An SL(4) web basis from hourglass plabic graphs

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Based on joint work with *Christian Gaetz, Oliver Pechenik, Stephan Pfannerer, and Jessica Striker* (submitted)

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Slides: https://www.jpswanson.org/talks/2024_Berkeley_webs.pdf

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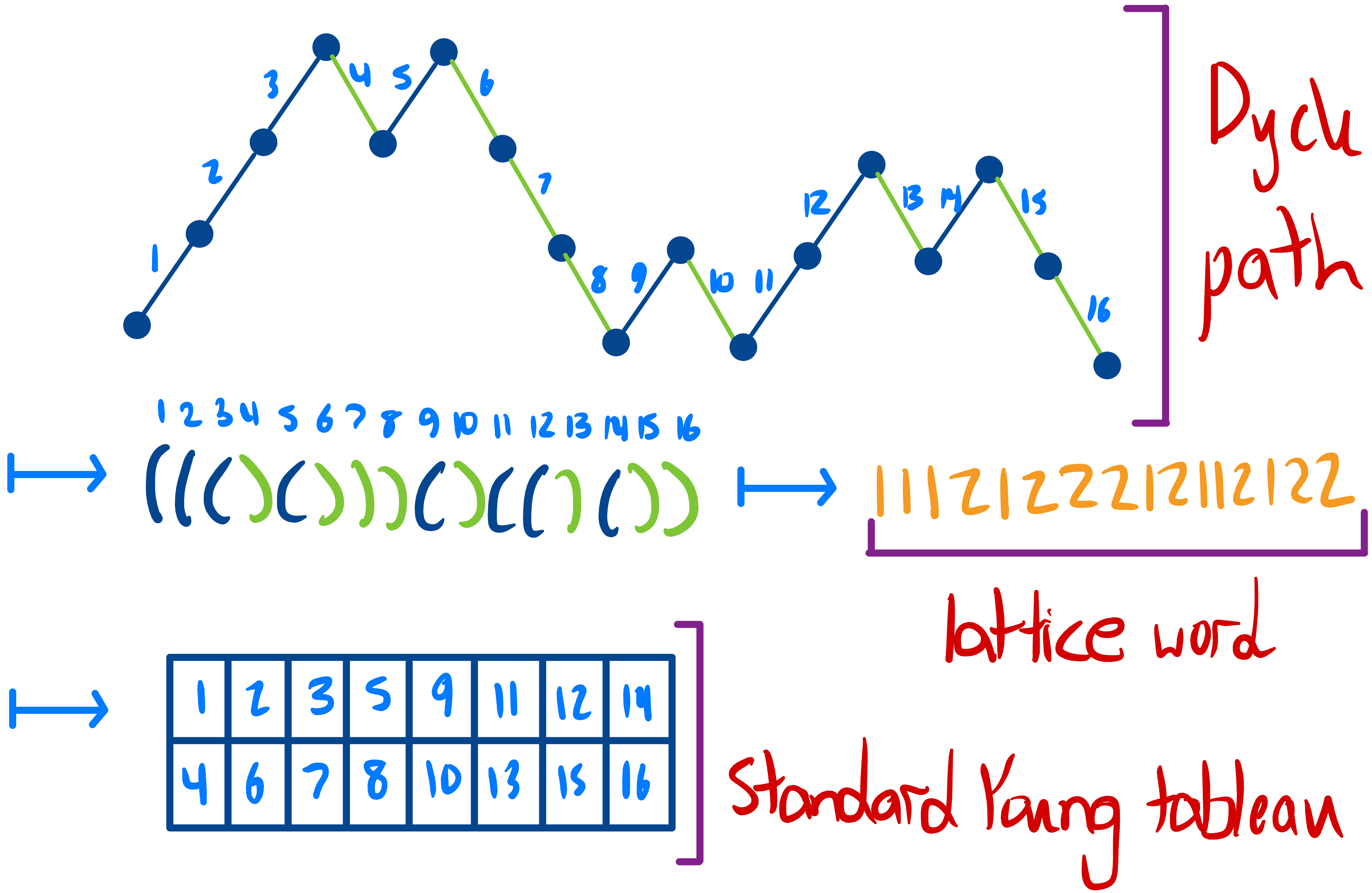
Outline

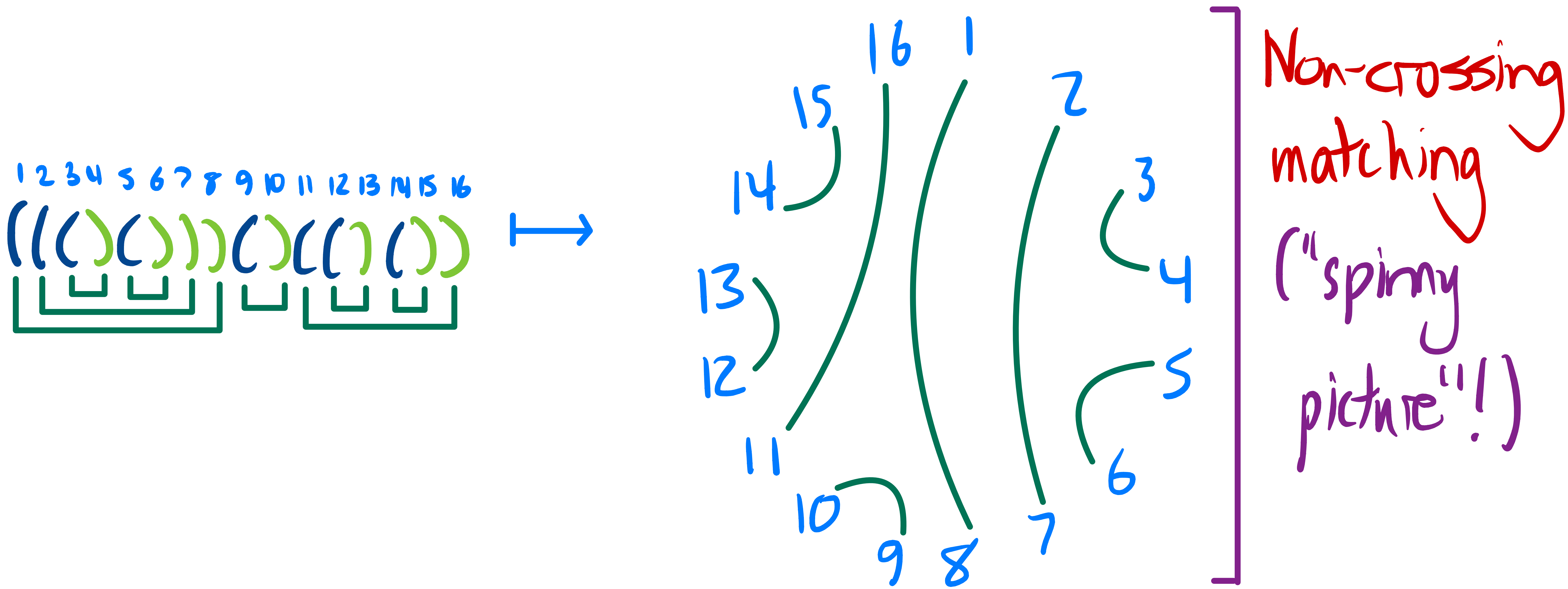
- Catalan combinatorics, 2-row webs
- 3-row web bases, plabic graphs, growth rules
- (New!) 4-row web bases, hourglass plabic graphs

