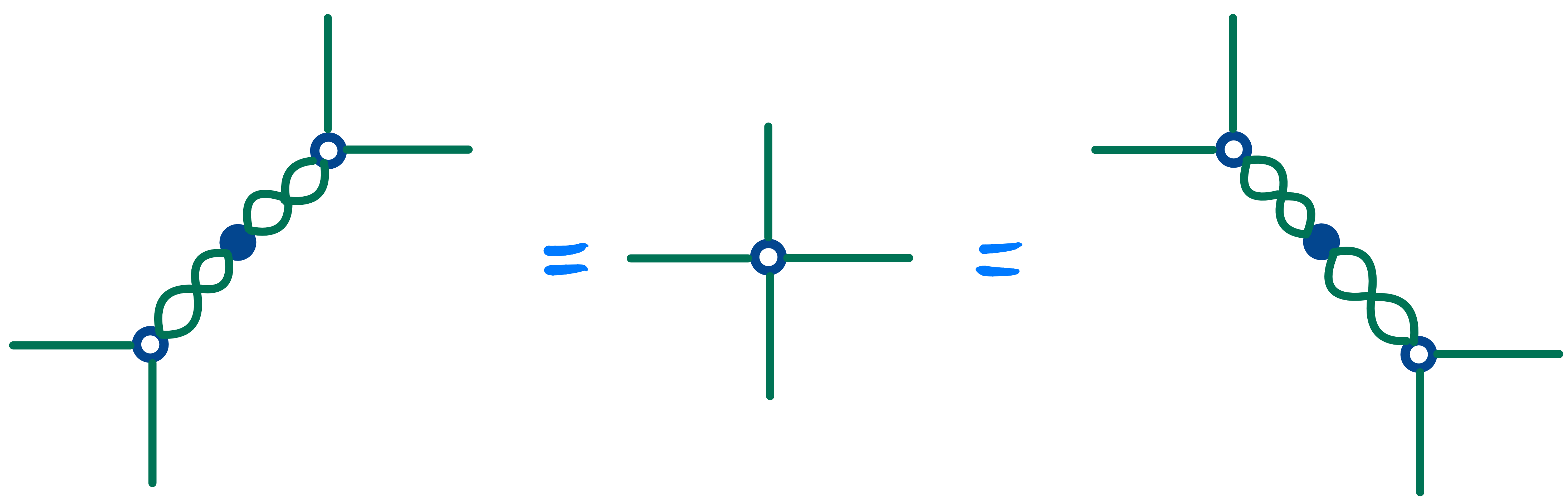
$\vdots$



in Δ_3

Pockets

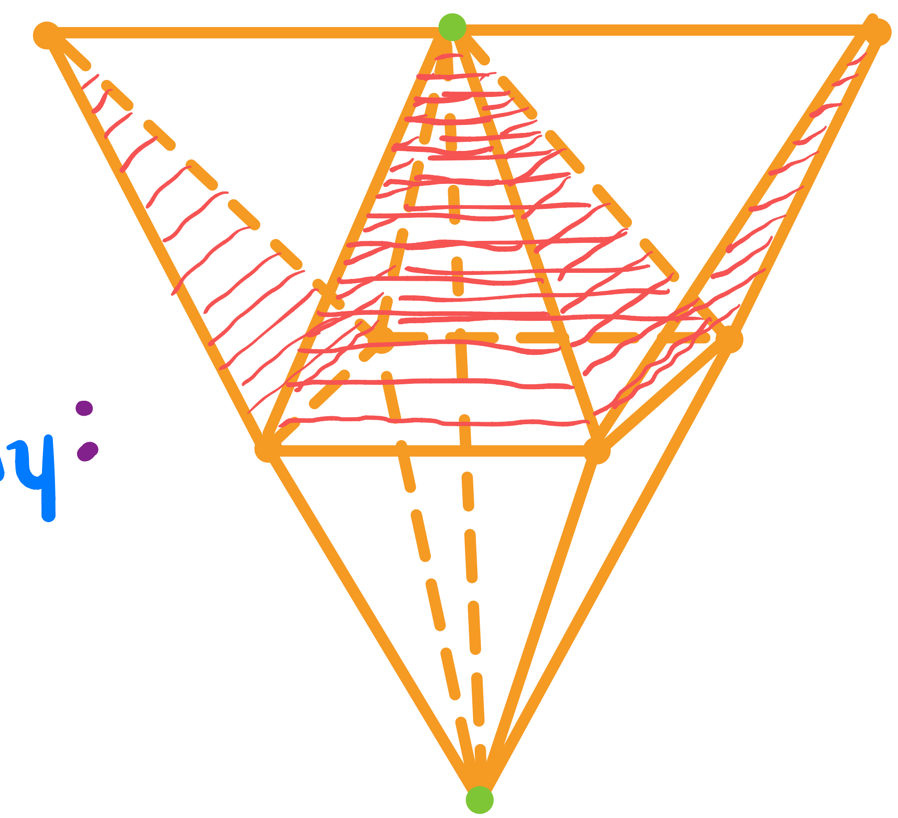
- We (GSSW) show there is a 3D analogue for the SL(4) web equivalence classes: pockets in Δ_4.
- Requires expanding sources/sinks via I=H moves:

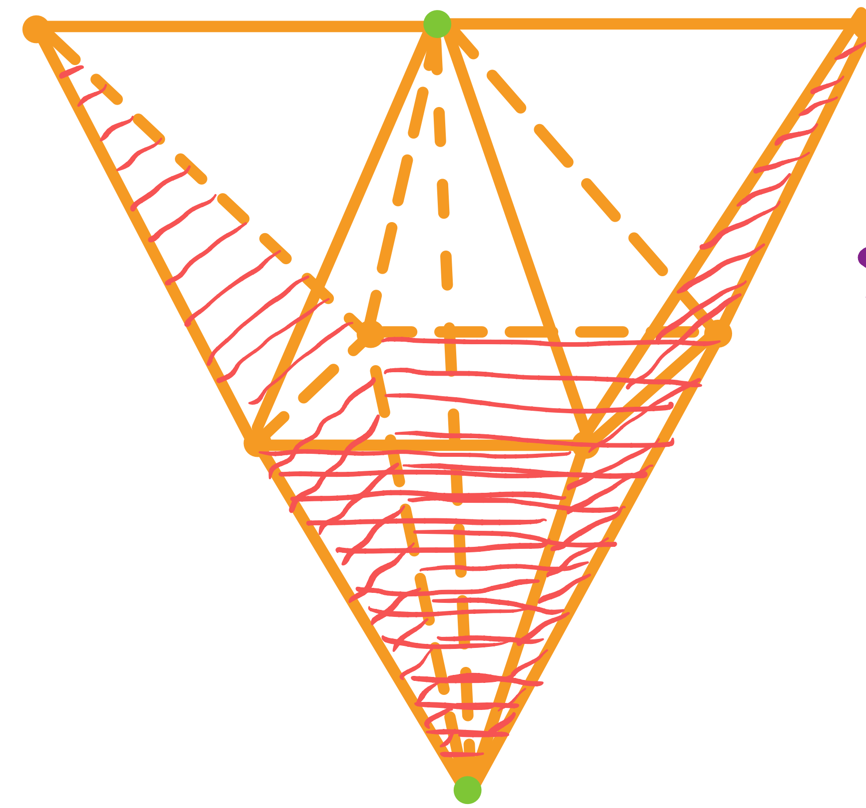


Pockets

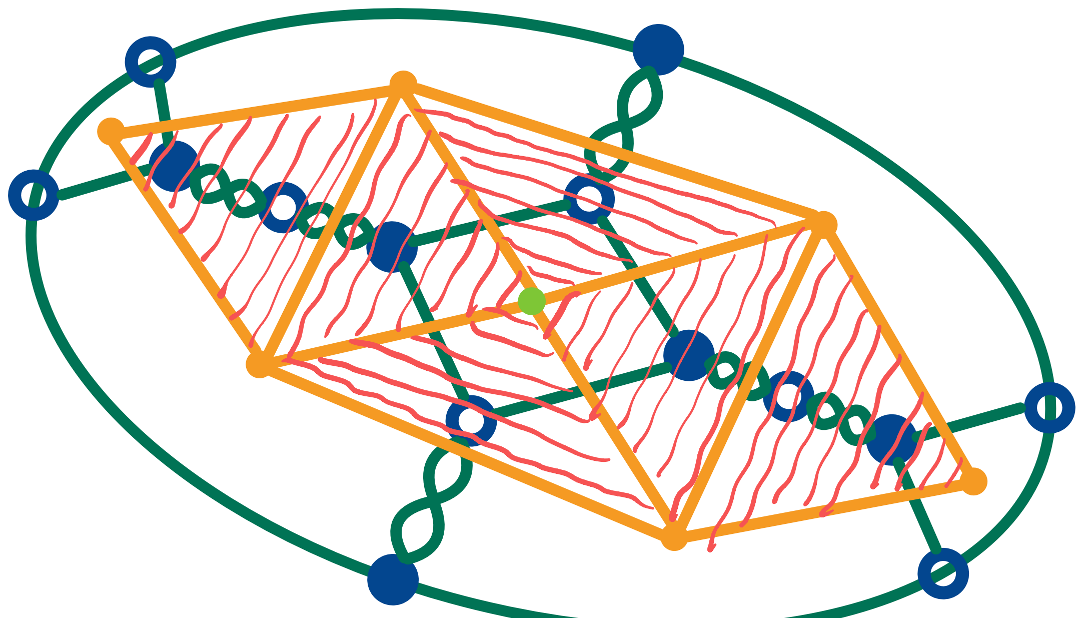
Ex

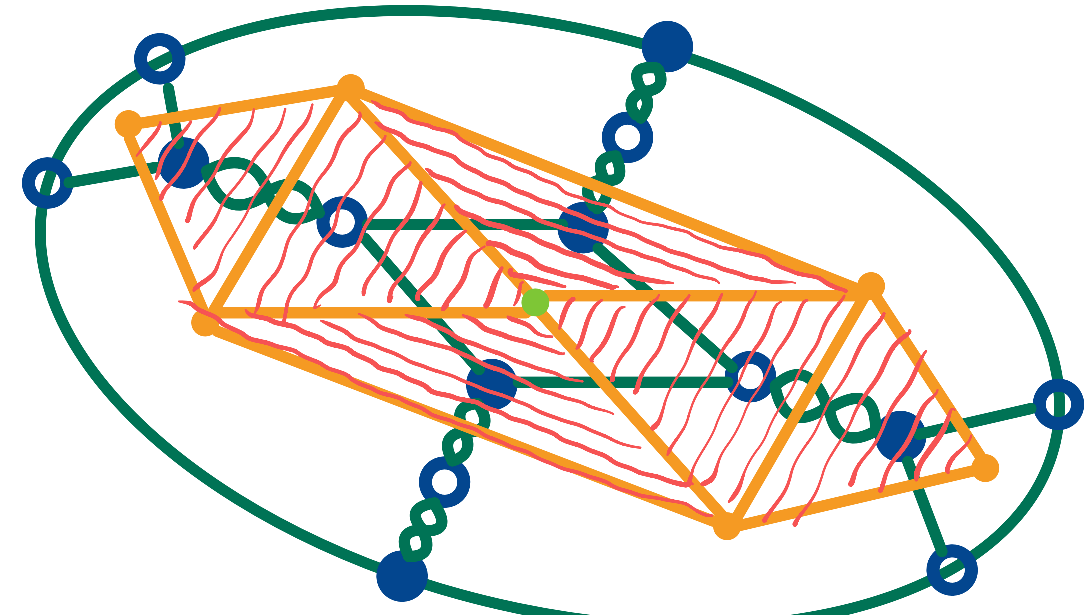
In Δ_4:





Pocket of square + benzene + IH class of $\overline{4}\{1,2\}\overline{3}\,\overline{2}\{3,4\}\overline{1}$

