Unifying lattices through hourglass plabic graphs

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Outline

- Hourglass plabic graphs

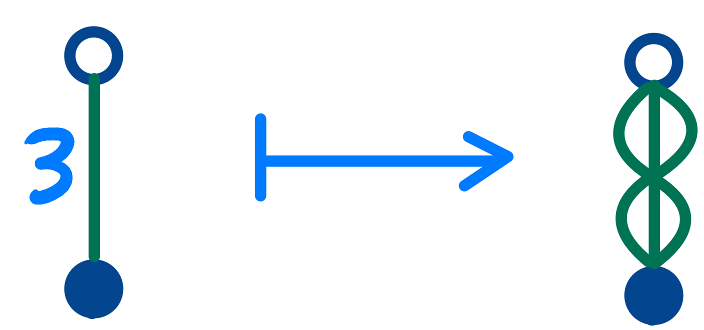
- ASM, PP, Tamari examples

- Pockets

Hourglass plabic graphs

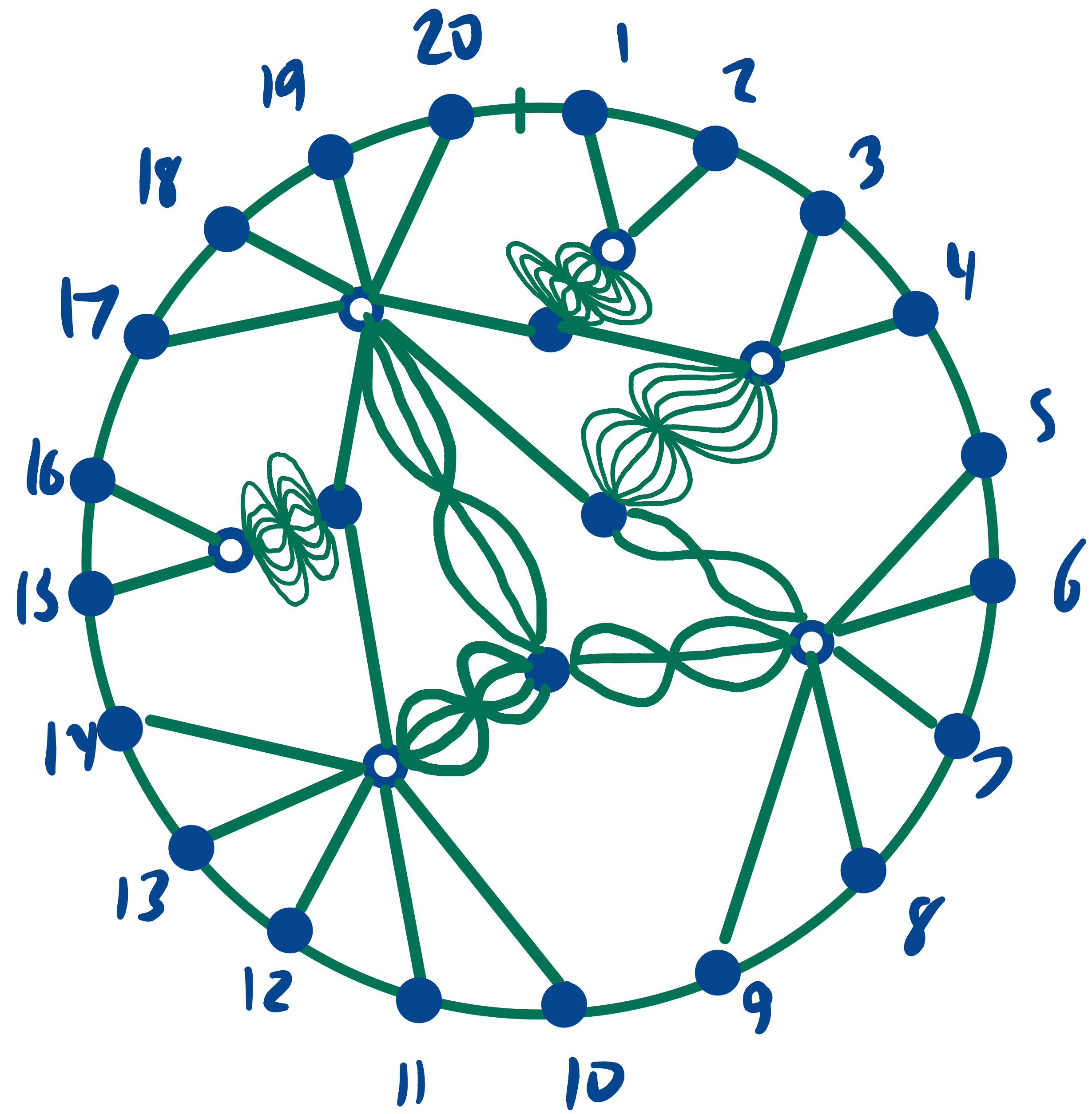
Def An r-hourglass plabic graph (r-HPG) is a planar bipartite graph embedded in a disk with edge weights in $[r]$ which sum to r around internal vertices, and boundary vertices have degree 1.

An edge with weight m is drawn as an m-hourglass:



Hourglass plabic graphs

Ex

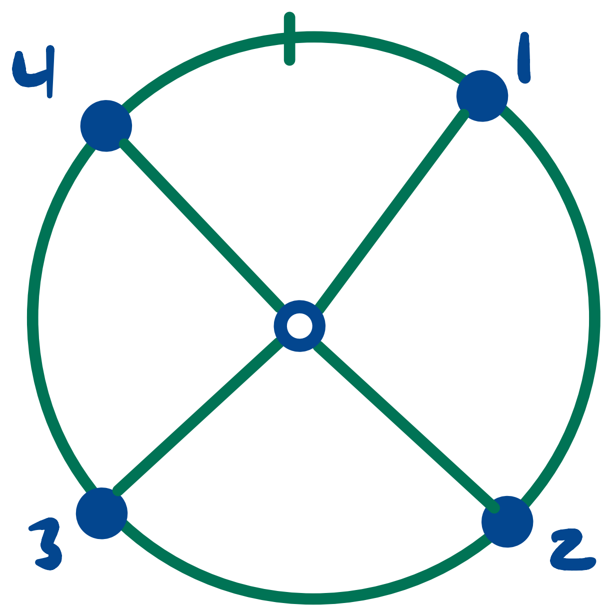


$(r=10)$

Hourglass plabic graphs

Q Why?

A • Encodes morphisms in $\mathrm{Hom}_{U_q(\mathfrak{sl}_r)}(V_q^{\otimes n}, \mathbb{C}(q))$

e.g. 

$$= \det\begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{pmatrix} \qquad (V_{q=1} = \mathbb{C}^4)$$

• Here  $\longleftrightarrow$ $\Lambda_q^3 V_q$ in domain.

• Effective calculation, e.g. link invariants

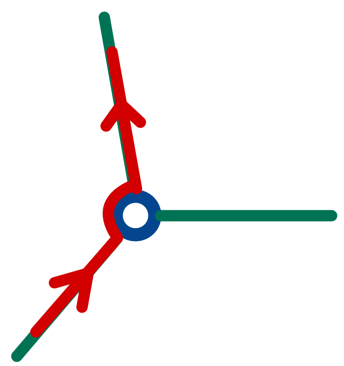
Hourglass plabic graphs

$\boxed{\text{A2}}$ Connects amazing combinatorial examples!

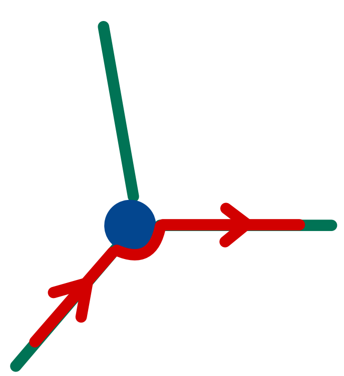
<u>Today</u>: ASMs, PPs, Tamari

Trip permutations

Recall Postnikov '06 introduced plabic graphs and their trip permutation:

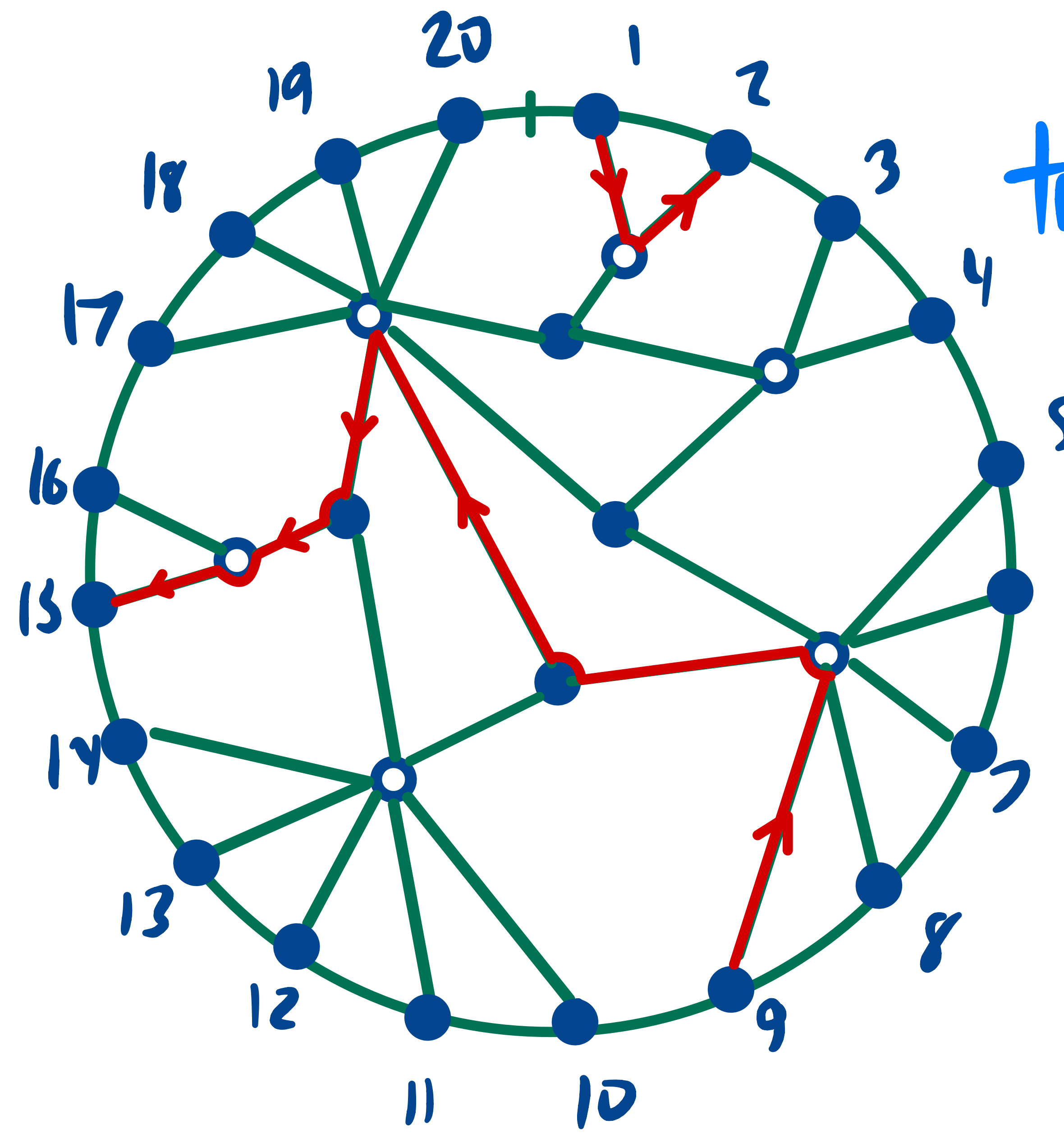


left at white



right at black

Ex

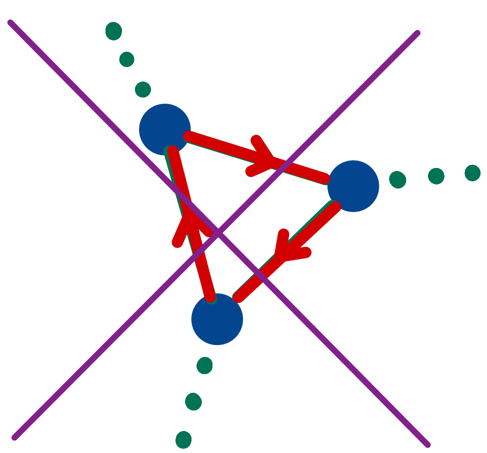


trip = $(\underline{1\ 2}\ 5\ 6\ 7\ 8\ \underline{9\ 15}\ 16)$
$(3\ 4\ 10\ 11\ 12\ 13\ 14\ 17\ 18\ 19\ 20)$

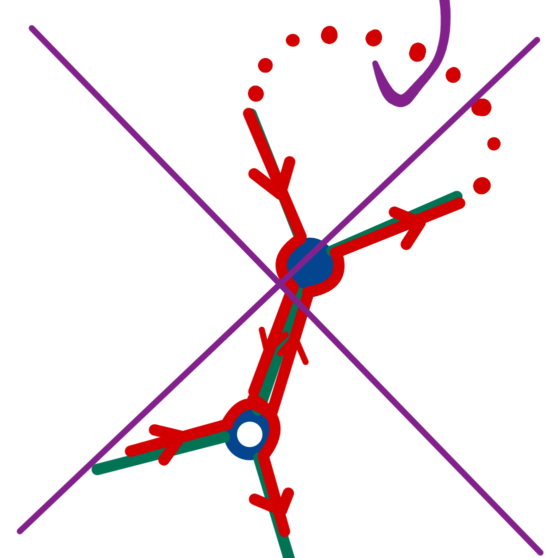
Trip permutations

Thm/Def (Postnikov '06) A (leafless, connected, fixed-point free) plabic graph is **reduced** if it has