Catalan objects

Some Catalan bijections:



Catalan objects



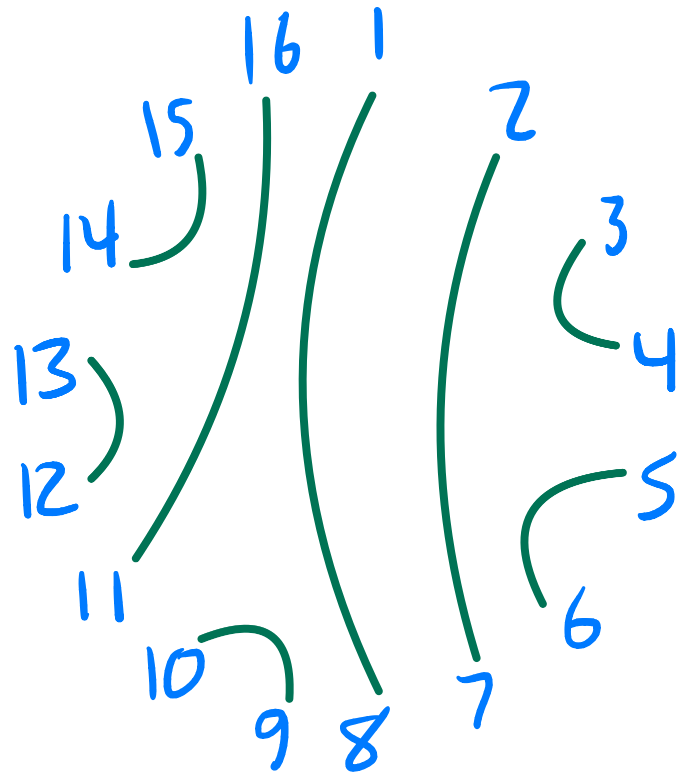
Catalan objects

Thm The bijection $NCM(2n) \xrightarrow{\sim} SYT(2 \times n)$ sends rotation to promotion, reflection to evacuation.

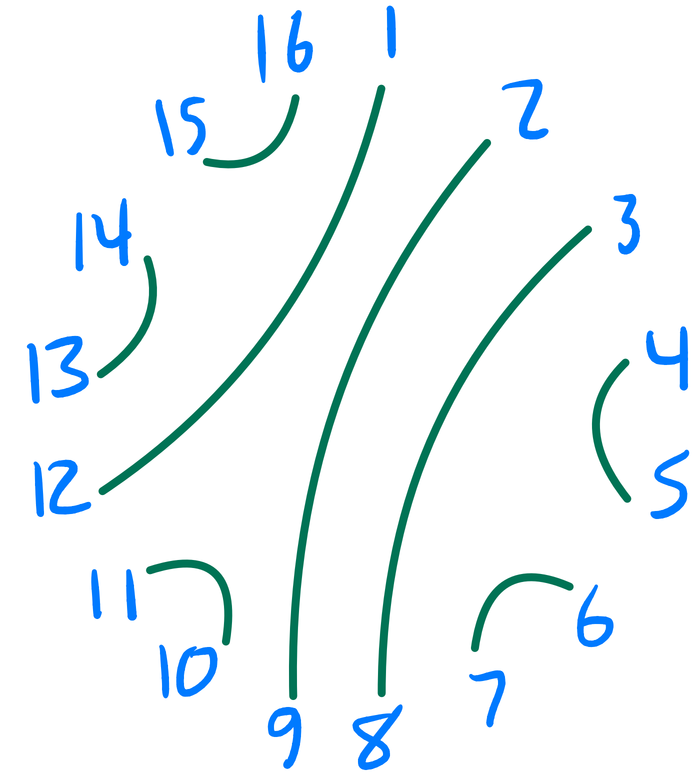
- "Hidden" dihedral action on $SYT(2 \times n)$!

Catalan objects

Ex



Rotate ⟼



↧

1	2	3	5	9	11	12	14
4	6	7	8	10	13	15	16

Promote ⟼

1	2	3	4	6	10	13	15
5	7	8	9	11	12	14	16

SL_2-Invariants

Let $V = \mathbb{C}^2$, $V_i \in \{V, V^*\}$.

Q What are the SL_2-invariants of $V_1 \otimes \cdots \otimes V_n$?

