Type B q-Stirling numbers

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Based on joint work with *Bruce Sagan* (arXiv: 2205.14078)
and *Nolan Wallach* (arXiv: 2109.03407)

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q-analogues and Stirling numbers in other types

Josh Swanson <jswanson@ucsd.edu> Jul 2, 2019, 5:26 PM

to Bruce

Hi Bruce,

• • •

Sagan, Bruce bsagan@msu.edu via michiganstate.onmicrosoft.com Jul 2, 2019, 5:57 PM

to Josh

Hi Josh,

• • •

Josh Swanson <jswanson@ucsd.edu> Jul 2, 2019, 6:15 PM

to Bruce

Thanks for the quick response! For what it's worth, my explicit recurrence for the type B q-Stirlings of the second kind is...

StirB_q(n, k) = StirB_q(n-1, k-1) + [2k+1]_q StirB_q(n-1, k)

with base cases StirB_q(0, k) = \delta_{k=0}.

• • •

I'd certainly be interested in a combinatorial interpretation!

Sagan, Bruce bsagan@msu.edu via michiganstate.onmicrosoft.com Jul 3, 2019, 6:15 AM
to Josh

Hi Josh,

. . .

Sagan, Bruce bsagan@msu.edu via michiganstate.onmicrosoft.com Jul 3, 2019, 11:48 AM
to Josh

Hi Josh,

Attached is a the combinatorial interpretation you asked for. I didn't put in any examples, but just ask if anyth

. . .

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q-Stirling numbers in type B

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ABSTRACT

Stirling numbers of the first and second kind, which respectively count permutations in the symmetric group and partitions of a set, have found extensive application in combinatorics, geometry, and algebra. We study analogues and q-analogues of these numbers corresponding to the Coxeter group of type B. In partic-

Outline

- Stirling variations
  - Classical type A
  - Classical q-analogue
- Type B q-Stirling numbers
- Enumerative results
- Super coinvariant algebra motivations

Overview

	Type A	Type B	Other Types
Classical	Stirling 1730	Zaslavski '81?	Zaslavski '81?
q-analogue	Carlitz '33 Gould '61	S.-Wallach '21 S.-Sagan '22 Today! Bagno-Garber '22+	S.-Sagan '23+

Variations on unsigned/signed Stirling numbers of the first kind and unordered/ordered Stirling numbers of the second kind

Classical Stirling Numbers

Def The (type A) Stirling numbers of the second kind are defined recursively by

$$S(n,k) = S(n-1,k-1) + kS(n-1,k)$$

$$S(0,k) = \delta_{0k}$$

(See A008277)

Ex

$n=0$:	1	0	0	0	0	0	...
$n=1$:	0	1	0	0	0	0	...
$n=2$:	0	1	1	0	0	0	...
$n=3$:	0	1	3	1	0	0	...
$n=4$:	0	1	7	6	1	0	...
$n=5$:	0	1	15	25	10	1	...

$= 7 + 3\cdot 6 = S(5,3)$

Classical Stirling Numbers

Def A (type A) set partition of $[n]=\{1,2,\ldots,n\}$, written $P \vdash [n]$,

is $P=\{\underbrace{B_1,\ldots,B_k}_{\text{blocks of } P}\}$

s.t.

$B_1 \cup \cdots \cup B_k = [n]$

$B_i \cap B_j = \emptyset$ for $i \neq j$

$B_i \neq \emptyset$ for all i

Ex $\{\{1,3\},\{5,6\},\{2,4\}\} \vdash [6]$

$\Updownarrow$

$\underbrace{13/24/56}_{\text{write in increasing order, say}} \in S([6],3)$

- Set $S([n],k)=\{P \vdash [n] : P \text{ has } k \text{ blocks}\}$

Classical Stirling Numbers

Thm $S(n,k) = \# S([n],k)$

Pf Remove n: if singleton, $k-1$ blocks, otherwise k copies with k blocks

Thm $S(n,k) = h_{n-k}(1,2,\dots,k) = \sum_{\gamma_1+\cdots+\gamma_k=n-k} 1^{\gamma_1}2^{\gamma_2}\cdots k^{\gamma_k}$

Pf $h_{n-k}(x_1,\dots,x_k) = h_{n-k}(x_1,\dots,x_{k-1}) + x_k h_{n-k-1}(x_1,\dots,x_k)$ with $x_i = i$

Ex $S(5,3) = 1^2 2^0 3^0 + 1^0 2^2 3^0 + 1^0 2^0 3^2 + 1^1 2^1 3^0 + 1^1 2^0 3^1 + 1^0 2^1 3^1$
$= 1 + 4 + 9 + 2 + 3 + 6 = 25$

Classical Stirling Numbers

$\boxed{\text{Def}}$ A (type A) <u>ordered</u> set partition of $[n]$, written $P \vDash [n]$, is $P = (B_1, \ldots, B_k)$ s.t. $\{B_1, \ldots, B_k\} \vdash [n]$.