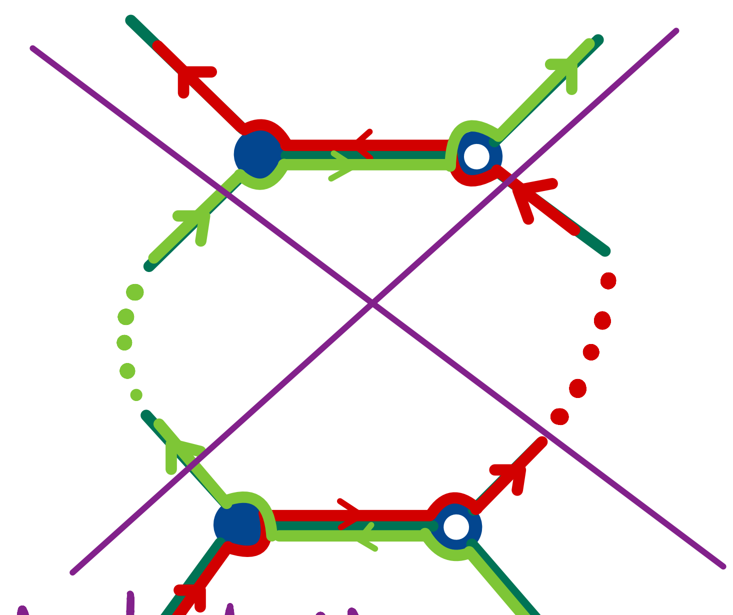
1. no round trips
2. no essential self-intersections
3. no bad double crossings



round trip



self-intersection

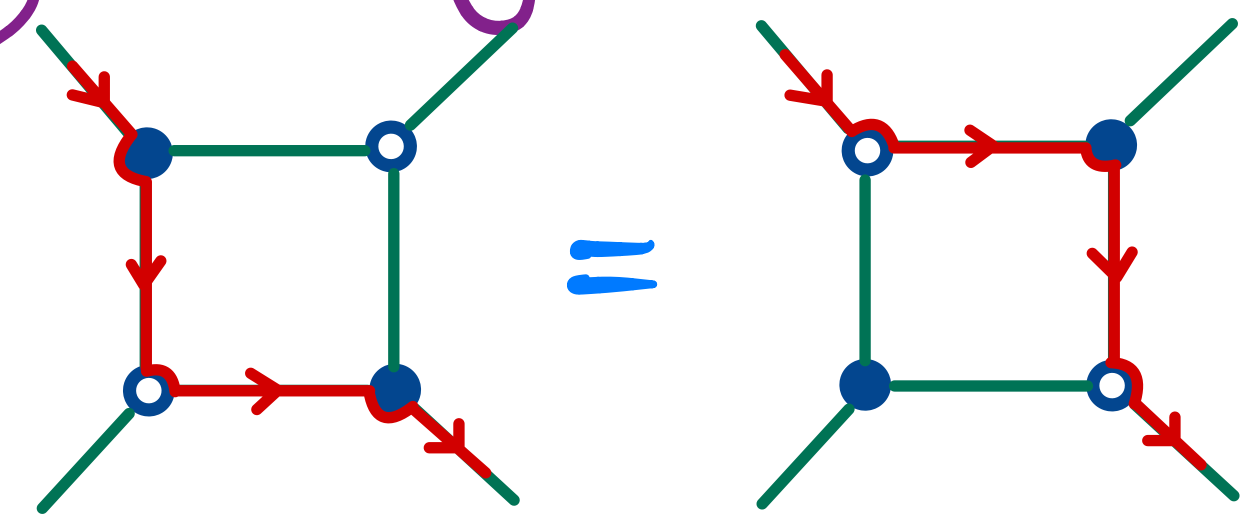


bad double crossing

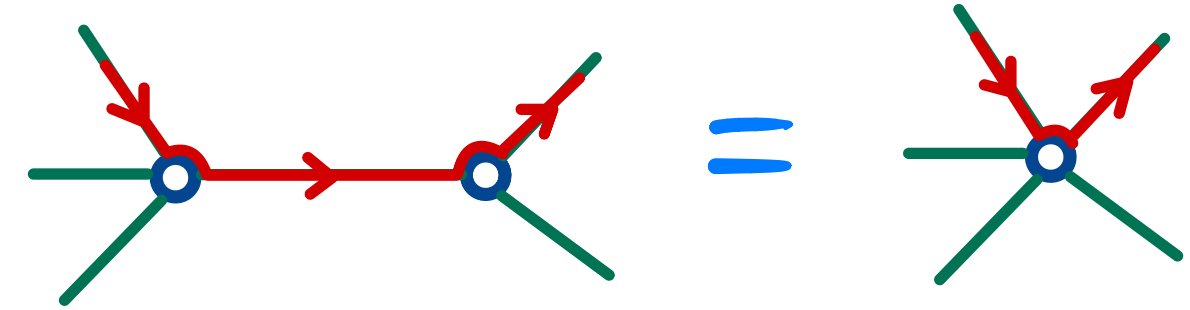
Trip permutations

Thm (Postnikov '06) Two such plabic graphs have the same trip if and only if they are related by moves:

M1 Square move:



M2 Edge contraction:

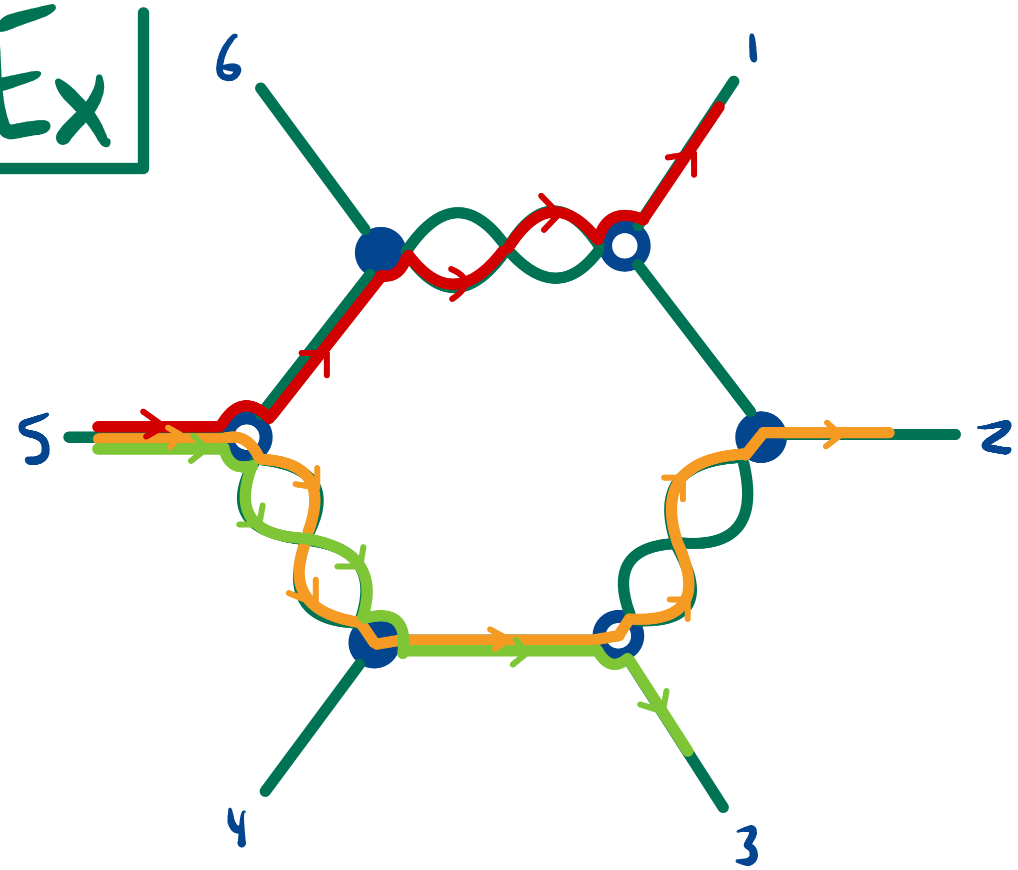


M3 Vertex removal:

Trip permutations

Def (GPPSS '23+) An r-hourglass plabic graph has trip permutations $\mathrm{trip}_1, \dots, \mathrm{trip}_{r-1}$ where trip_i takes the ith left at white and ith right at black:

Ex



$(r=4)$

$= \mathrm{trip}_1 = (1\,3\,5)(6\,4\,2)$

$= \mathrm{trip}_2 = (1\,4)(2\,5)(3\,6)$

$= \mathrm{trip}_3 = (5\,3\,1)(2\,4\,6)$

Note

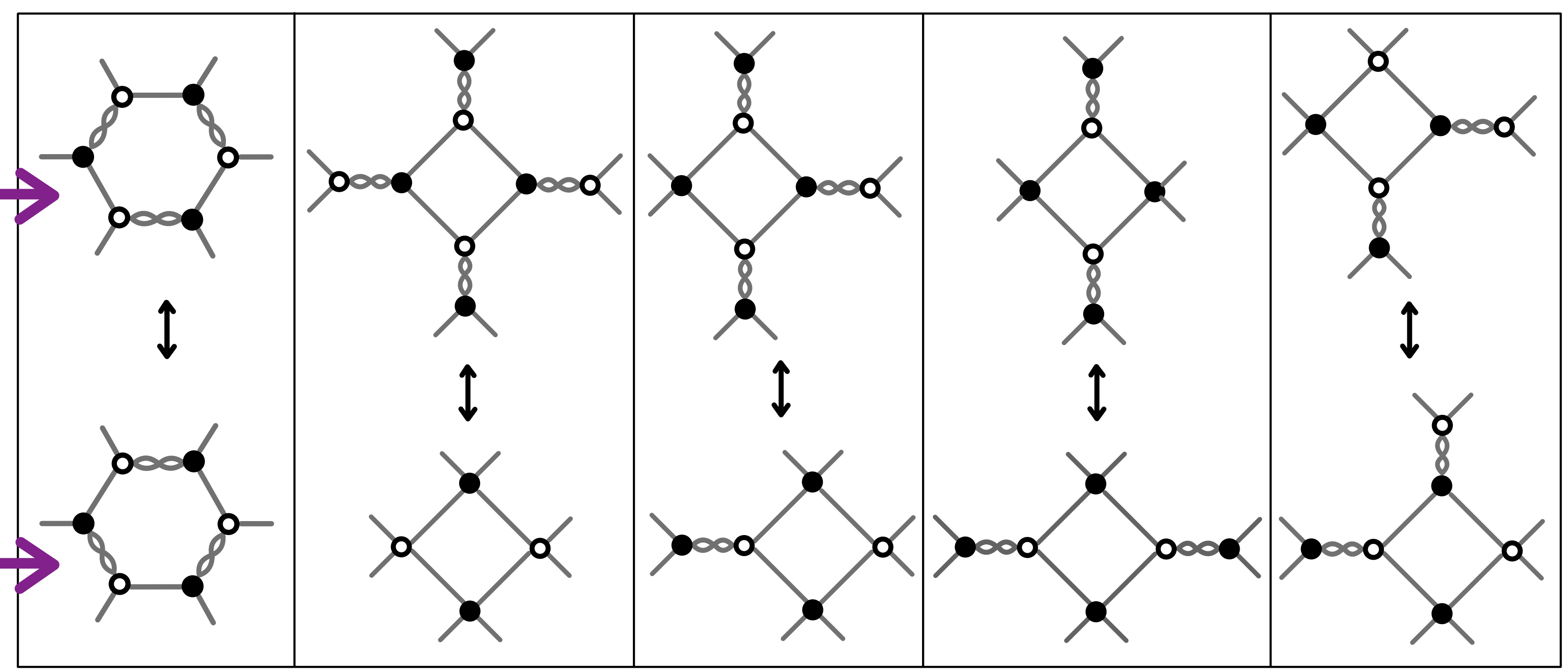
$\mathrm{trip}_i = \mathrm{trip}_{r-i}^{-1}$!

r=4 moves

Thm (GPPSS '23+) Two contracted, fully reduced 4-hourglass plabic graphs have the same $\mathrm{trip}_1, \mathrm{trip}_2, \mathrm{trip}_3$ if and only if they are related by

"top"

"bottom"



benzene move

square moves

r=4 moves

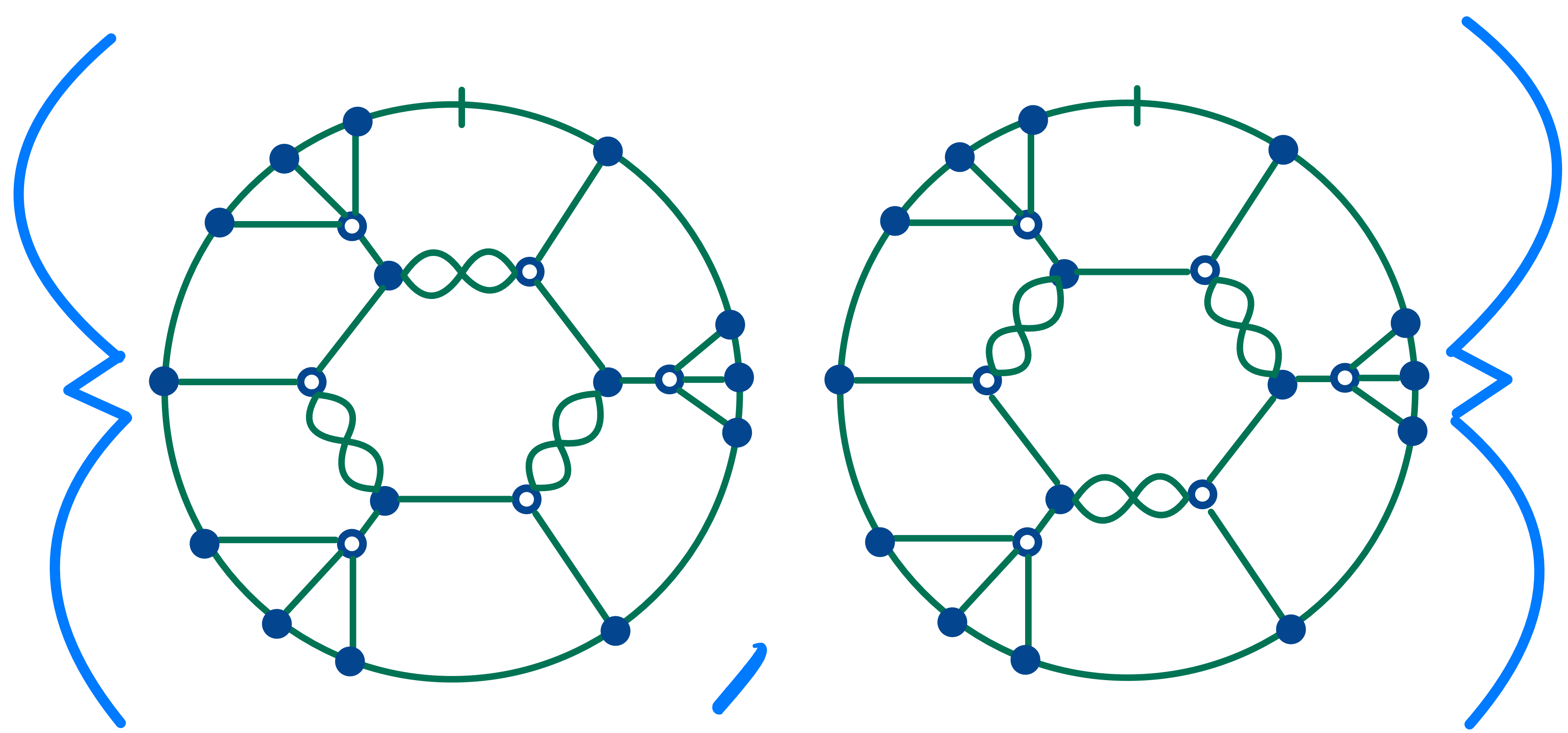
Thm (GPPSS '23+) There is a bijection between $SYT(4\times c)$ and such move-classes. It sends $\mathrm{prom}_i(T)$ to $\mathrm{trip}_i(G)$.