That is, identify $\mathrm{Hom}_{SL_2}(V_1 \otimes \cdots \otimes V_n, \mathbb{C}) \subset \mathbb{C}[x_{ij}; y_{kl}]$.

Ex $V \otimes V$: $\det\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = x_{11}x_{22} - x_{12}x_{21}$ is unique invariant

$\begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix} \begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix}$ $\quad \det(g \cdot X) = \det(g)\det(X) = \det(X)$

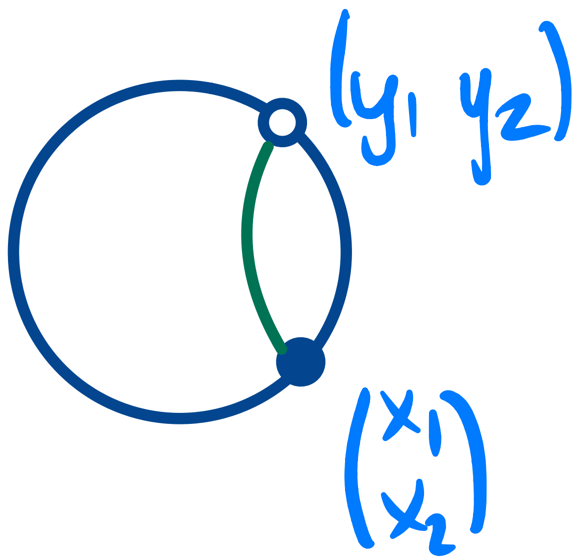
Ex $V^* \otimes V$: pairing $\langle y, x \rangle = x_1 y_1 + x_2 y_2$ is invariant

$(y_1, y_2) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $\quad \langle y g^{-1}, g x \rangle = \langle y, x \rangle$

SL_2-Tensor diagrams

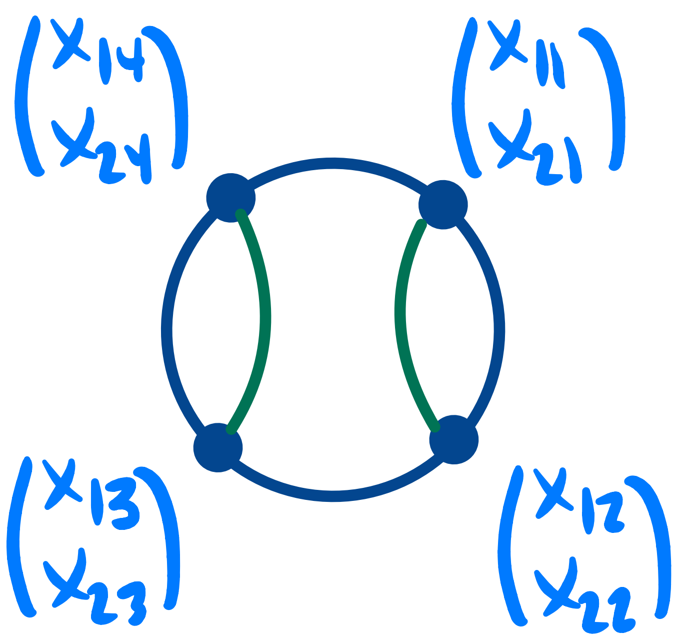
- Encode morphisms/invariants in diagrams: $V \longleftrightarrow \bullet$
  $V^* \longleftrightarrow \circ$

Ex



$$= x_1 y_1 + x_2 y_2 \in \mathrm{Inv}(V^* \otimes V)$$

Ex

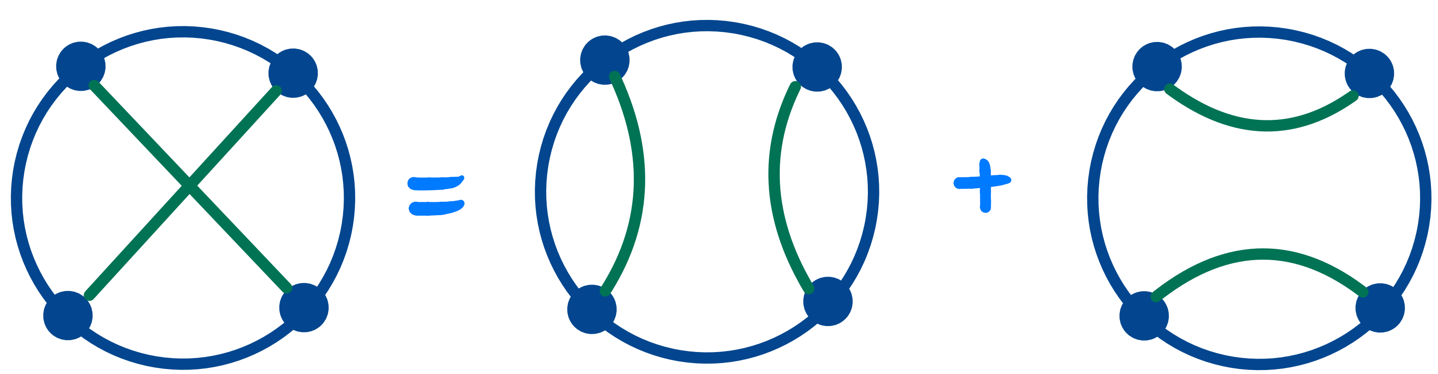


$$= \det\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \cdot \det\begin{pmatrix} x_{13} & x_{14} \\ x_{23} & x_{24} \end{pmatrix} \in \mathrm{Inv}(V^{\otimes 4})$$