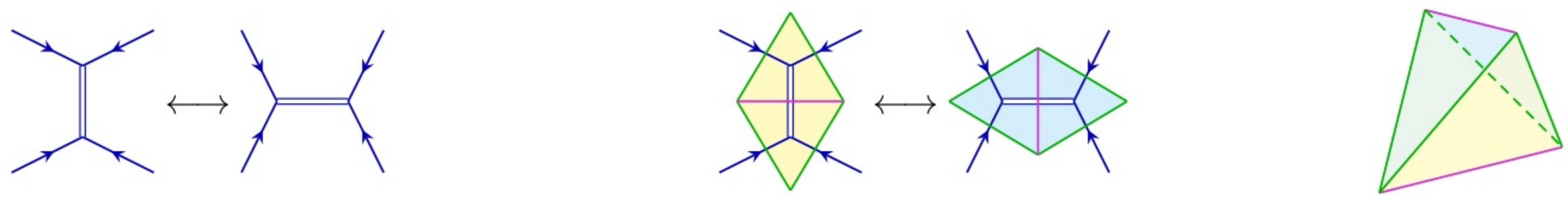




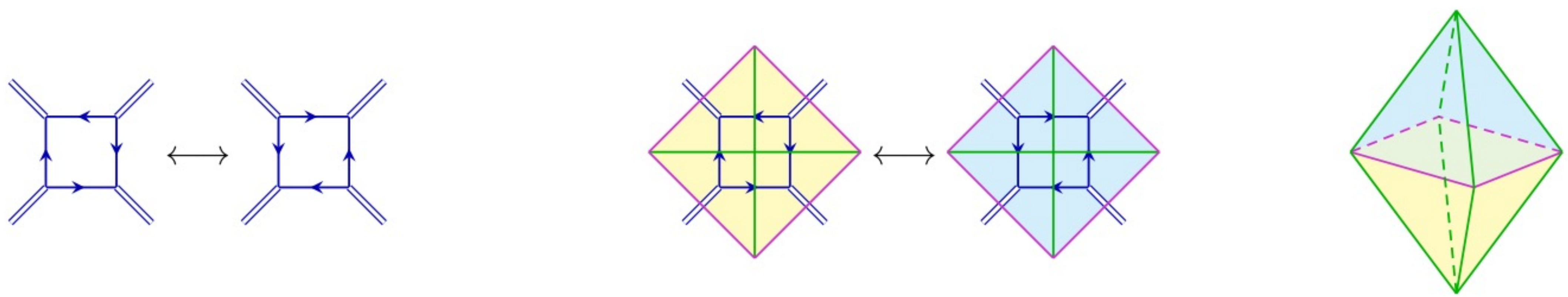
+6 more

Pockets

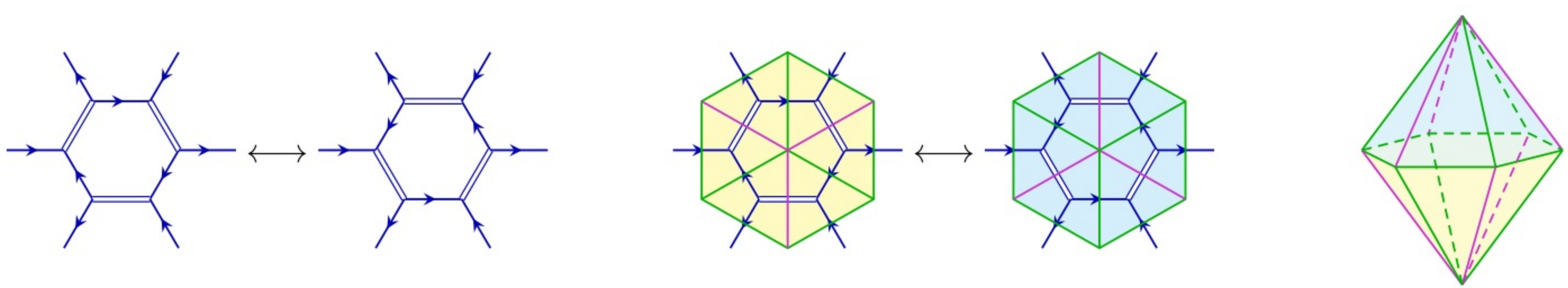
- Build pockets from 4-HPG basis web move classes:



IH move between webs — Dual diskoids — tetrahedron



Square move between webs — Dual diskoids — octahedron



Benzene move between webs — Dual diskoids — dodecahedron

Pockets

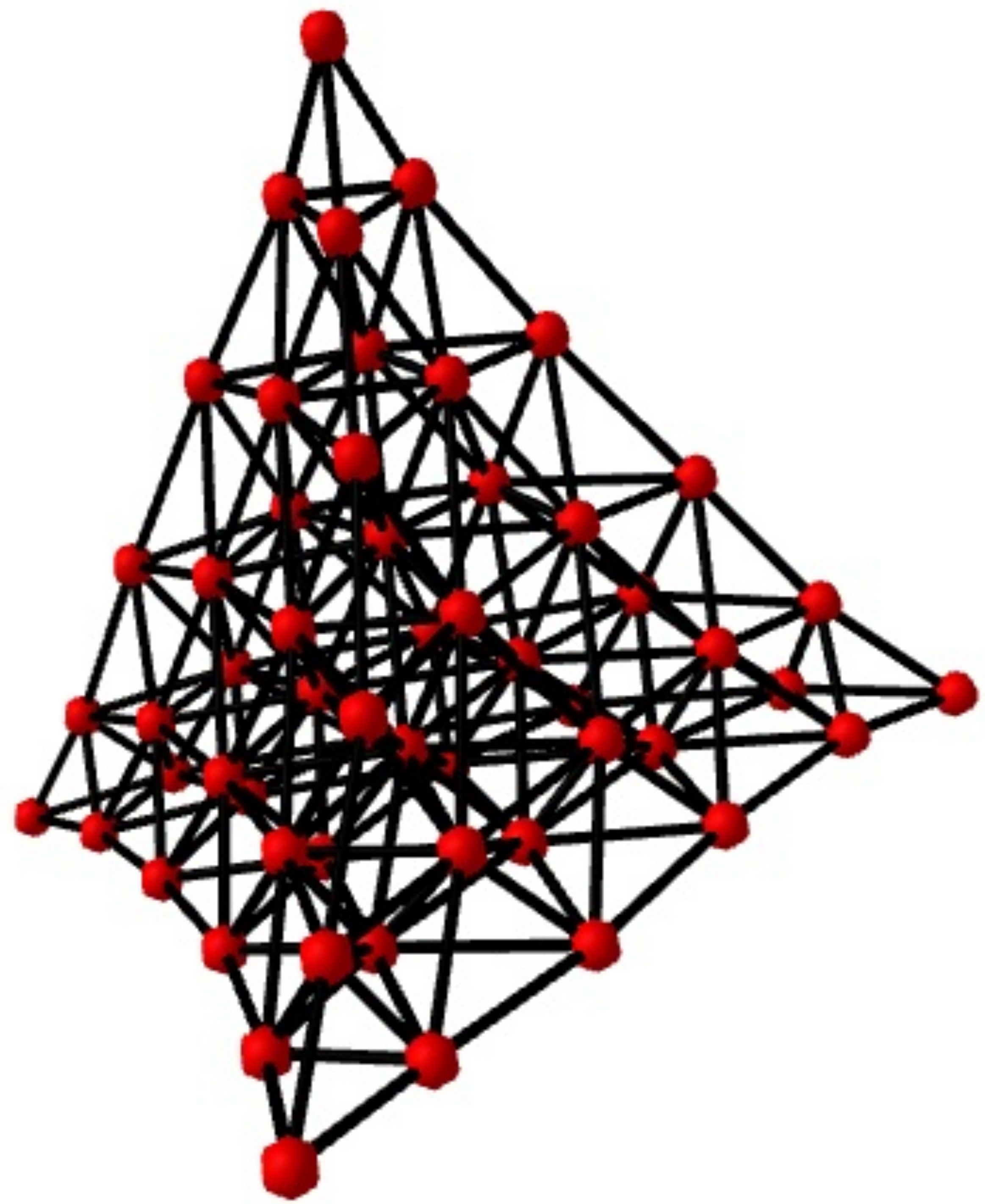
Thm (GSSW '24+) The pocket $P = P(T)$ is CAT(0) and a singular interval bundle over the closed disk.
The simplicial sections of P are in bijection with the move-class corresponding to T.

Thm (GSSW '24+) Given an irreducible component of a Satake fiber of $\Delta = \Delta(SL_4^\vee)$ indexed by T, there is a dense open set U s.t. every point of U extends uniquely to a configuration $P(T) \hookrightarrow \Delta$ which preserves distances.