Ex

$T=$

1	2	6
3	5	10
4	7	11
8	9	12

$\longmapsto$

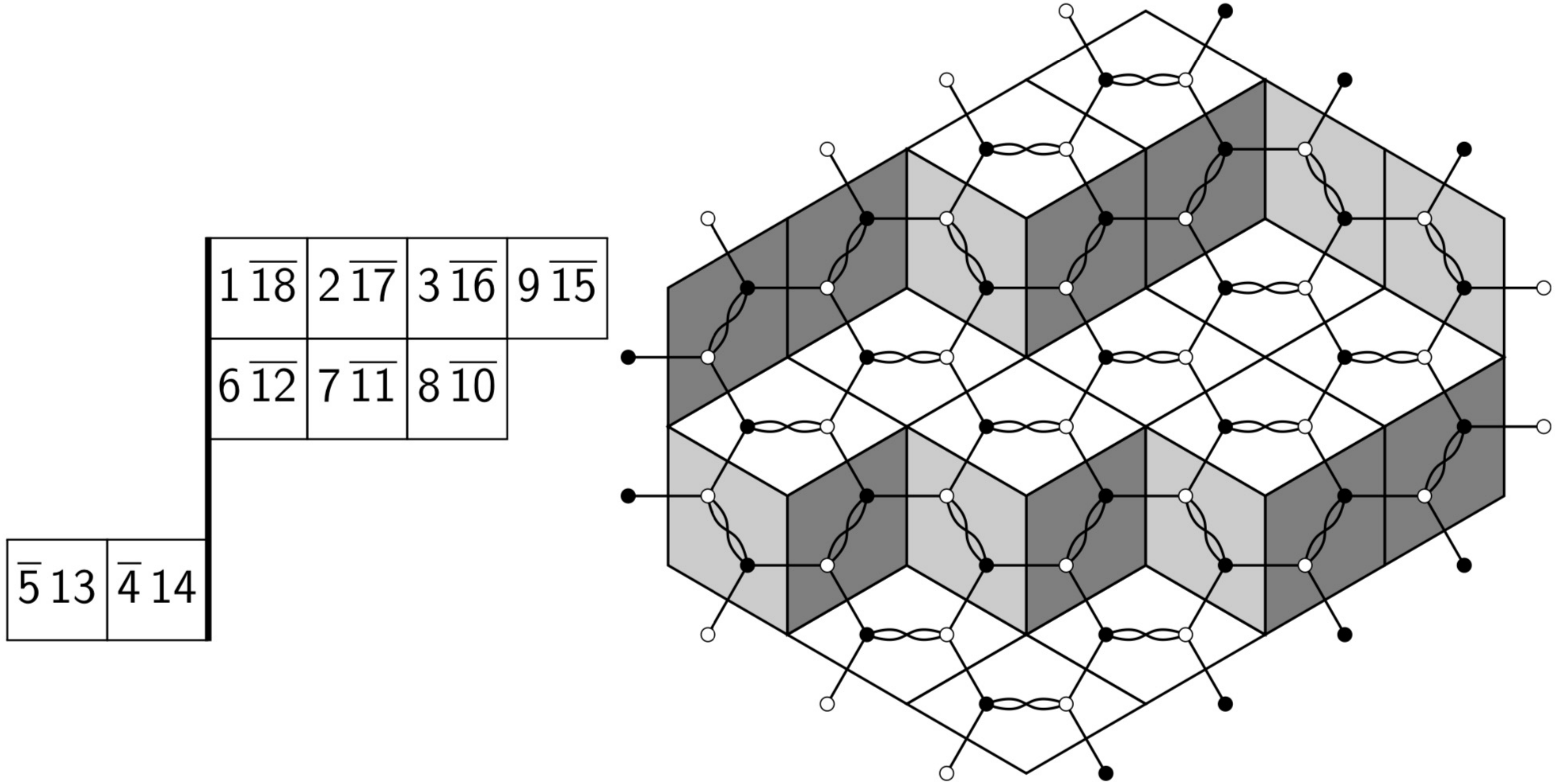


PP lattice

Prop (GPPSS '23+) Let $a, b, c \geq 1$ with $a \geq c$.
The $r=4$ move-equivalence class associated to the (oscillating) tableau with lattice word $L = 1^a \bar{4}^b 2^c \bar{1}^{a-c} \bar{2}^c 4^b \bar{1}^c$ is in bijection with plane partitions in an $a \times b \times c$ box.

PP lattice

Ex



Note
- trip_1 does not see benzene moves (no square moves)
- trip_2 does
- Poset: bottom < top

ASM lattice

$\boxed{\text{Prop}}$ (GPPSS '23+) Let $n \geq 1$.

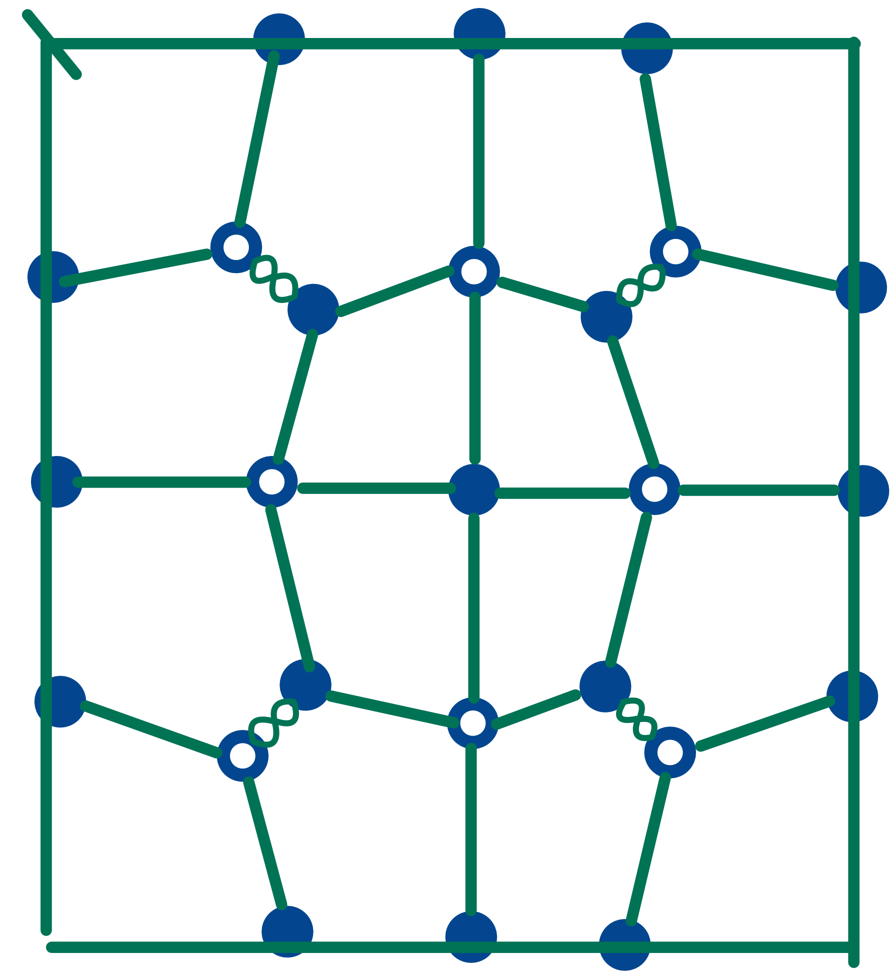
The $r=4$ move-equivalence class associated to the tableau with lattice word $L = 1^n 2^n 3^n 4^n$ is in bijection with $n \times n$ alternating sign matrices.

ASM lattice

Ex

1	2	3
4	5	6
7	8	9
10	11	12

$\mapsto$ class of



ASM:

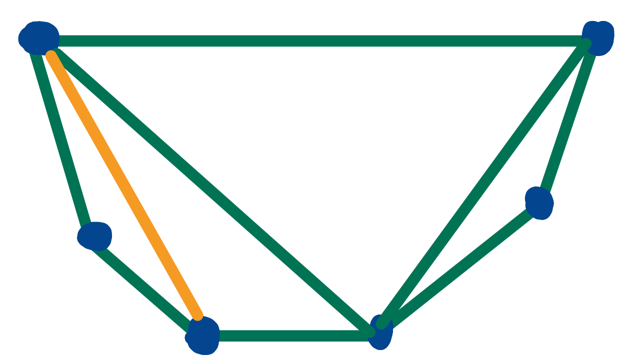
$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Note

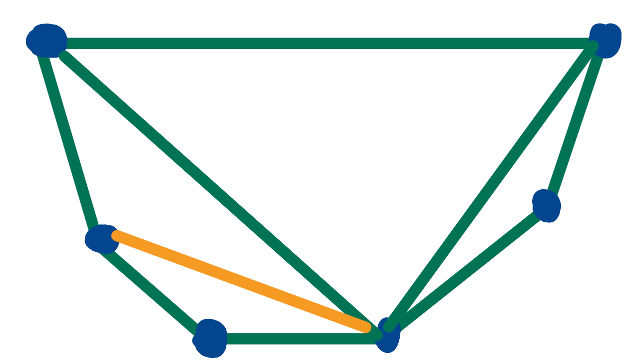
- trip_2 does not see square moves (no benzene moves)
- trip_1 does
- Poset: increase by $\begin{matrix} -1 & 1 \\ 1 & -1 \end{matrix}$

Tamori lattice

Recall | The Tamori lattice is the set of triangulations of an r-gon, with covering relations given by flips that increase the slope of the diagonal:



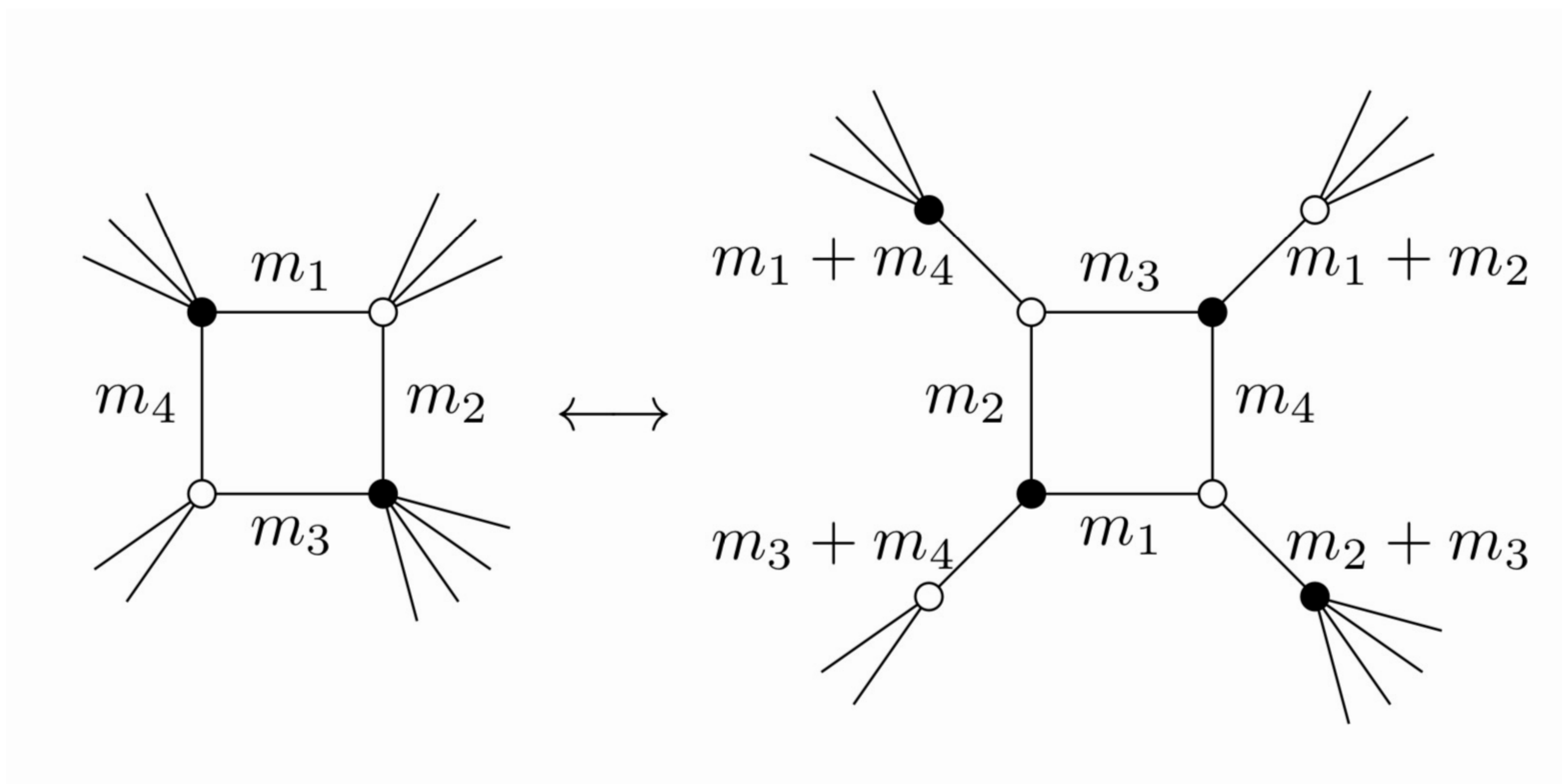
$<$



General square moves

$\boxed{\text{Def}}$ (See Cautis-Kamnitzer-Morrison '14, Fraser-Lam-Le '19, Fraser '23)

The r-HPG square move is:



Fraser 2-column webs

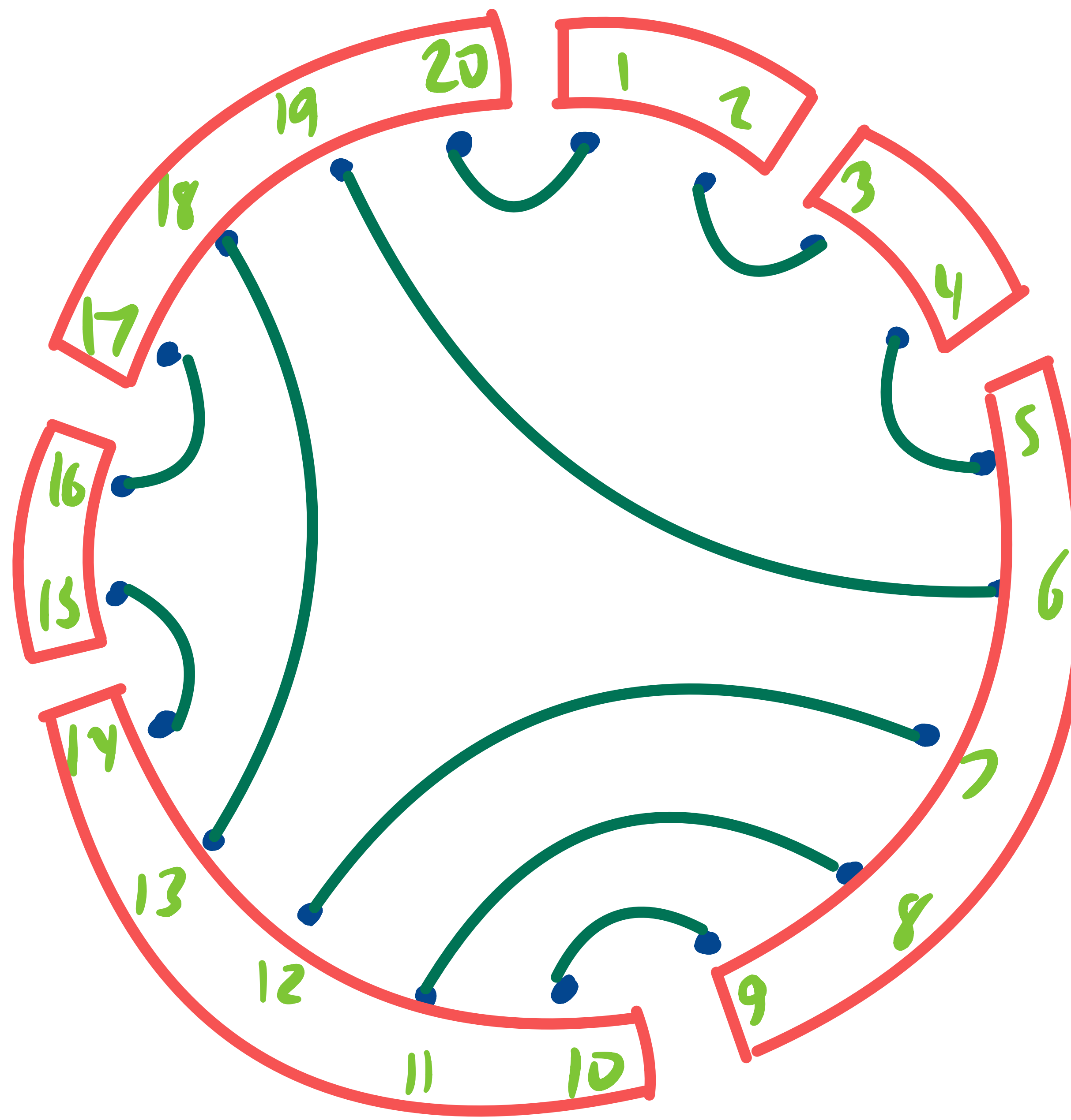
- Fraser '23 constructed a web basis (up to square moves) indexed by $SYT(r \times 2)$.

Thm (GPPSS '23+) Fraser's construction is a bijection between $SYT(r \times 2)$ and square move equivalence classes of contracted, fully reduced r-HPG's with b internal black vertices and $b+2$ internal white vertices. It sends $\mathrm{prom}_i(T)$ to $\mathrm{trip}_i(W)$.

Fraser 2-column webs

Ex

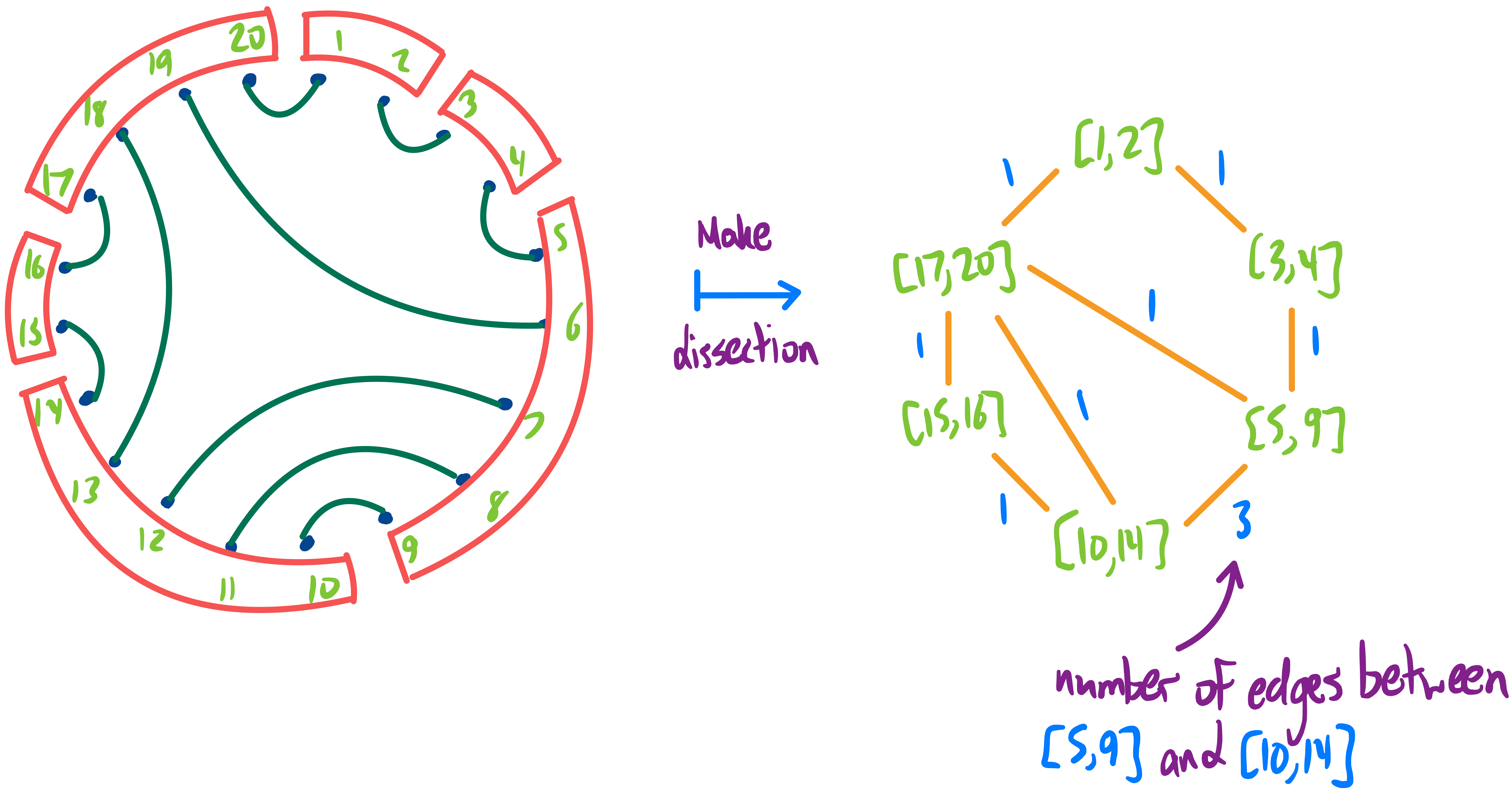
$$T = \begin{array}{cc} 1 & 3 \\ 2 & 5 \\ 4 & 10 \\ 6 & 11 \\ 7 & 12 \\ 8 & 15 \\ 9 & 17 \\ 13 & 18 \\ 14 & 19 \\ 16 & 20 \end{array} \xmapsto{\text{Catalan}}$$



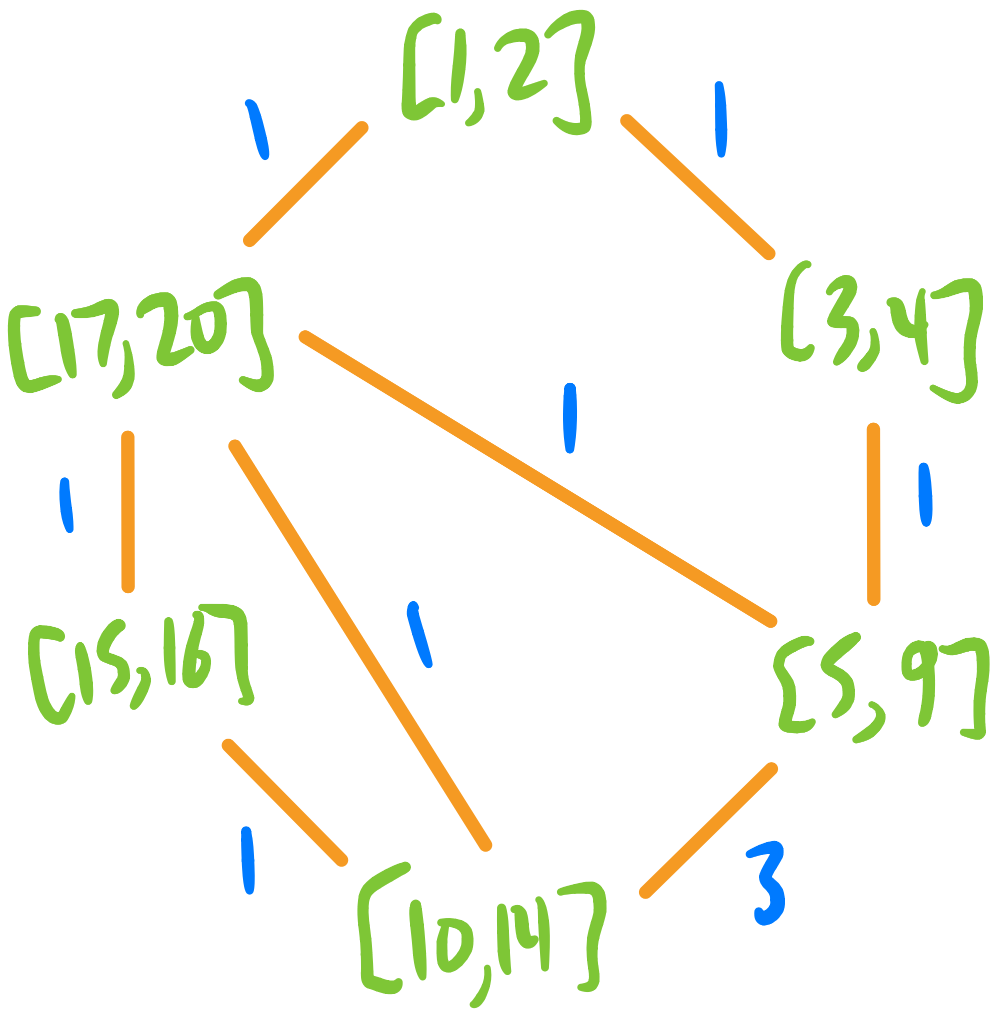
"color sets" are broken by "short arcs" where i and i+1 (mod 2k) are connected (i.e. cyclic ascents)

(()()(((()))(()())))