SL_2-Tensor diagrams

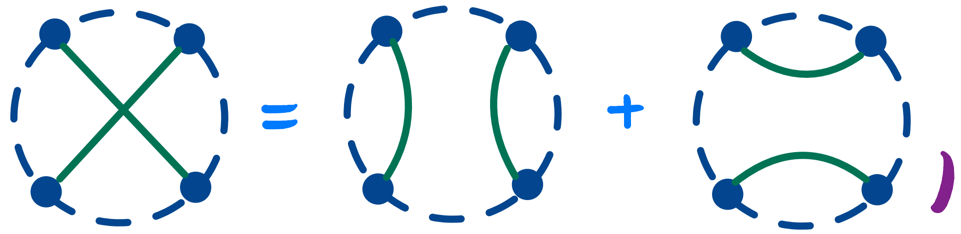
- Encode morphisms/invariants in diagrams: $V \longleftrightarrow \bullet$, $V^* \longleftrightarrow \circ$

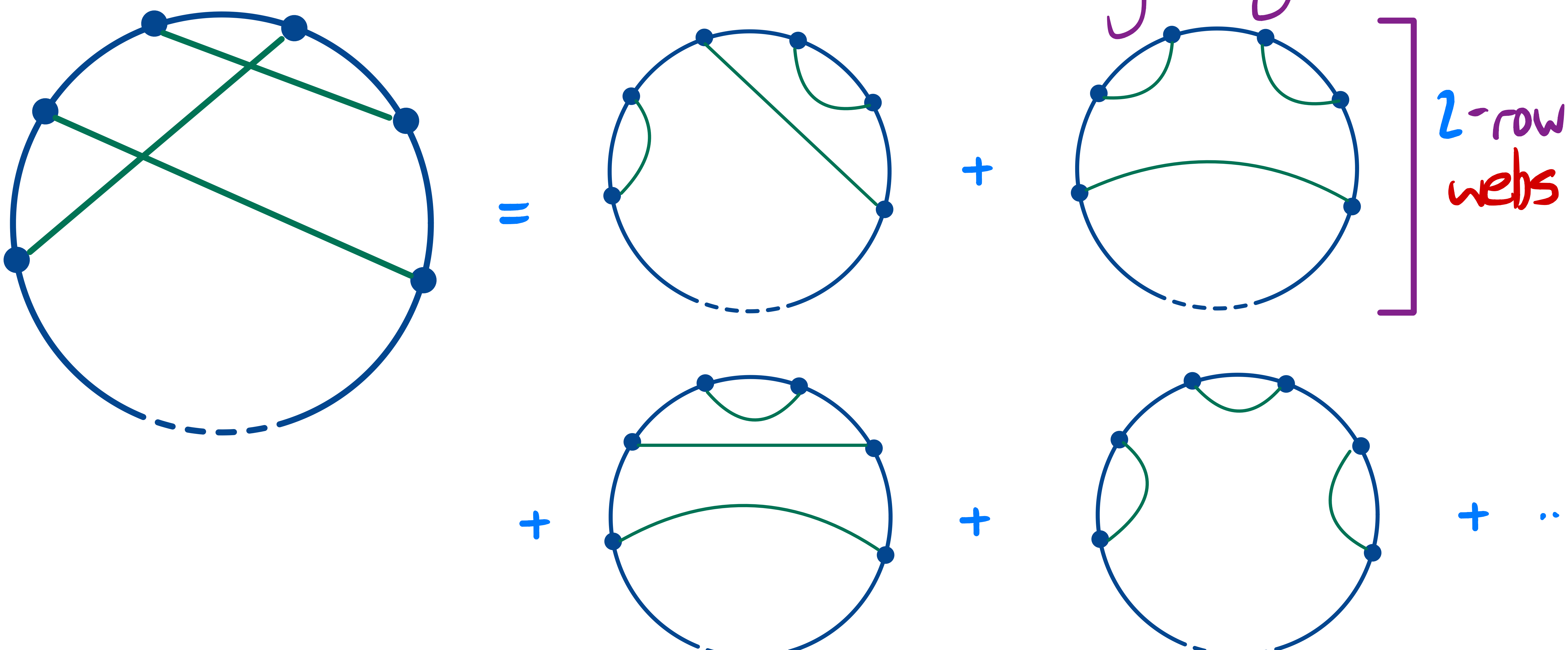
Ex Plücker relations:



$$(x_{11}x_{23} - x_{21}x_{13})(x_{12}x_{24} - x_{22}x_{14})$$
$$=$$
$$(x_{11}x_{22} - x_{21}x_{12})(x_{13}x_{24} - x_{23}x_{14})$$
$$+$$
$$(x_{11}x_{24} - x_{21}x_{14})(x_{12}x_{23} - x_{22}x_{13})$$

Temperly-Lieb basis

- Using , can reduce any matching diagram to a linear combination of matching diagrams:



Temperley-Lieb basis

Thm The noncrossing 2-row webs are a basis for

$$\mathrm{Inv}_{SL_2}(V_1 \otimes \cdots \otimes V_n) \qquad (V_i \in \{V, V^*\})$$

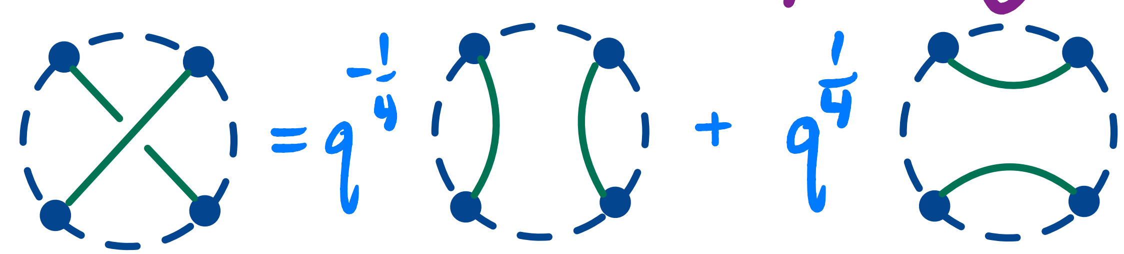
Pf
- **Spanning:** diagrams span by classical invariant theory, noncrossing by uncrossing rule.
- **Independence:** by Pieri rule,

$$\dim \mathrm{Inv}_{SL_2}(V^n) = \#SYT(2 \times \tfrac{n}{2})$$

Quantum Link Invariants

The quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$ is an algebra deforming the (universal enveloping algebra of) $\mathfrak{sl}_2$.