Pockets

Ex Pocket of 5×5 ASM:

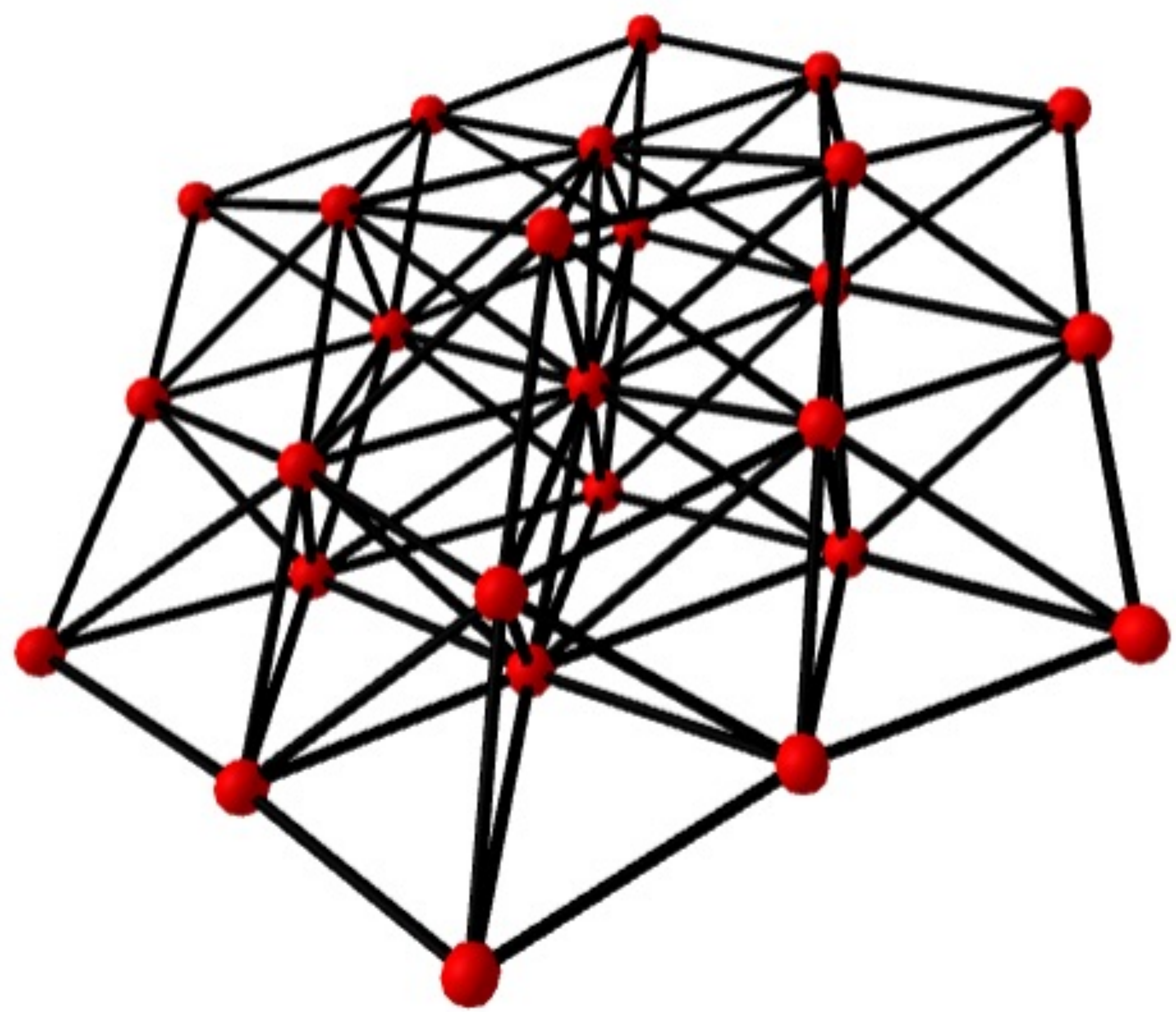
$L = 1^5 2^5 3^5 4^5$



Note Related to height functions, octahedral recurrence, tilings of the Aztec diamond, distributive lattices