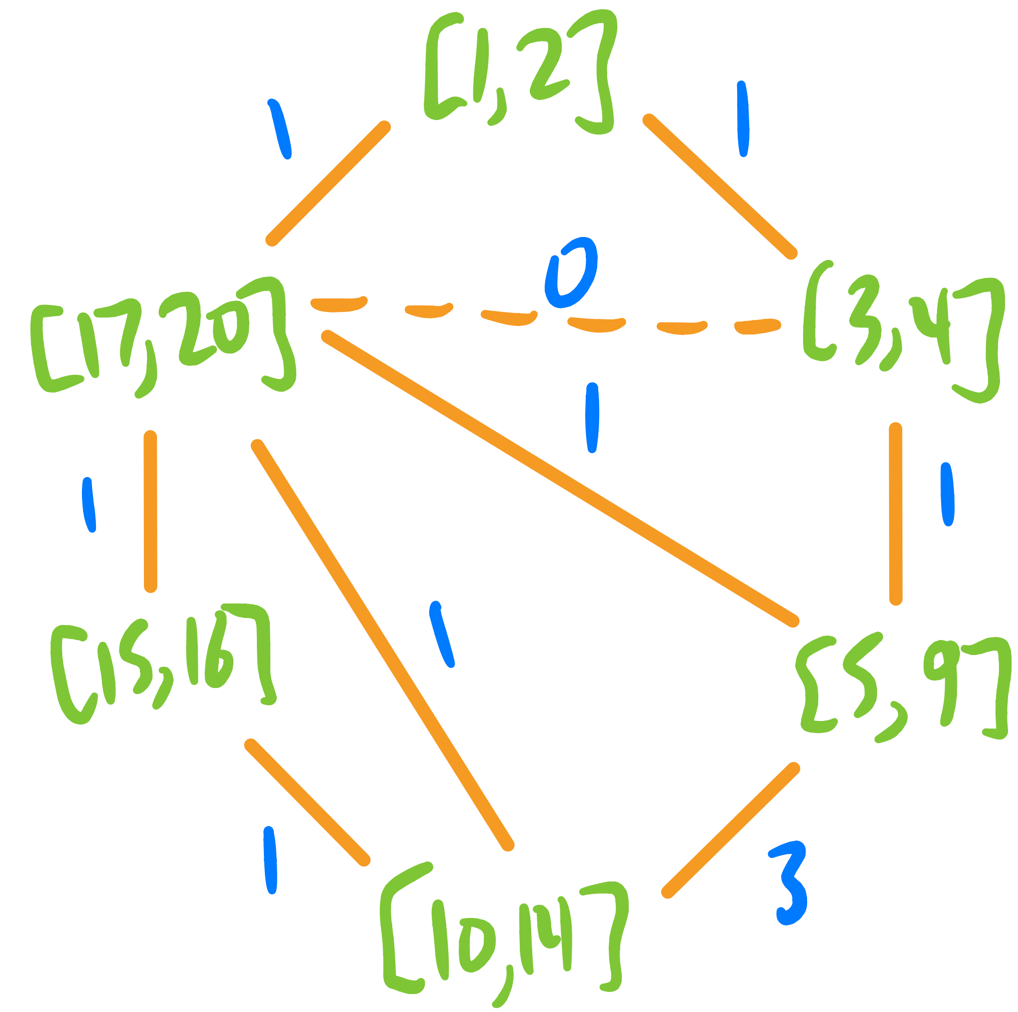
Fraser 2-column webs



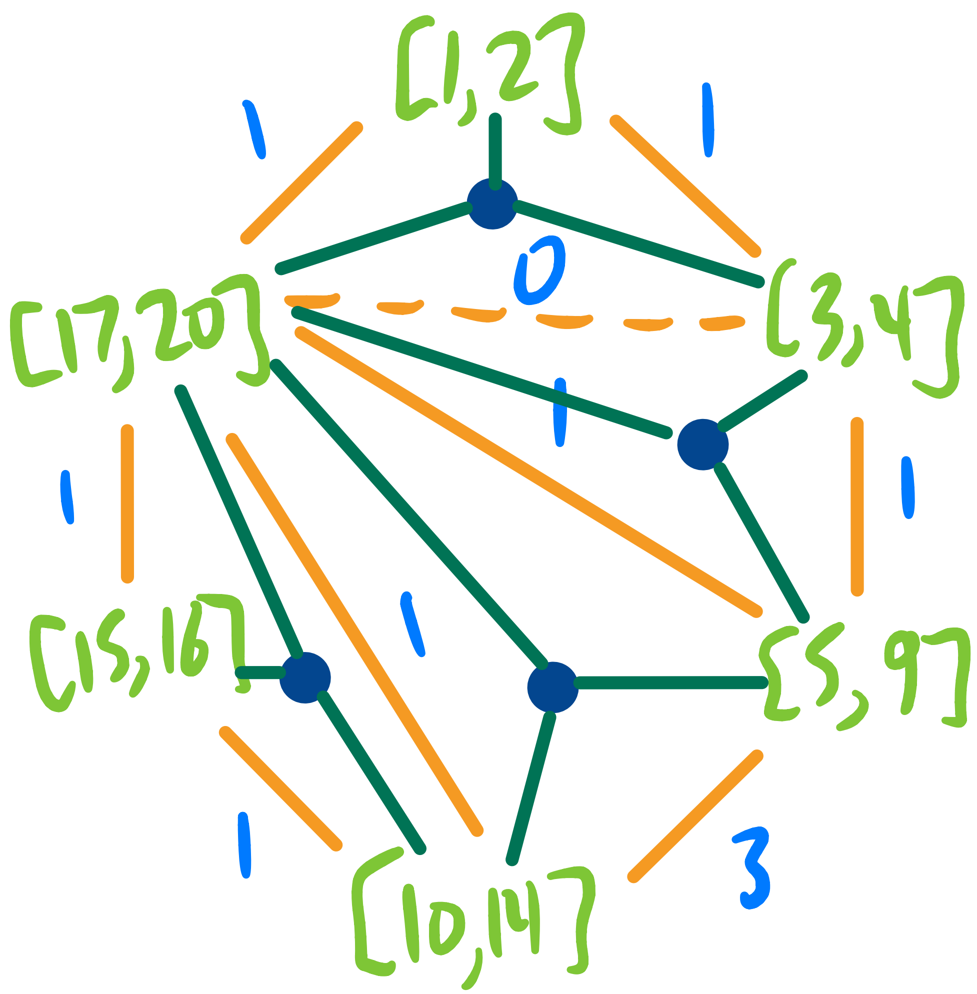
Fraser 2-column webs



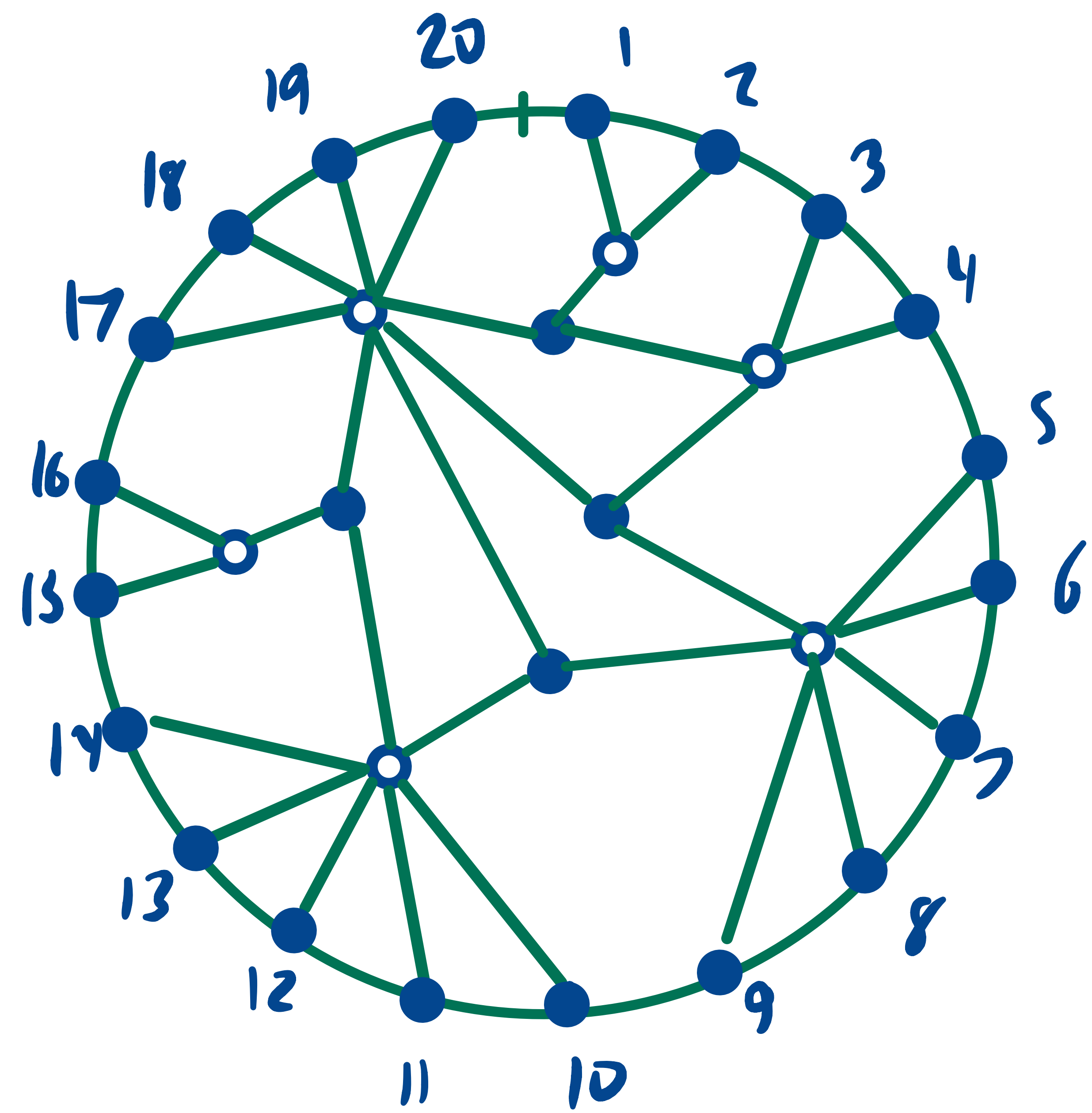
Make
↦
triangulation
(not unique,
all related by
"flips" though)



Fraser 2-column webs

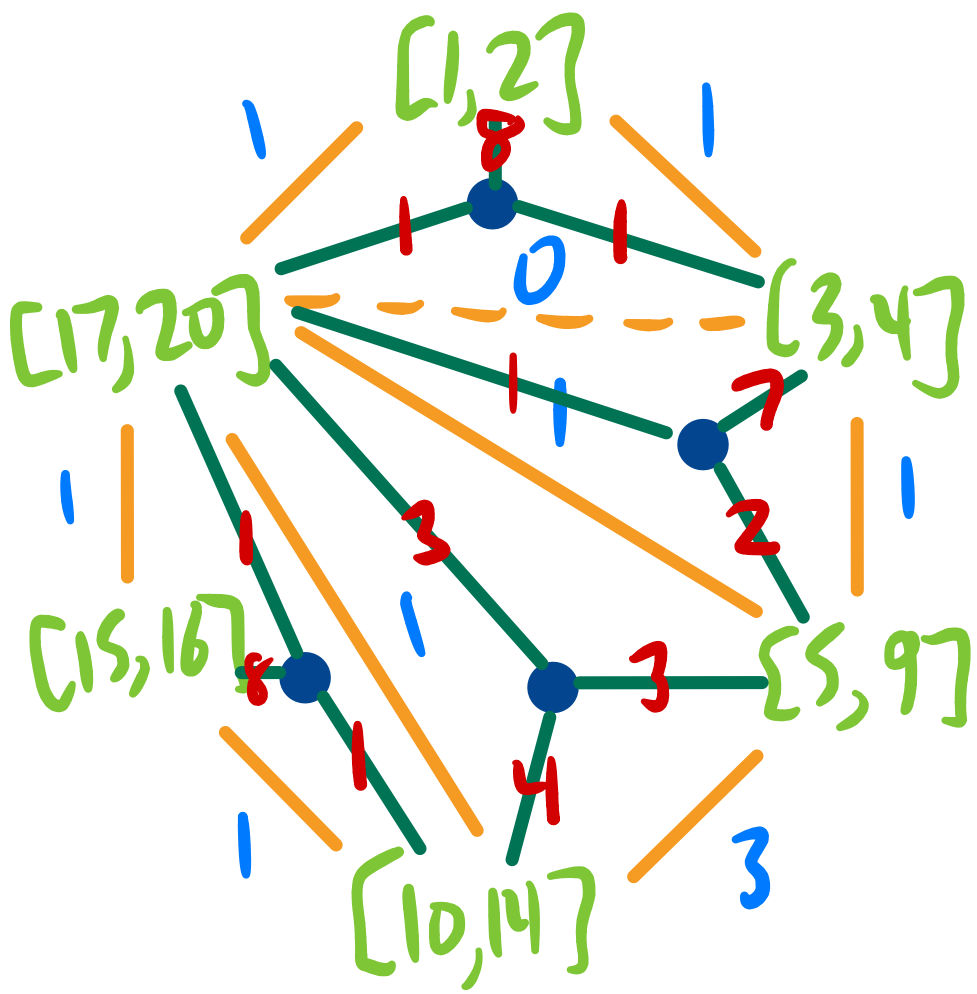


Make $\longmapsto$ bipartite graph

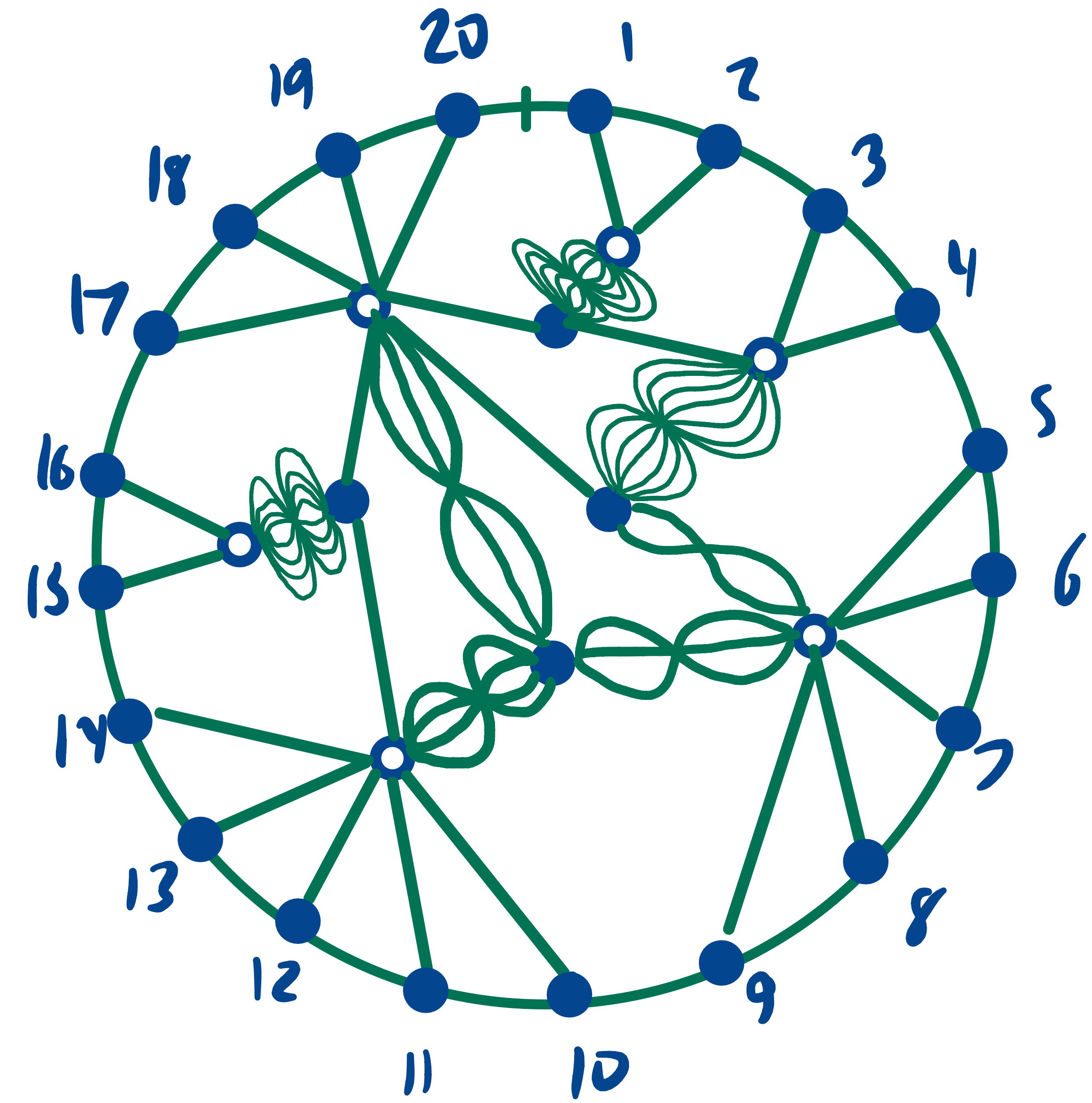


Expand intervals back out, connect them to unfilled vertices, which connect to internal black vertices from triangles