- Sending $q \to 1$ recovers the usual case.
- $\mathrm{Inv}(V_1 \otimes \cdots \otimes V_n)$ still has a Temperly-Lieb web basis:



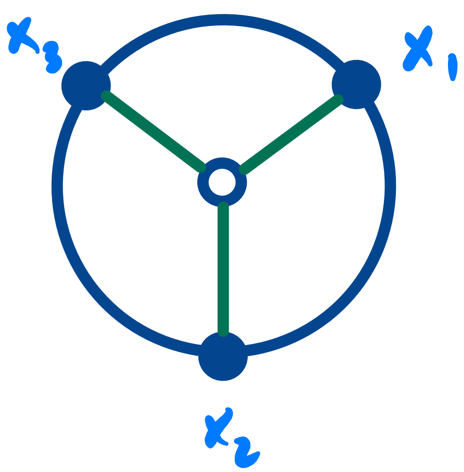
- Projections of knots/links/tangles become polynomials in q; refined version of Jones polynomial

SL_3-Tensor diagrams

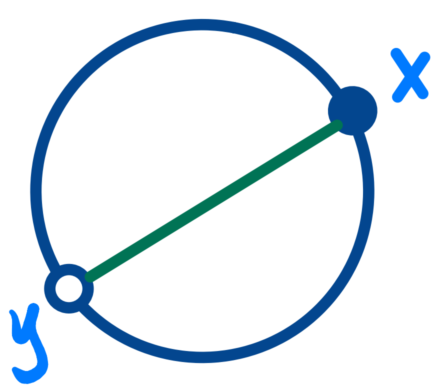
- Let $V = \mathbb{C}^3$, $V_i \in \{V, V^*\}$.

Q Is there a web basis for $\mathrm{Inv}_{SL_3}(V_1 \otimes \cdots \otimes V_n)$?

Ex



$$= \det\begin{pmatrix} | & | & | \\ x_1 & x_2 & x_3 \\ | & | & | \end{pmatrix}$$



$$= \langle y, x \rangle$$

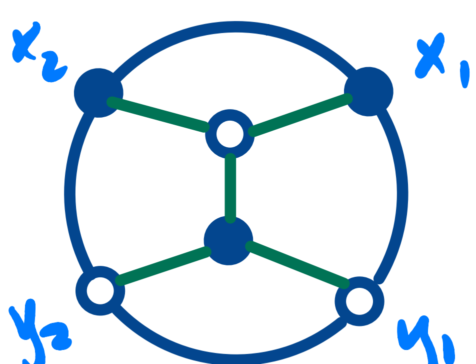
There's more! →

SL_3-Tensor diagrams

- Let $V = \mathbb{C}^3$, $V_i \in \{V, V^*\}$.

Q Is there a web basis for $\mathrm{Inv}_{SL_3}(V_1 \otimes \cdots \otimes V_n)$?

Note $\Lambda^2 V \cong V^*$ since $\Lambda^3 V = \det V = 1$ over SL_3.
$\Lambda^2 V^* \cong V$

Ex 

$= \det\begin{pmatrix} | & | & | \\ x_1 & x_2 & y_1 \wedge y_2 \\ | & | & | \end{pmatrix}$