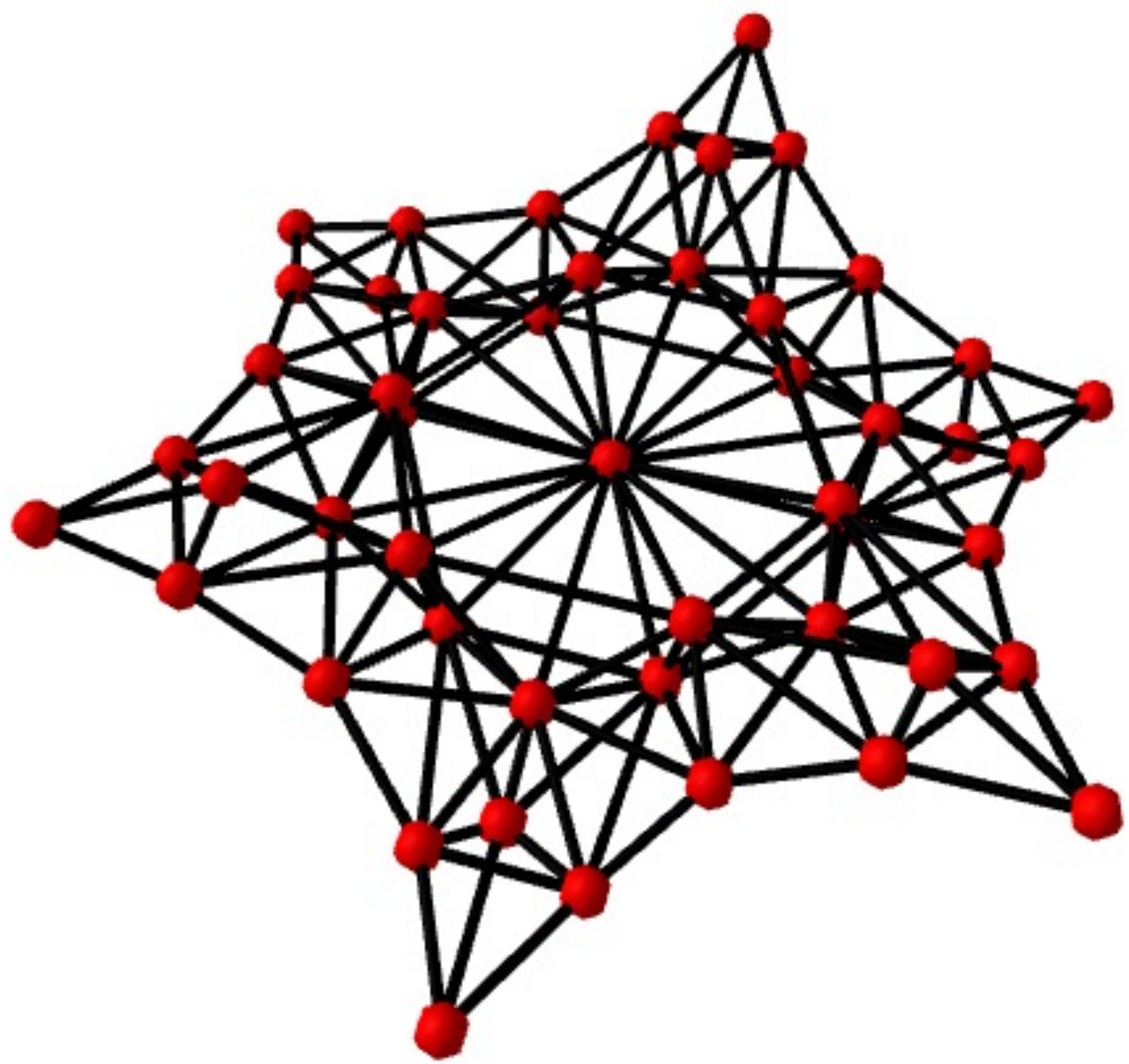
Pockets

Ex Pocket of 2×2×2 PP:



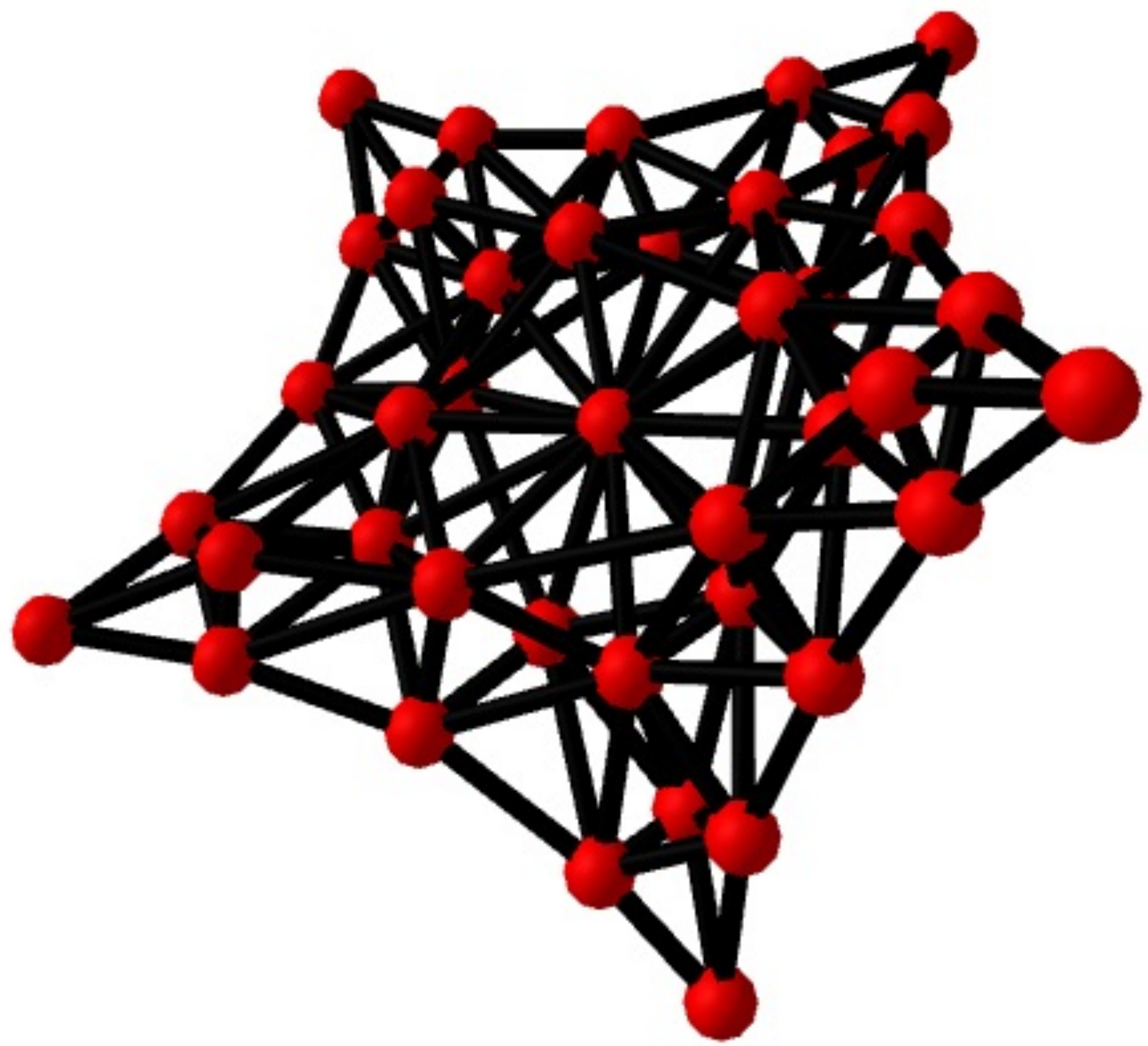
Pockets

Ex | Pocket of a "chained hexagon":



Pockets

Ex Pocket of a "chained pentagon":



Note Not realizable in $\mathbb{R}^3$

Future Work

- Understand Prom.
  - Characterize image — Web duality?
- General SL_r case
  - Mostly know how to do $r=5$
- Higher growth rates
- ASM/IP connections
- Code for webs

THANKS!