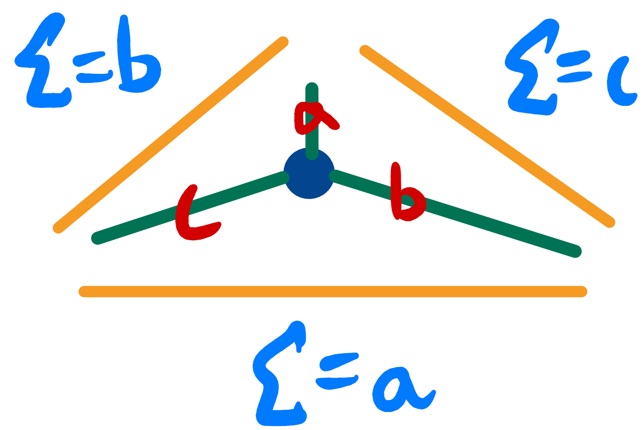
Fraser 2-column webs



Add
hourglass
edges



Edge multiplicities:



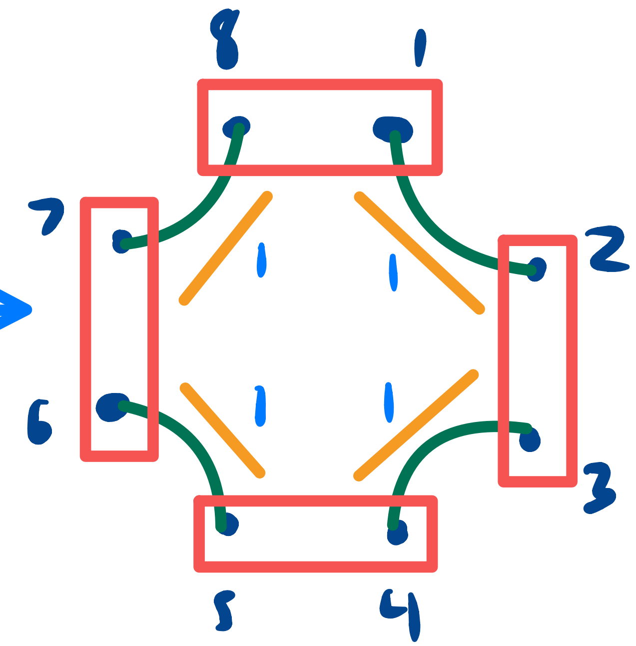
Fraser 2-column webs

Prop (GPPSS '23+) Let $r \geq 1$.
The move-equivalence class associated to the tableau with lattice word $L = 1122\cdots rr$ is in bijection with the Tamari lattice of triangulations of an r-gon.

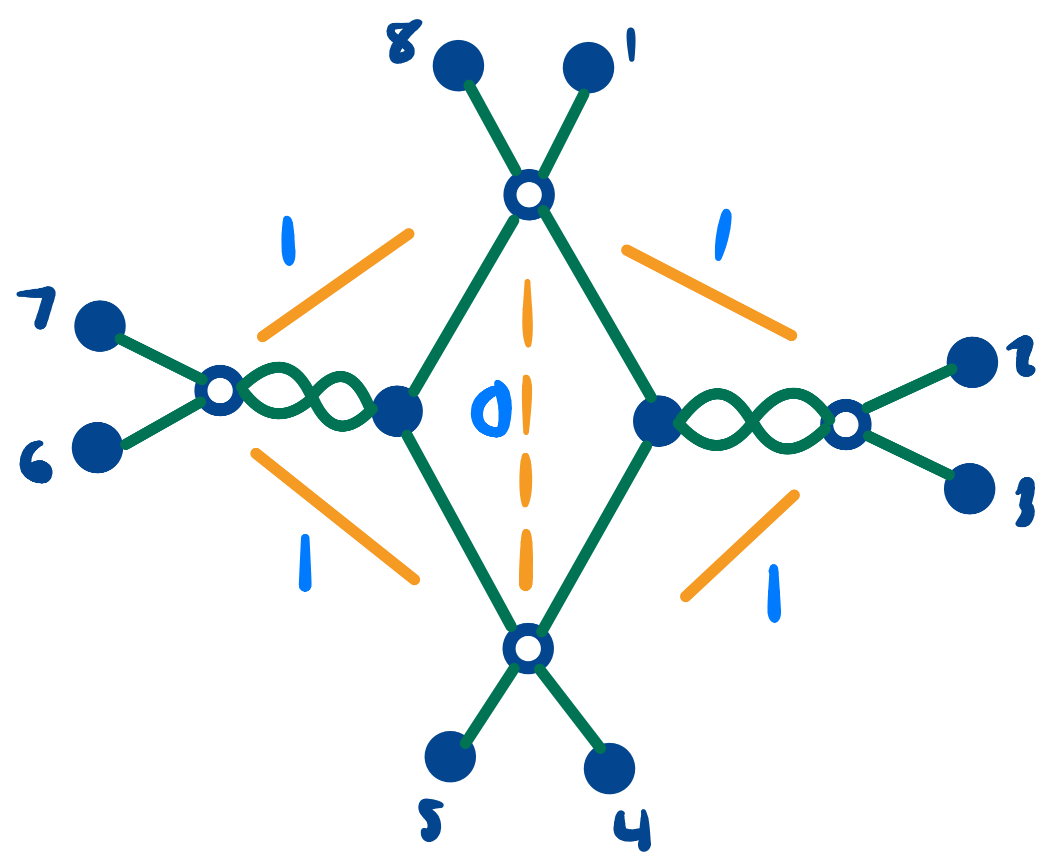
Fraser 2-column webs

Ex

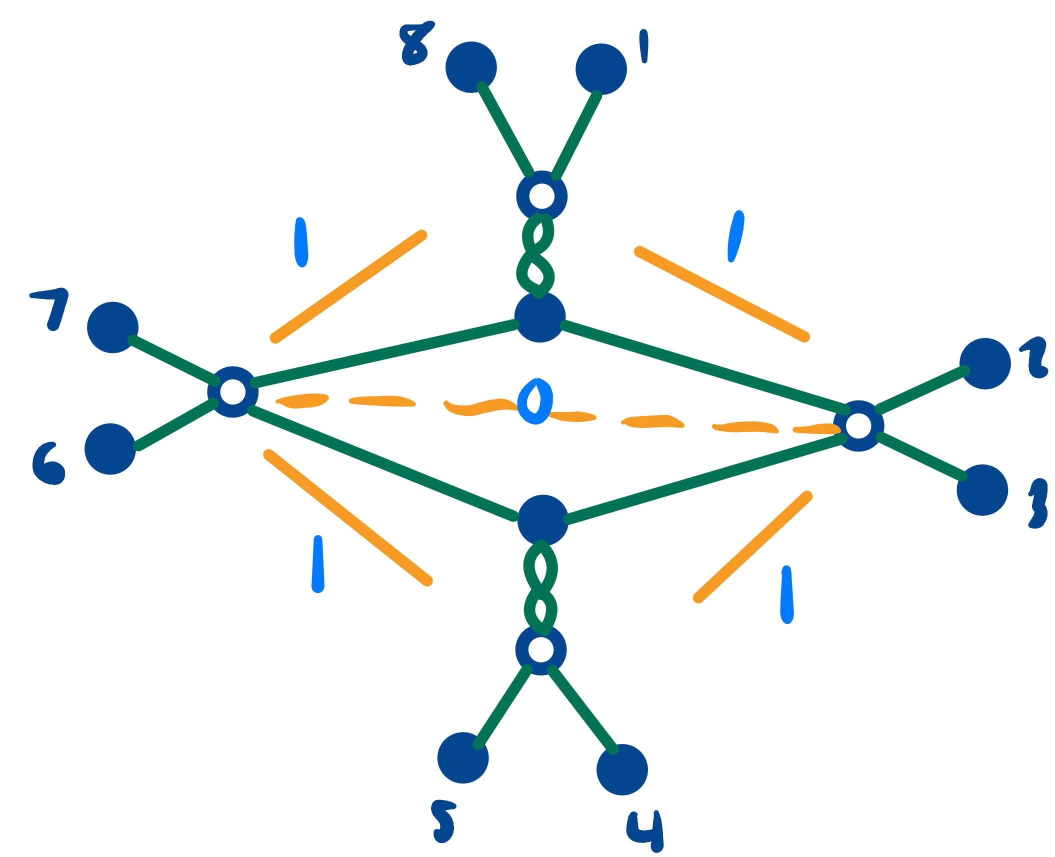
$$T = \begin{array}{cc} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{array} \mapsto ()()()() \mapsto$$



$\mapsto$



or



Pockets

- Fontaine-Kamnitzer-Kuperberg '13 embedded (dual) SL(3) basis webs inside the affine building. Related to geometric Satake correspondence.
- Following a suggestion of Kuperberg, <u>ongoing work</u> of GSS + Haihan Wu '24+ associates SL(4) basis classes with 3D "pockets".
- Beautiful connections to tilings of the Aztec diamond, crystals, and more.

Affine Grassmannians

$\boxed{\text{Def}}$ The affine Grassmannian of $SL(r)^\vee$ is

$$\mathrm{AffGr}_r = PGL_r(\mathbb{C}((t)))/PGL_r(\mathbb{C}[[t]]).$$

$\boxed{\text{Fact}}$ There is a notion of distance d on AffGr_r with values that are dominant $SL(r)$ weights:

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r) \in \mathbb{Z}^r / \langle 1, \dots, 1 \rangle$$

Affine Grassmannians

- Why? The double cosets

$$GL_r(\mathbb{C}[[t]])\backslash GL_r(\mathbb{C}((t)))/GL_r(\mathbb{C}[[t]])$$

have canonical representatives

$$t^\lambda = \begin{pmatrix} t^{\lambda_1} & & & \\ & t^{\lambda_2} & & \\ & & \ddots & \\ & & & t^{\lambda_r} \end{pmatrix}.$$

- Similarly for PGL_r.
- Use $d(p,q)=\lambda \Leftrightarrow p^{-1}q \in H t^\lambda H$.

<u>e.g.</u> $d(p,p)=0 \qquad d(p,q)=d(gp,gq) \qquad d(q,p)=-\mathrm{rev}(d(p,q))$

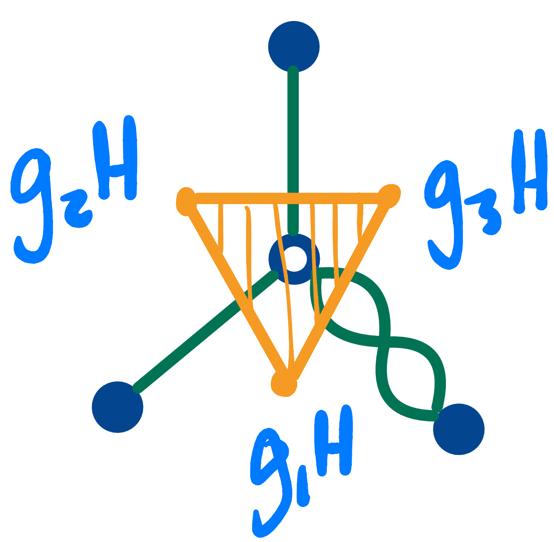
Affine Buildings

- The fundamental weights of $SL(r)$ are $\omega_i = (1^i, 0^{r-i})$
  $\omega_i^* = (0^{r-i}, -1^i) \equiv \omega_{r-i}$.
- Corresponds to $\Lambda^i V$ and $\Lambda^i V^*$.
- Essentially generates $SL(r)$-representation theory (e.g. Karoubi envelope...)

Affine Buildings

Def The affine building on AffGr_r is the simplicial complex Δ_r whose vertices are the points of AffGr_r and whose simplices are collections of points all of whose distances are fundamental weights.

Ex



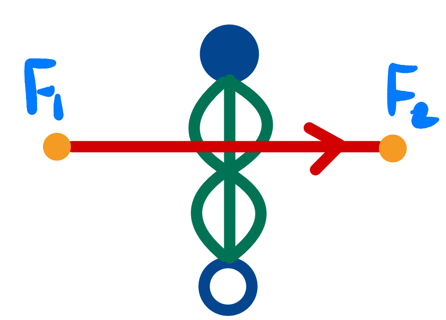
<u>If</u>

$$d(g_1,g_2)=w_1$$
$$d(g_2,g_3)=w_1$$
$$d(g_3,g_1)=w_2$$

<u>then</u> $\{g_1H, g_2H, g_3H\} \in \Delta_4$

Affine Buildings

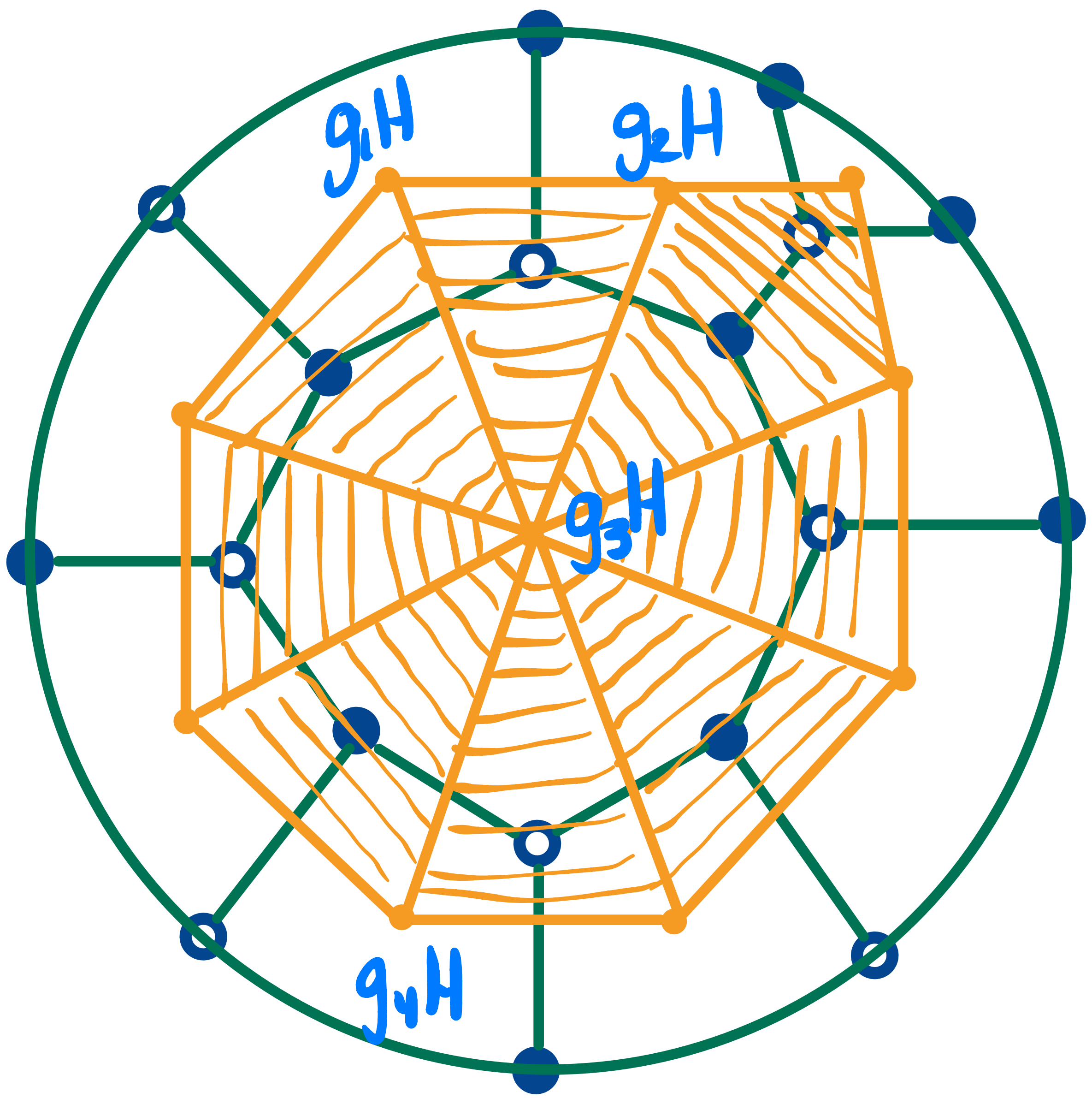
Thm (Fontaine-Kamnitzer-Kuperberg '13) The duals of non-elliptic $SL(3)$ basis webs can be embedded in Δ_3. For faces F_1, F_2, the distance between the corresponding vertices in Δ is the geodesic distance in Δ (or the embedding).