- Set $S^o([n],k) = \{P \vDash [n]$ where P has k blocks$\}$
- The (type A) <u>ordered</u> Stirling numbers of the second kind are

$$S^o(n,k) = k!\,S(n,k)$$
$$= \#S^o([n],k)$$

$\boxed{\text{Ex}}$ $13/56/24 \in S^o([6],3)$ and $S^o(5,3) = 3!\,S(5,3) = 6 \cdot 25 = 150$

Classical q-Stirling Numbers

Def The q-integers are defined by

$$[k] = 1+q+\cdots+q^{k-1} = \frac{1-q^k}{1-q}$$

Have $\lim_{q\to 1} [k] = k$.

Def ((Carlitz '33; Gould '61)

The (type A) q-Stirling numbers of the second kind are defined recursively by

$$\left.\begin{aligned} S[n,k] &= S[n-1,k-1] + [k]S[n-1,k] \\ S[0,k] &= \delta_{0k} \end{aligned}\right] \Rightarrow S[n,k]\big|_{q=1} = S(n,k)$$

Classical q-Stirling Numbers

Ex

	$k=0$:	$k=1$:	$k=2$:	$k=3$:	$k=4$:	$k=5$:
$n=0$:	1	0	0	0	0	0 ...
$n=1$:	0	1	0	0	0	0 ...
$n=2$:	0	1	1	0	0	0 ...
$n=3$:	0	1	$2+q$	1	0	0 ...
$n=4$:	0	1	$3+3q+q^2$	$3+2q+q^2$	1	0 ...
$n=5$:	0	1	$4+6q+4q^2+q^3$	$\boxed{6+8q+7q^2+3q^3+q^4}$	$4+3q+2q^2+q^3$	1 ...

$$=(3+3q+q^2)+(1+q+q^2)(3+2q+q^2)=S[5,3]$$

Classical q-Stirling Numbers

Thm $S[n,k] = h_{n-k}([1],[2],\dots,[k]) = \sum_{\gamma_1+\dots+\gamma_k=n-k} [1]^{\gamma_1}[2]^{\gamma_2}\cdots[k]^{\gamma_k}$

Pf Same!

Thm (Milne '82; Garsia-Remmel '86; Wachs-White '91; ...)

There are statistics $\mathrm{stat}: S([n],k) \to \mathbb{Z}_{\geq 0}$ such that

$$S[n,k] = \sum_{p \in S([n],k)} q^{\mathrm{stat}(p)}$$

Classical q-Stirling Numbers

- For instance, set

$$\mathrm{inv}(P) = \#\{(a, B_j) : a \in B_i \text{ for some } i<j \text{ and } a > \min B_j\}$$

Then $\displaystyle\sum_{P\in S([n],k)} q^{\mathrm{inv}(P)} = S[n,k]$

$\boxed{\text{Ex}}$ $145/27/36$ has $\mathrm{inv}=5$

Classical q-Stirling Numbers

Def The (type A) ordered q-Stirling numbers of the second kind are

$$S^o[n,k] = [k]!\,S[n,k]$$

(The q-Factorial is $[k]! = [k][k-1]\cdots[1]$)

- Encountered in Carlitz '33 originally!
- Statistics and additional work by Steingrímson '01, Haglund-Rhoades-Shimozono '18, ...

e.g. $$\sum_{P \in S^o([n],k)} q^{\mathrm{inv}(P)} = S^o[n,k]$$

Classical Type B Stirling Numbers

Def The type B Stirling numbers of the second kind are defined recursively by

$$S_B(n,k) = S_B(n-1,k-1) + (2k+1)\,S_B(n-1,k)$$

where $S_B(0,k) = \delta_{0k}$

(See A039755)

Ex

$n=0$:	1	0	0	0	0	0	…
$n=1$:	1	1	0	0	0	0	…
$n=2$:	1	4	1	0	0	0	…
$n=3$:	1	13	9	1	0	0	…
$n=4$:	1	40	58	16	1	0	…
$n=5$:	1	121	330	170	25	1	…

$= 58 + 7\cdot 16 = S_B(5,3)$

Classical Type B Stirling Numbers

Def A ***type B* set partition** of $\langle n\rangle = \{-n, -n+1, \dots, -1, 0, 1, \dots, n-1, n\}$, written $P \vdash \langle n\rangle$, is a set partition of $\langle n\rangle$ such that

$$B \in P \ni -B = \{-b : b \in B\} \in P$$

and $B \neq -B$ unless $0 \in B$.