SL_3-Tensor diagrams

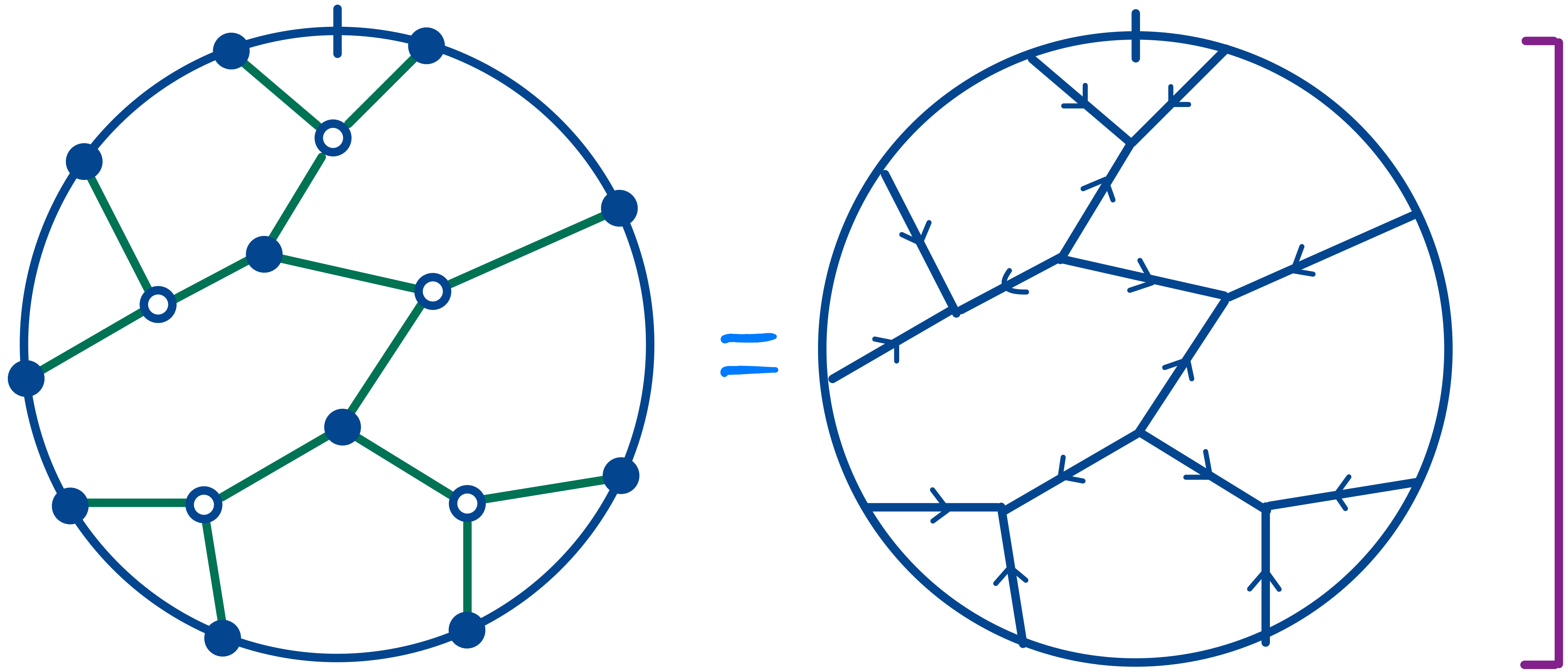
Def 3-row webs are

- planar graphs embedded in a disk
- trivalent interior vertices, univalent boundary vertices
- bipartite
- marked "initial" boundary vertex

They encode $\mathrm{Inv}_{SL_3}(V_1 \otimes \cdots \otimes V_n)$.

SL_3-Tensor diagrams

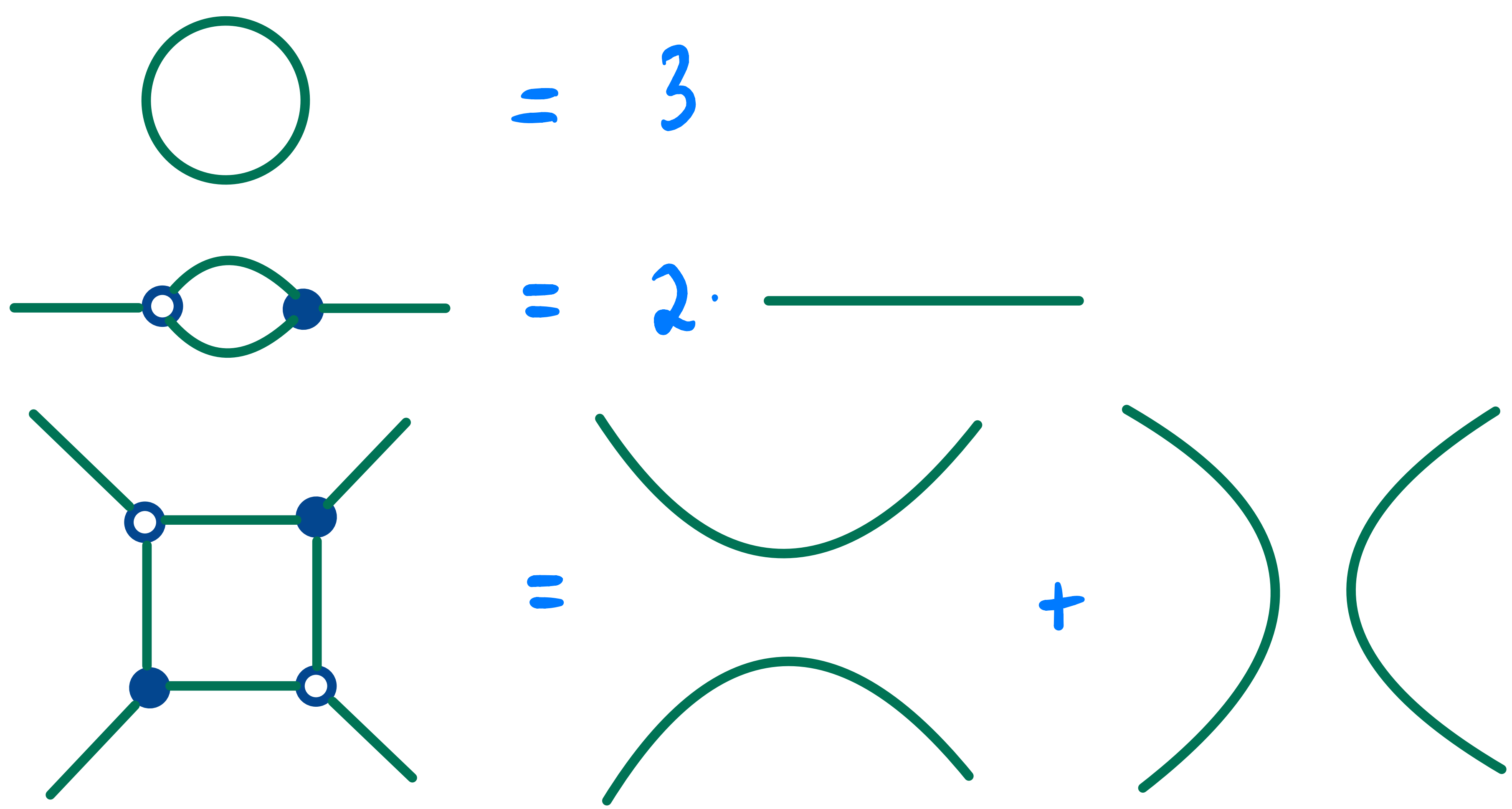
Ex



"Spinny picture"!