$d(\phi(F_1), \phi(F_2)) = \omega_3$

Affine Buildings

Ex There exist $g_1, g_2, g_3, g_4, \dots$ s.t.



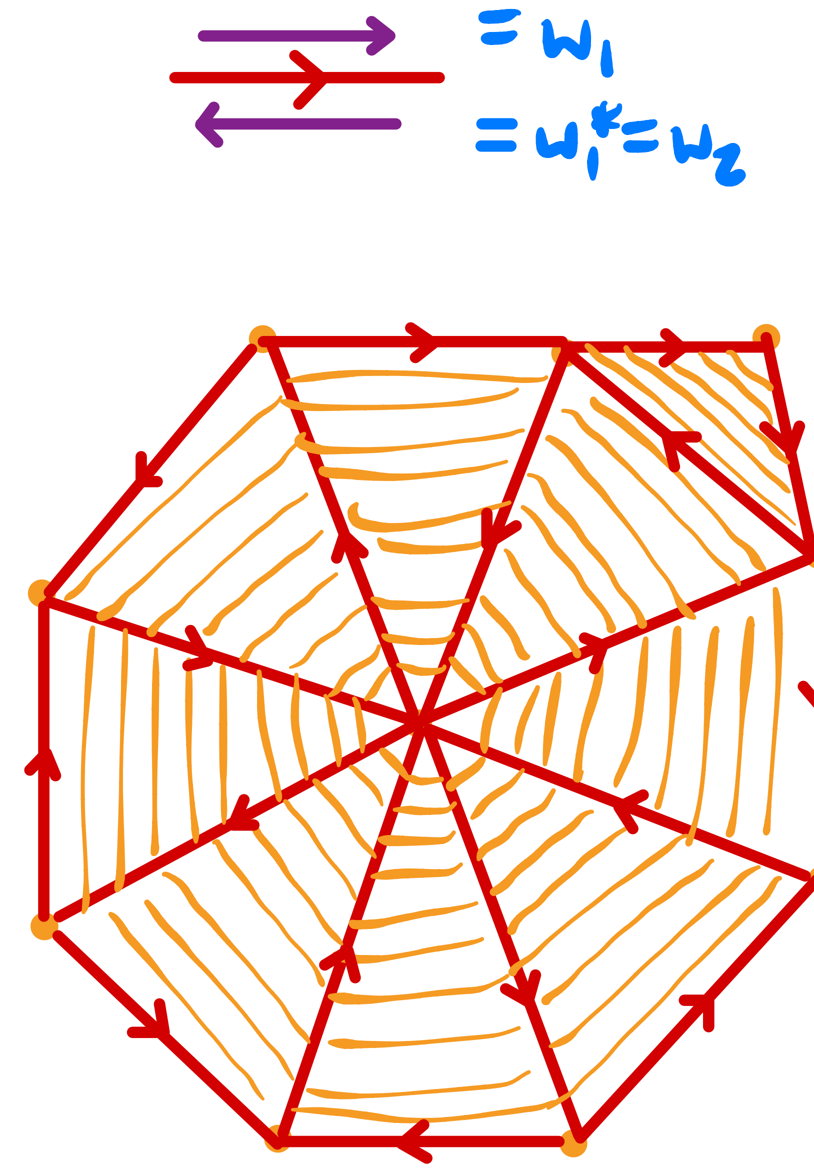
$d(g_1, g_2) = w_1$

$d(g_2, g_3) = w_1$

$\vdots$

$d(g_4, g_1) = 2w_1$

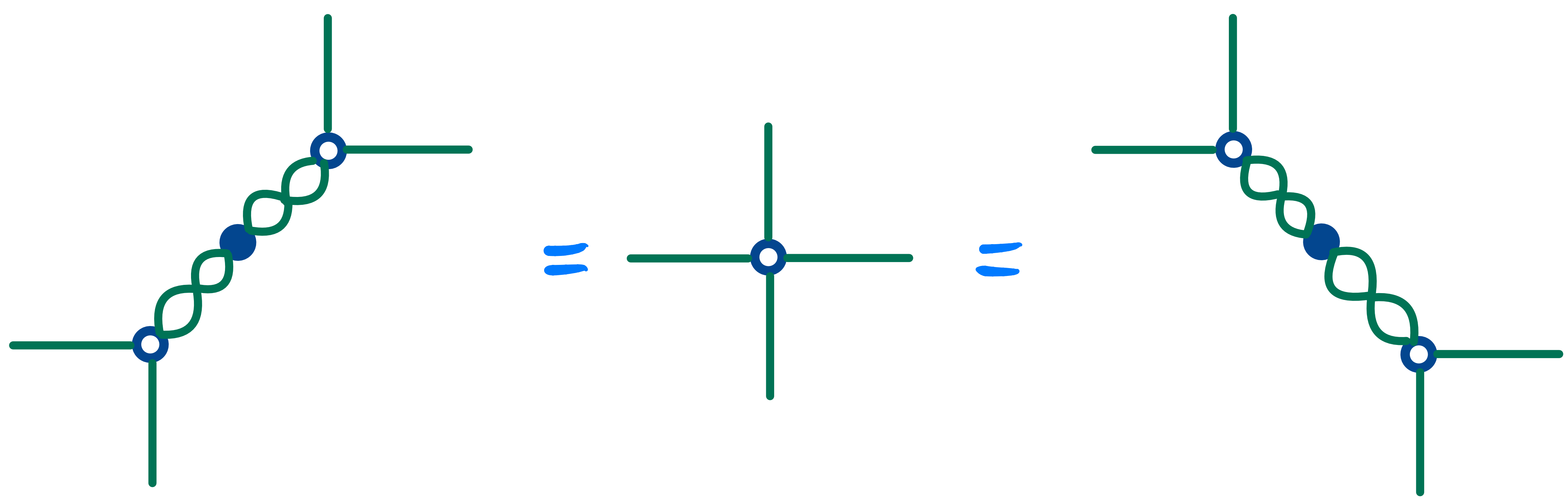
$\vdots$



in Δ_3

Pockets

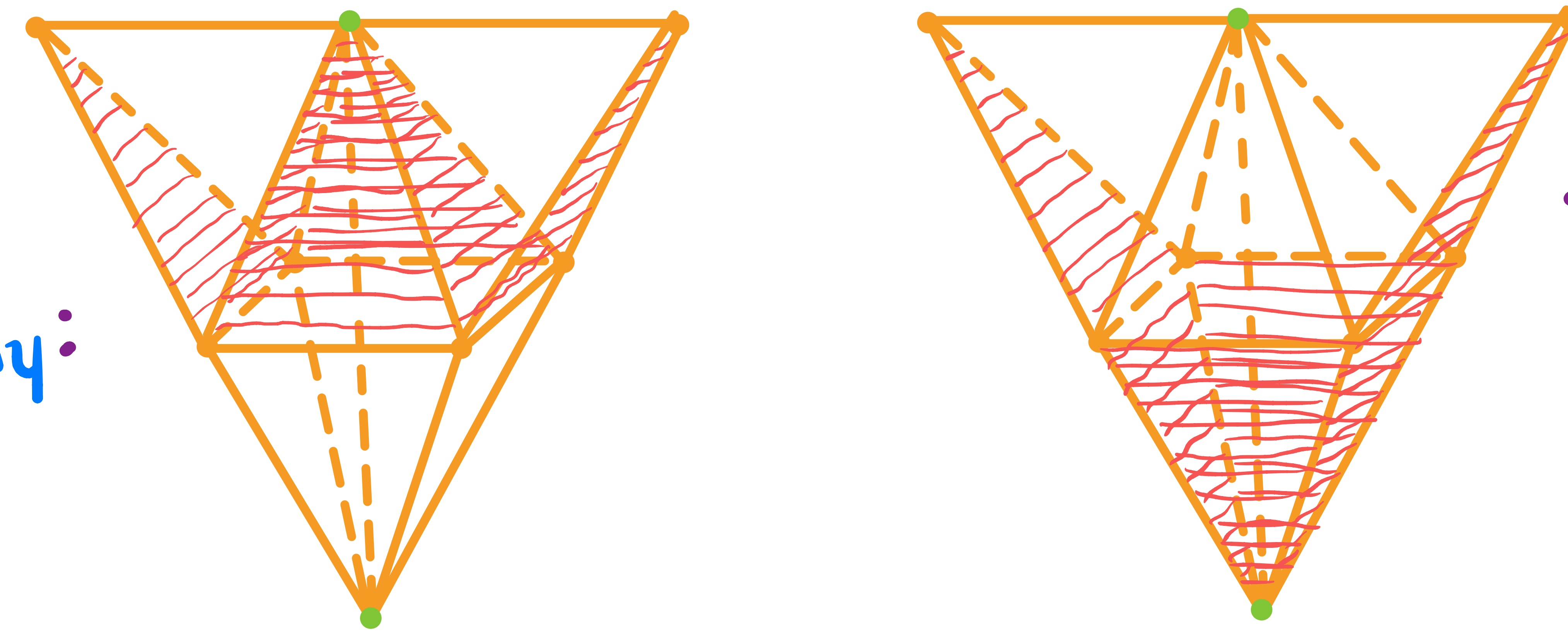
- We (GSS+Wu) show there is a 3D analogue for the SL(4) web equivalence classes: pockets in Δ_4.
- Requires expanding sources/sinks via I=H moves:



Pockets

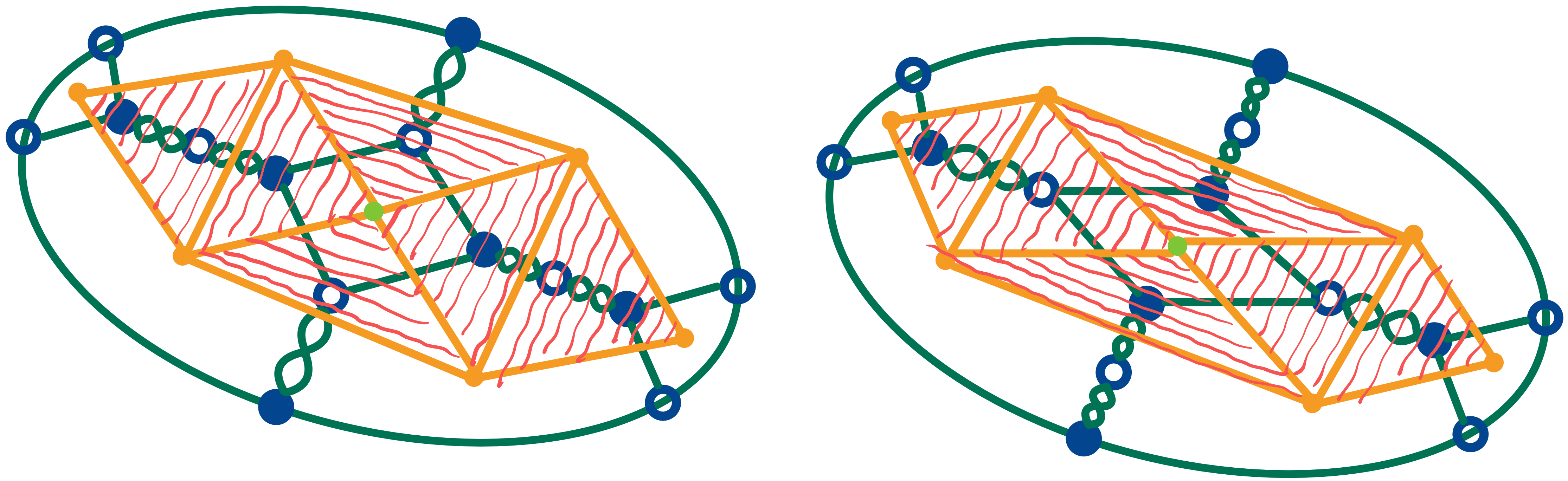
Ex

In Δ_4:



Pocket of square + benzene + IH class of

$\bar{4}\{1,2\}\bar{3}\,\bar{2}\{3,4\}\bar{1}$

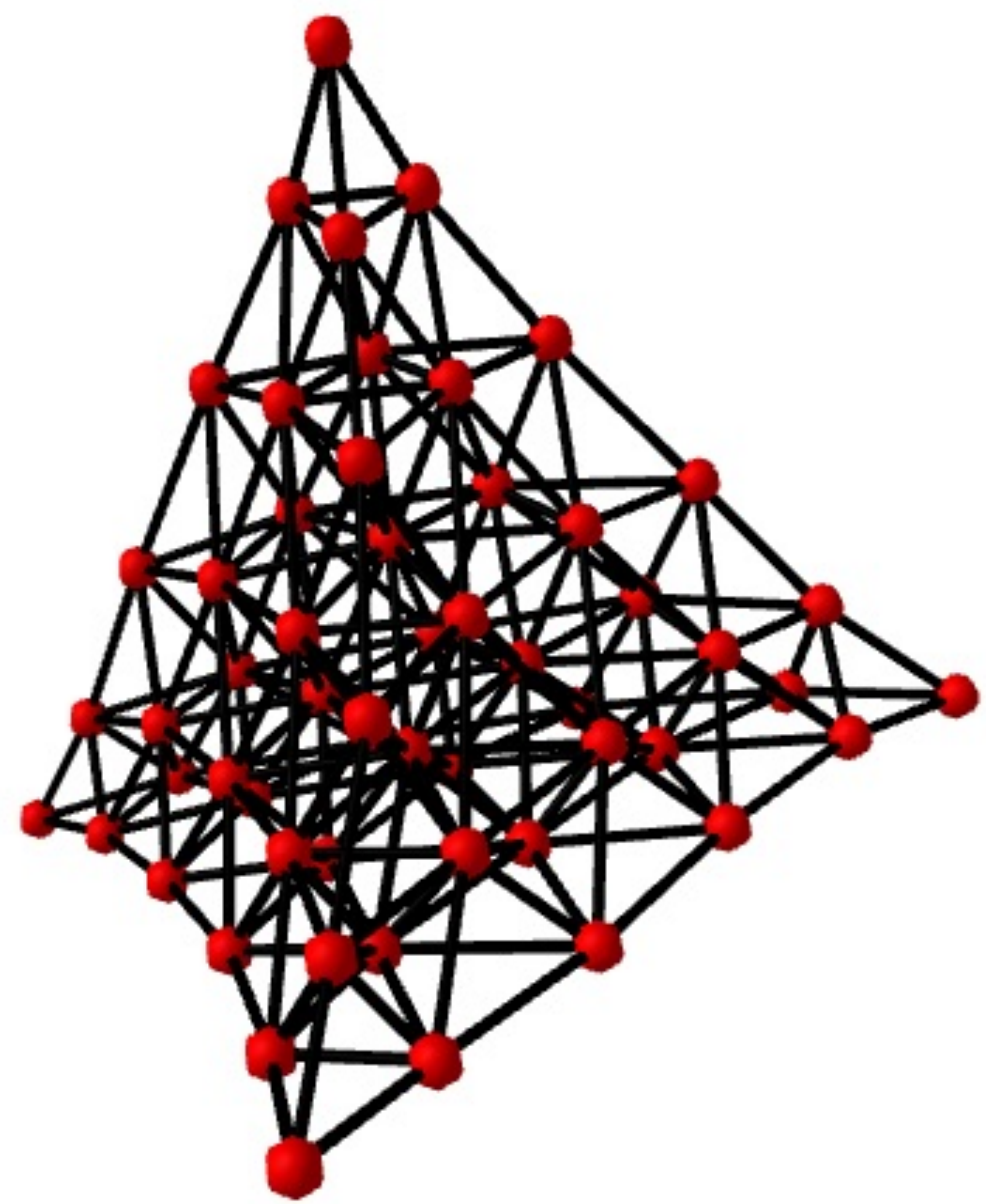


+6 more

Pockets

Ex Pocket of 5×5 ASM:

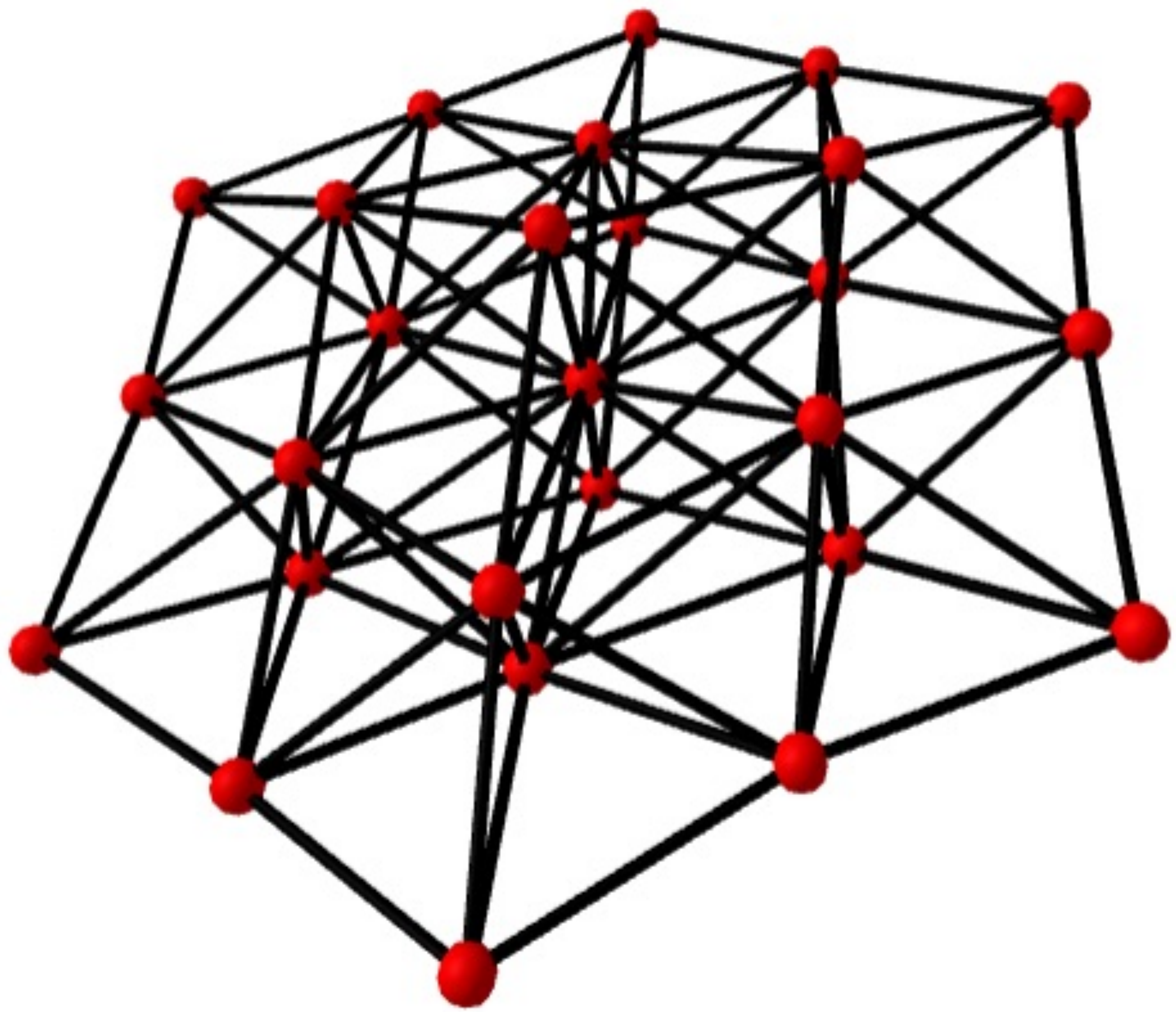
$L = 1^5 2^5 3^5 4^5$



Note Related to height functions, octahedral recurrence, tilings of the Aztec diamond, distributive lattices

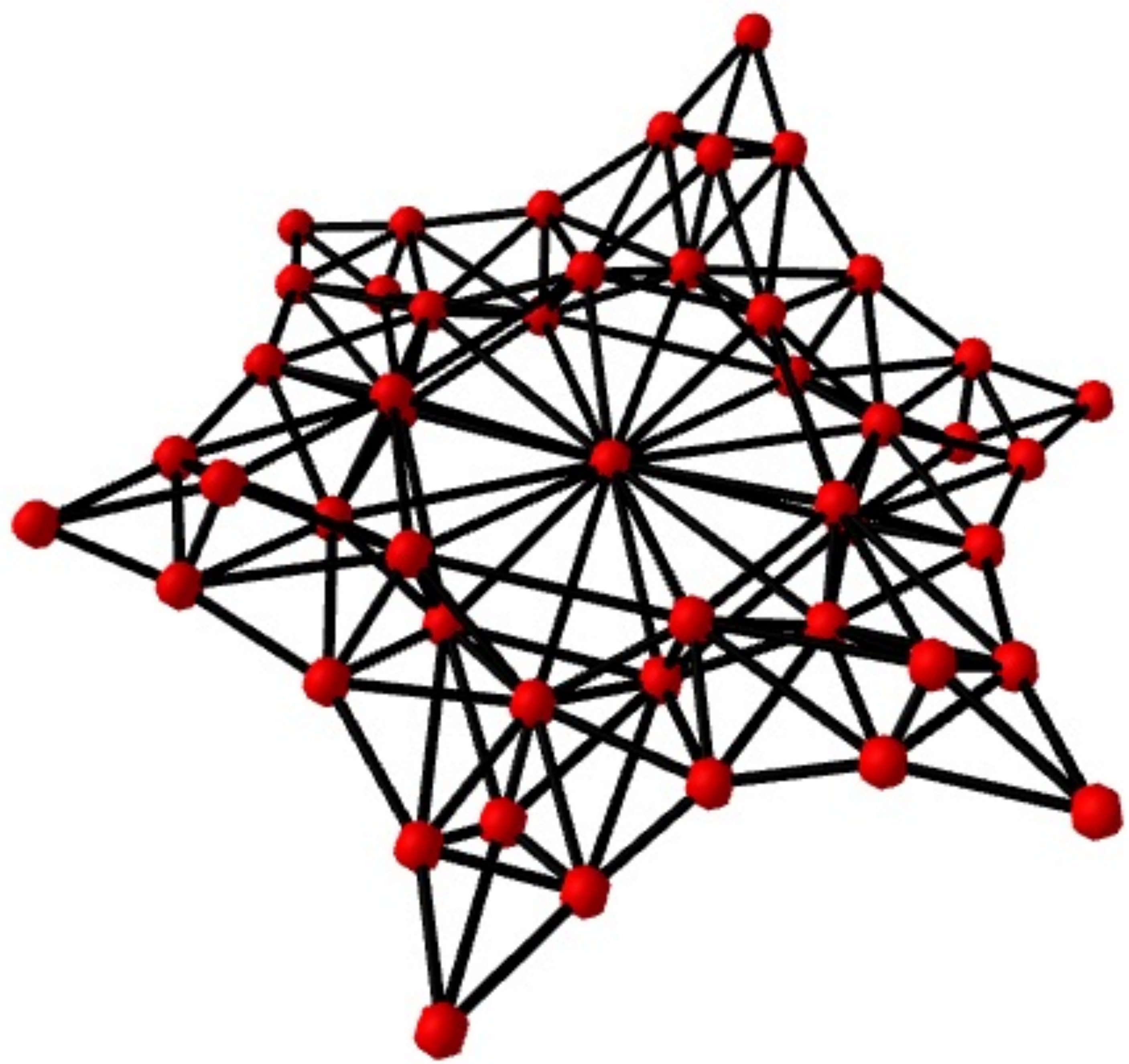
Pockets

Ex Pocket of $2 \times 2 \times 2$ PP:



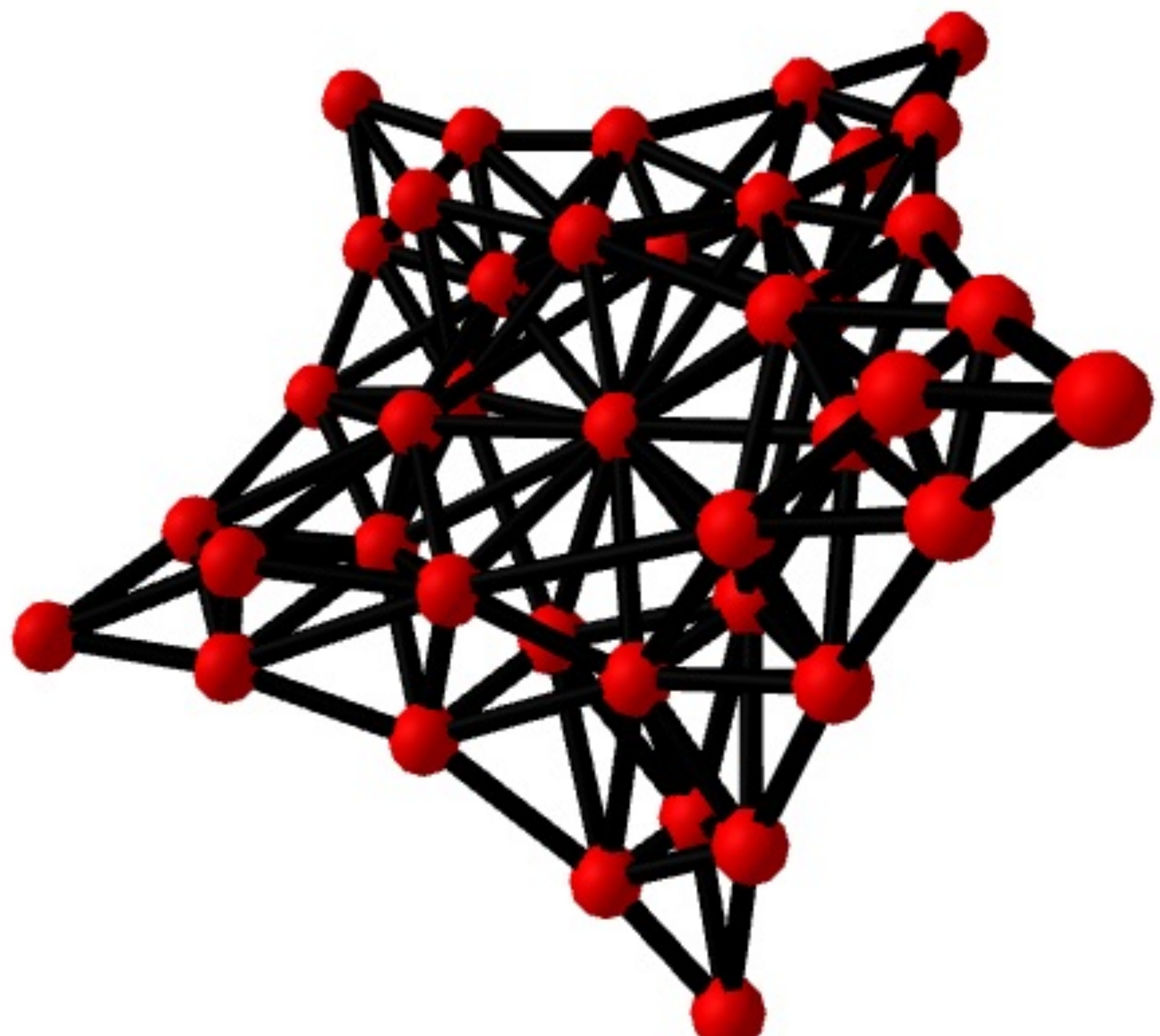
Pockets

Ex Pocket of a "chained hexagon":



Pockets

Ex Pocket of a "chained pentagon":



Note Not realizable in $\mathbb{R}^3$

THANKS!