Ex $\{\{4\}, \{1,-3\}, \{-2,0,2\}, \{-4\}, \{-1,3\}\}$

$$\updownarrow$$

$$\bar{2}02 \,/\, \frac{\bar{1}3}{1\bar{3}} \,/\, \frac{\bar{4}}{4} \in S_B(\langle 4\rangle, 2)$$

increasing minima, say

Why? Naturally indexes type B_n hyperplane arrangement intersection lattice

- Set $S_B(\langle n\rangle, k) = \{P \vdash \langle n\rangle : P \text{ has } 2k+1 \text{ blocks}\}$

Classical Type B Stirling Numbers

Thm $S_B(n,k) = \# S_B(\langle n \rangle, k)$

Pf Remove $\pm n$: if singleton, $2k-1$ blocks, otherwise $2k+1$ copies with $2k+1$ blocks

Thm $S_B(n,k) = h_{n-k}(1,3,\dots,2k+1) = \sum_{\gamma_0+\dots+\gamma_k=n-k} 1^{\gamma_0} 3^{\gamma_1} \cdots (2k+1)^{\gamma_k}$

Pf Same!

Classical Type B Stirling Numbers

Def A *type B ordered set partition* of $\langle n \rangle$, written $P \vDash \langle n \rangle$, is $P = (B_0, B_1, \ldots, B_{2k})$ s.t. $\{B_0, \ldots, B_k\} \vdash \langle n \rangle$ s.t. $0 \in B_0$ and $B_{2i} = -B_{2i-1}$ for $i \geq 1$

- Set $S_B^o(\langle n \rangle, k) = \{P \vDash \langle n \rangle$ where P has $2k+1$ blocks$\}$
- The *type B ordered Stirling numbers of the second kind* are

$$S_B^o(n,k) = 2^k k! \, S_B(n,k)$$
$$= \# S_B^o([n], k)$$

Ex $\bar{2}02 / \frac{\bar{4}}{4} / \frac{1\bar{3}}{\bar{1}3} \in S_B^o(\langle 4 \rangle, 2)$ and $S_B^o(5,3) = 8 \cdot 3! \cdot 170 = 8160$

New Type B q-Stirling Numbers

Def (S.-Wallach '21; Bagno-Garber '22+)

The type B q-Stirling numbers of the second kind are defined recursively by

$$\left.\begin{aligned} S_B[n,k] &= S_B[n-1,k-1] + [2k+1]\,S_B[n-1,k] \\ S_B[0,k] &= \delta_{0k} \end{aligned}\right] \Rightarrow S_B[n,k]\big|_{q=1} = S_B(n,k)$$

Why? Type B super coinvariant algebra Hilbert series data; later

New Type B q-Stirling Numbers

Ex

	k=0:	k=1:	k=2:	k=3:	
n=0:	1	0	0	0	...
n=1:	1	1	0	0	...
n=2:	1	$2+q+q^2$	1	0	...
n=3:	1	$3+3q+4q^2+2q^3+q^4$	$\boxed{3+2q+2q^2+q^3+q^4}$	1	...

$$= (2+q+q^2) + (1+q+q^2+q^3+q^4)\cdot 1 = S_B[3,2]$$

New Type B q-Stirling Numbers

Thm $S_B[n,k] = h_{n-k}([1],[3],\ldots,[2k+1]) = \sum_{\gamma_0+\cdots+\gamma_k=n-k} [1]^{\gamma_0}[3]^{\gamma_1}\cdots[2k+1]^{\gamma_k}$

PF Same!

Thm (S.-Sagan '22; Bagno-Garber '22+)

There are statistics $\mathrm{stat}: S(\langle n\rangle, k) \to \mathbb{Z}_{\geq 0}$ such that

$$S_B[n,k] = \sum_{p \in S_B(\langle n\rangle,k)} q^{\mathrm{stat}(p)}$$

New Type B q-Stirling Numbers

- Our inversion-style type B set partition statistic is...

Def (S.-Sagan '22)

For $P \vdash \langle n \rangle$, set

$$\mathrm{inv}(P) = \#\{(a, B_j) : a \in B_i \text{ for some } i < j \text{ and } a > \min |B_j|\}$$

- Then $\displaystyle\sum_{P \in S_B(\langle n \rangle, k)} q^{\mathrm{inv}(P)} = S_B[n,k]$.

Ex $\bar{2}02 / \begin{smallmatrix}\bar{1}3\\1\bar{3}\end{smallmatrix} / \begin{smallmatrix}\bar{4}\\4\end{smallmatrix}$ has $\mathrm{inv} = 3$

New Type B q-Stirling Numbers

Def The type B ordered q-Stirling numbers of the second kind are

$$S_B^o[n,k] = [2k]!!\, S_B[n,k]$$
$$= [2]^k [k]_{q^2}!\, S_B[n,k]$$

where the q-double factorial is

$$[n]!! = \begin{cases} [n][n-2]\cdots[2] & \text{if } n \geq 2 \text{ is even} \\ [n][n-2]\cdots[1] & \text{if } n \geq 1 \text{ is odd} \\ 1 & \text{if } n=0 \text{ or } n=-1 \end{cases}$$

- In Fact, $\displaystyle\sum_{p \in S_B^o(\langle n \rangle, k)} q^{inv(p)} = S_B^o[n,k]$! We have a maj version too.

Enumerative Results

Thm (Classical)

$$x^n = \sum_{k=0}^{n} S(n,k)\, x(x-1)\cdots(x-k+1)$$

(Bala '15, Bagno-Biagioli-Garber '19)

$$x^n = \sum_{k=0}^{n} S_B(n,k)(x-1)(x-3)\cdots(x-2k+1)$$

(S.-Sagan '22)

$$x^n = \sum_{k=0}^{n} S_B[n,k](x-[1])(x-[3])\cdots(x-[2k-1])$$

Enumerative Results

Thm (Classical)

$$\sum_{n\geq 0} S(n,k)\, x^n = \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}$$

(Bagno-Garber '22+)

$$\sum_{n\geq 0} S_B(n,k)\, x^n = \frac{x^k}{(1-x)(1-3x)\cdots(1-(2k+1)x)}$$

(S.-Sagan '22)

$$\sum_{n\geq 0} S_B[n,k]\, x^n = \frac{x^k}{(1-[1]x)(1-[3]x)\cdots(1-[2k+1]x)}$$

Enumerative Results

$\boxed{\text{Thm}}$ (Classical) $\displaystyle\sum_{n\geq 0} S^0(n,k)\frac{x^n}{n!} = (e^x-1)^k$

(See S.-Sagan '22)

$$\sum_{n\geq 0} S^0[n,k]\frac{x^n}{[n]!} = \sum_{i=0}^{k} \frac{(-1)^{k-i}}{q^{\binom{i}{2}+i(k-i)}} \begin{bmatrix} k \\ i \end{bmatrix} \exp_q([i]x)$$

where $\begin{bmatrix} n \\ k \end{bmatrix} = \dfrac{[n]!}{[k]![n-k]!}$ and $\exp_q(x) = \displaystyle\sum_{n\geq 0} \frac{x^n}{[n]!}$

- Proof uses q-differential equations, $D_q F(x) = \dfrac{F(x)-F(qx)}{x-qx}$

Enumerative Results

Thm (?)

$$\sum_{n\geq 0} S_B^0(n,k)\frac{x^n}{n!} = e^x(e^{2x}-1)^k$$

(S.-Sagan '22)

$$\sum_{n\geq 0} S_B^0[n,k]\frac{x^n}{[n]!} = \sum_{i=0}^{k} \frac{(-1)^{k-i}}{q^{i^2+(2i+1)(k-i)}} \begin{bmatrix} k \\ i \end{bmatrix}_{q^2} \exp_q([2i+1]x)$$

- Have additional generating function identities and <u>first kind</u> versions, <u>e.g.</u> $\sum_{n,k\geq 0} c[n,k]\frac{x^n}{[n]!}y^k = \exp_q[-y\log_q[1-x]]$

Enumerative Results

$\boxed{\text{Thm}}$ (Conjectured in S.-Wallach '21, proved in S.-Sagan '22)

$$\sum_{k=0}^{n} (-q)^{n-k} S^o[n,k] = 1$$

and

$$\sum_{k=0}^{n} (-q)^{n-k} S^o_B[n,k] = 1$$

- Algebraic and combinatorial proofs
- Conjectured description of $\sum_{k=0}^{n} (-q^2)^{n-k} S^o[n,k] - 1$ and more

Open Problems

Conj For each n,k, the coefficients of $S[n,k]$ (and $c[n,k]$) are log-concave, hence unimodal.

Conj For each n,k, the coefficients of $S_B[n,k]$ (and $c_B[n,k]$) are parity-log-concave, hence unimodal.

Ex $S_B[6,4] = 15 + 24q + 34q^2 + 38q^3 + 43q^4 + 42q^5 + 43q^6 + 38q^7 + 35q^8$
$+ 26q^9 + 20q^{10} + 14q^{11} + 10q^{12} + 6q^{13} + 4q^{14} + 2q^{15} + q^{16}$

Coinvariant Algebras

Thm (Newton) $\mathbb{Q}[x_1, \ldots, x_n]^{S_n} = \mathbb{Q}[e_1, \ldots, e_n]$ where $e_k = \underbrace{\sum_{1 \le i_1 < \cdots < i_k \le n} x_{i_1} \cdots x_{i_k}}_{\text{elementary symmetric polynomial}}$

and $\sigma(x_i) = x_{\sigma(i)}$

Thm (Hilbert) $\langle \mathbb{Q}[x_1, \ldots, x_n]_+^{S_n} \rangle = \langle e_1, \ldots, e_n \rangle$

Def The coinvariant algebra of S_n is

$$R_n = \frac{\mathbb{Q}[x_1, \ldots, x_n]}{\langle e_1, \ldots, e_n \rangle}$$

Coinvariant Algebras

Thm (Borel) $R_n \cong \underbrace{H^*}_{\text{singular cohomology}}(\underbrace{Fl_n}_{\text{complete flag manifold}})$

Thm (Chevalley) $R_n \cong \mathbb{Q}S_n$

$\Rightarrow \dim R_n = n!$

Thm (Artin) $\{x_1^{\alpha_1} \cdots x_n^{\alpha_n} : 0 \le \alpha_i < i\}$ descends to a basis for R_n

Cor $$\mathrm{Hilb}(R_n; q) = \sum_{d \ge 0} \dim(R_n)_d \cdot q^d = \underbrace{1 \cdot (1+q) \cdot (1+q+q^2) \cdots \underbrace{(1+q+\cdots+q^{n-1})}_{[n]_q}}_{[n]_q!}$$

Super Coinvariant Algebras

- Superspace is $\underbrace{\mathbb{Q}[x_1, -, x_n, \theta_1, -, \theta_n]}_{Sym(x_1,-,x_n)\otimes\Lambda(\theta_1,-,\theta_n)}$ where $\theta_i\theta_j = -\theta_j\theta_i$ anti-commute (and $x_i\theta_j = \theta_j x_i$, $x_ix_j = x_jx_i$)

- S_n acts diagonally: $\sigma(x_i) = x_{\sigma(i)}$, $\sigma(\theta_i) = \theta_{\sigma(i)}$

Super Coinvariant Algebras

$\boxed{\text{Def}}$ The super coinvariant algebra of S_n is

$$SR_n = \mathbb{Q}[\underline{x}_n, \underline{\theta}_n] / \langle \mathbb{Q}[\underline{x}_n, \underline{\theta}_n]^{S_n}_{+} \rangle$$

$\boxed{\text{Conj}}$ (Zabrocki '19) $\mathrm{Hilb}(SR_n; q, z) = \sum_{k=0}^{n} S^{\circ}[n,k]\, z^{n-k}$

$\boxed{\text{Thm}}$ (Rhoades-Wilson '23+) Hilbert series is true!

Super Coinvariant Algebras

- Can define SR_G for any pseudoreflection group G.

Conj (Wallach-S. '21)

$$\mathrm{Hilb}(SR_{B_n}; q, z) = \sum_{k=0}^{n} S_B^{\circ}[n,k]\, z^{n-k}$$

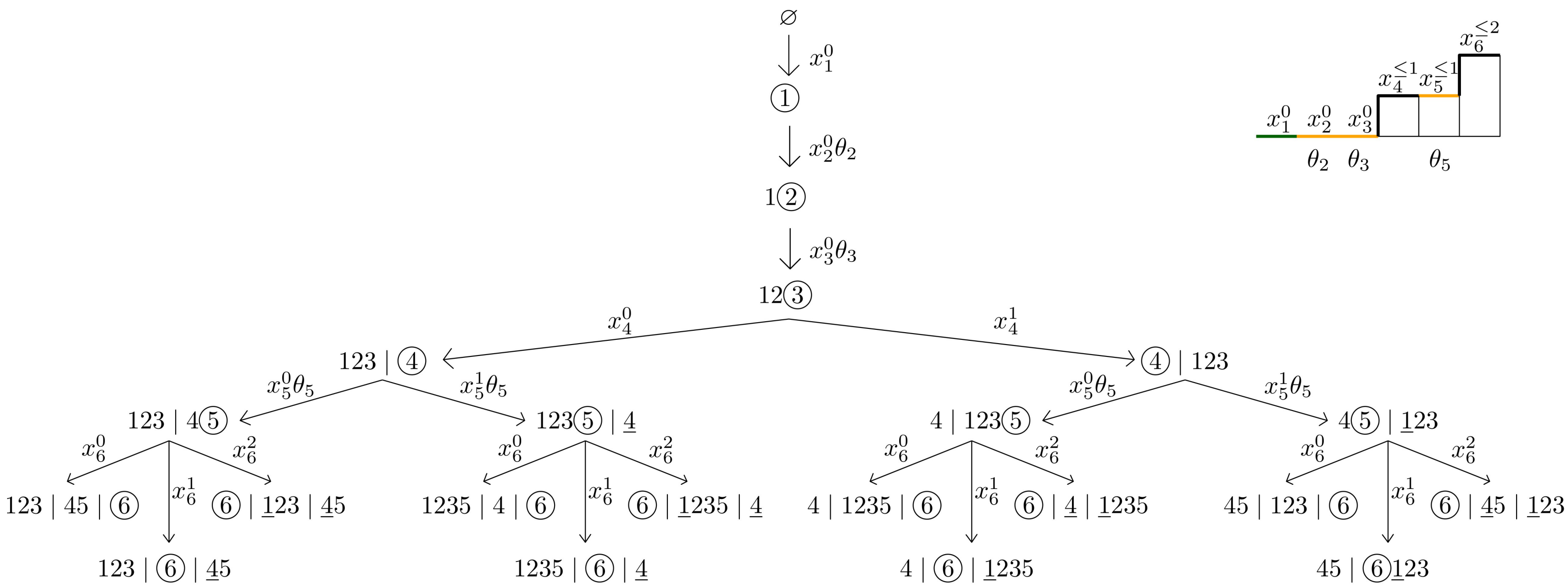
- Alternating sum identities arise from...

Thm (Wallach-S.) $\mathrm{Hilb}(SR_G; q, -q) = 1$

- Have $\mathrm{Hilb}(SR_G; q, -q^{e_i^*}) - 1 = \chi(SR_G, d_i)$ for "generalized exterior derivatives" d_i

Super Coinvariant Algebras

- Conjectured "Artin"-style basis in types A and B
  - naturally motivates inv on PF[n] and PF⟨n⟩!



∅
x_1^0
①
$x_2^0\theta_2$
1②
$x_3^0\theta_3$
12③
x_4^0
x_4^1
123 | ④
④ | 123
$x_5^0\theta_5$
$x_5^1\theta_5$
$x_5^0\theta_5$
$x_5^1\theta_5$
123 | 4⑤
123⑤ | $\underline{4}$
4 | 123⑤
4⑤ | $\underline{1}$23
x_6^0
x_6^1
x_6^2
123 | 45 | ⑥
⑥ | $\underline{1}$23 | $\underline{4}$5
123 | ⑥ | $\underline{4}$5
1235 | 4 | ⑥
⑥ | $\underline{1}$235 | $\underline{4}$
1235 | ⑥ | $\underline{4}$
4 | 1235 | ⑥
⑥ | $\underline{4}$ | $\underline{1}$235
4 | ⑥ | $\underline{1}$235
45 | 123 | ⑥
⑥ | $\underline{4}$5 | $\underline{1}$23
45 | ⑥$\underline{1}$23
x_1^0 x_2^0 x_3^0
$x_4^{\le 1}$ $x_5^{\le 1}$ $x_6^{\le 2}$
θ_2 θ_3 θ_5

THANKS!