SL_3-Web basis

Thm (Kuperberg) The generating SL_3-web relations are



SL_3-Web basis

Thm (Kuperberg; Kuperberg-Khovanov)
Call an SL_3-web non-elliptic if it is connected and it has no internal 2-faces or 4-faces.
The non-elliptic webs form a basis of

$$\mathrm{Inv}_{SL_3}(V_1 \otimes \cdots \otimes V_n). \qquad (V_i \in \{V, V^*\})$$

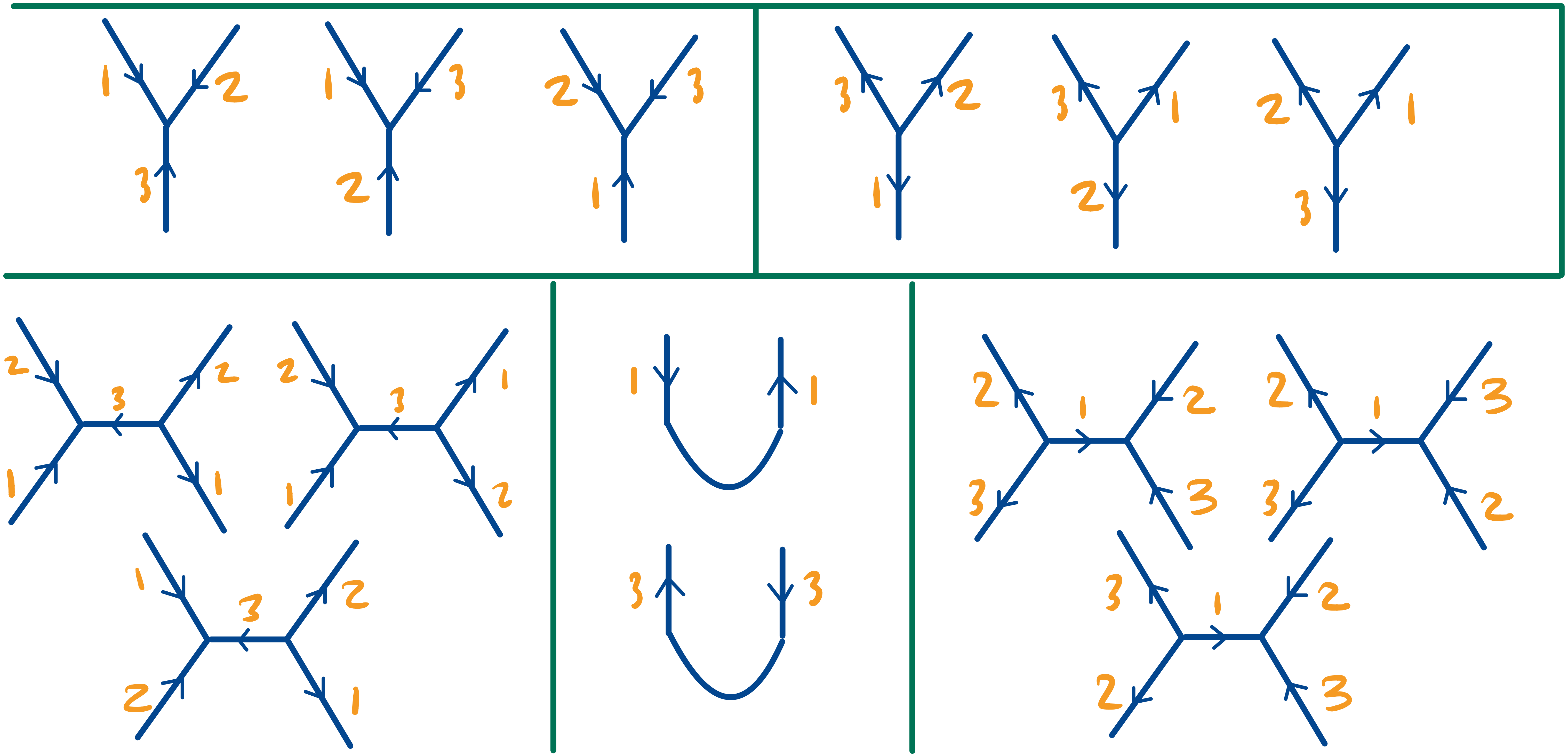
SL_3-Web basis

Pf] • **Spanning:** similar to SL_2 case.

• **Independence:** bijection to $SYT(3 \times \frac{n}{3})$ using growth rules. (Other descriptions of this bijection have since been found.)

SL_3-Web basis

Kuperberg-Khovanov growth rules:



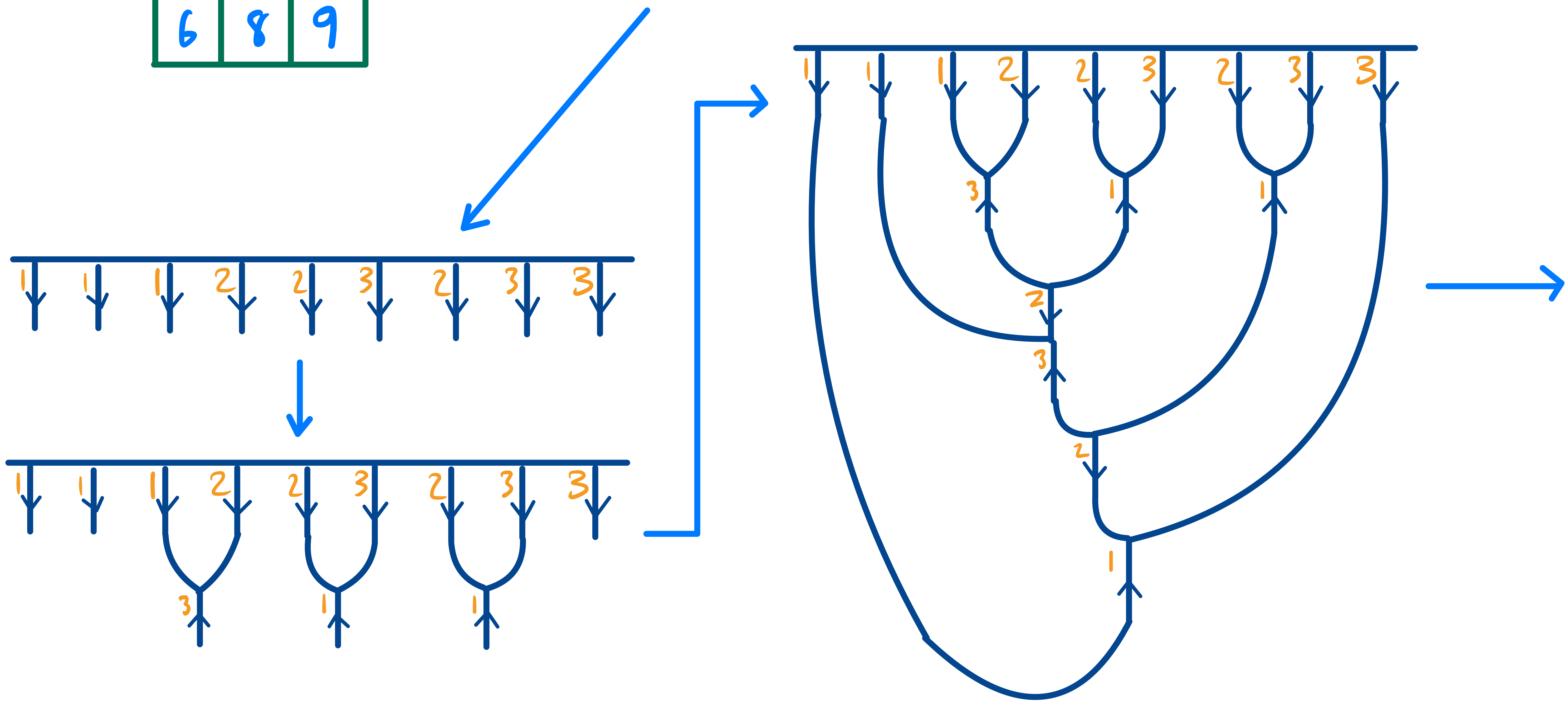
SL_3-Web basis

Ex

$T =$

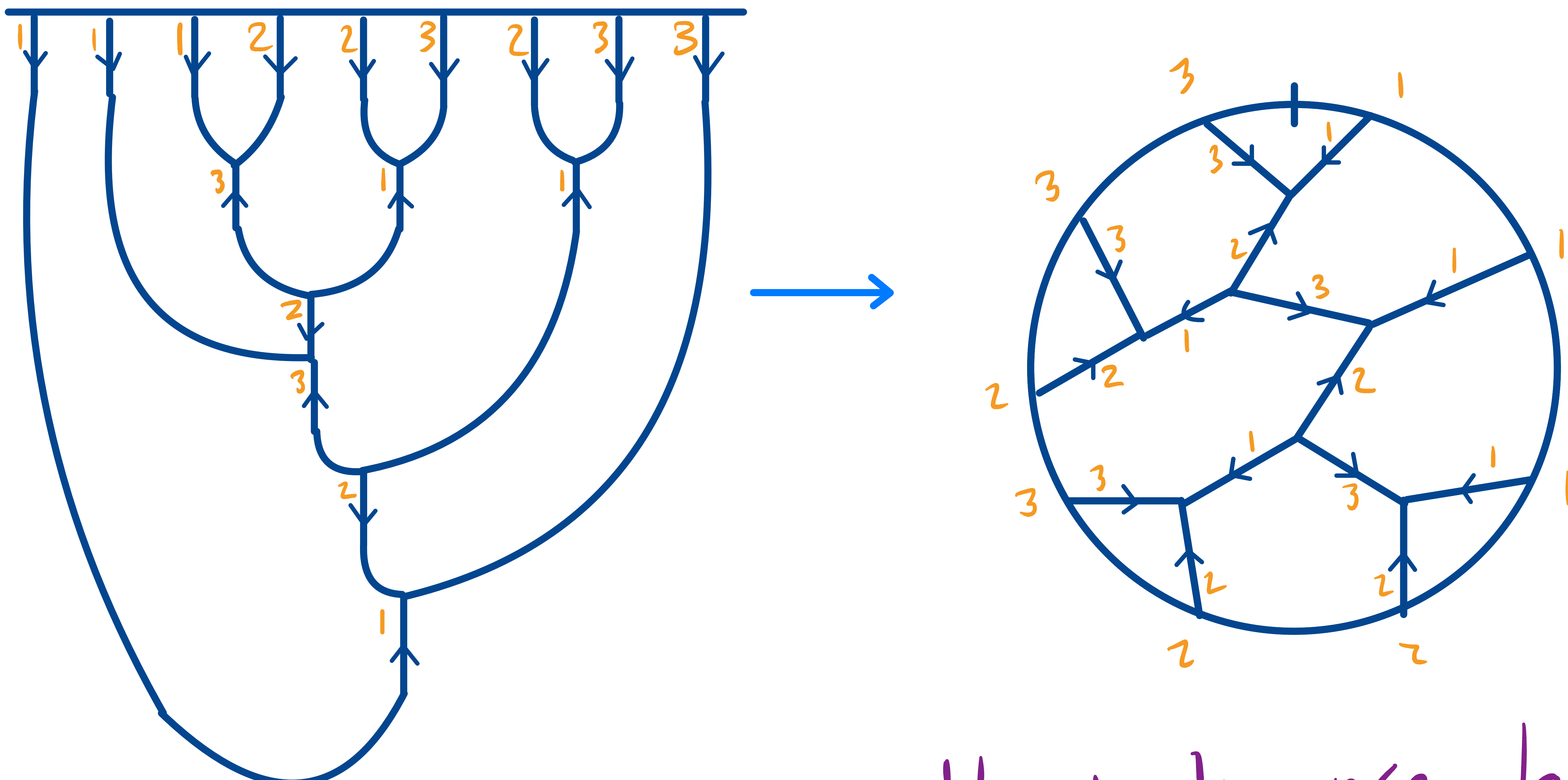
1	2	3
4	5	7
6	8	9

$\longrightarrow$ 111223233



SL_3-Web basis

$(T \to 111223233)$



Now just erase labels!

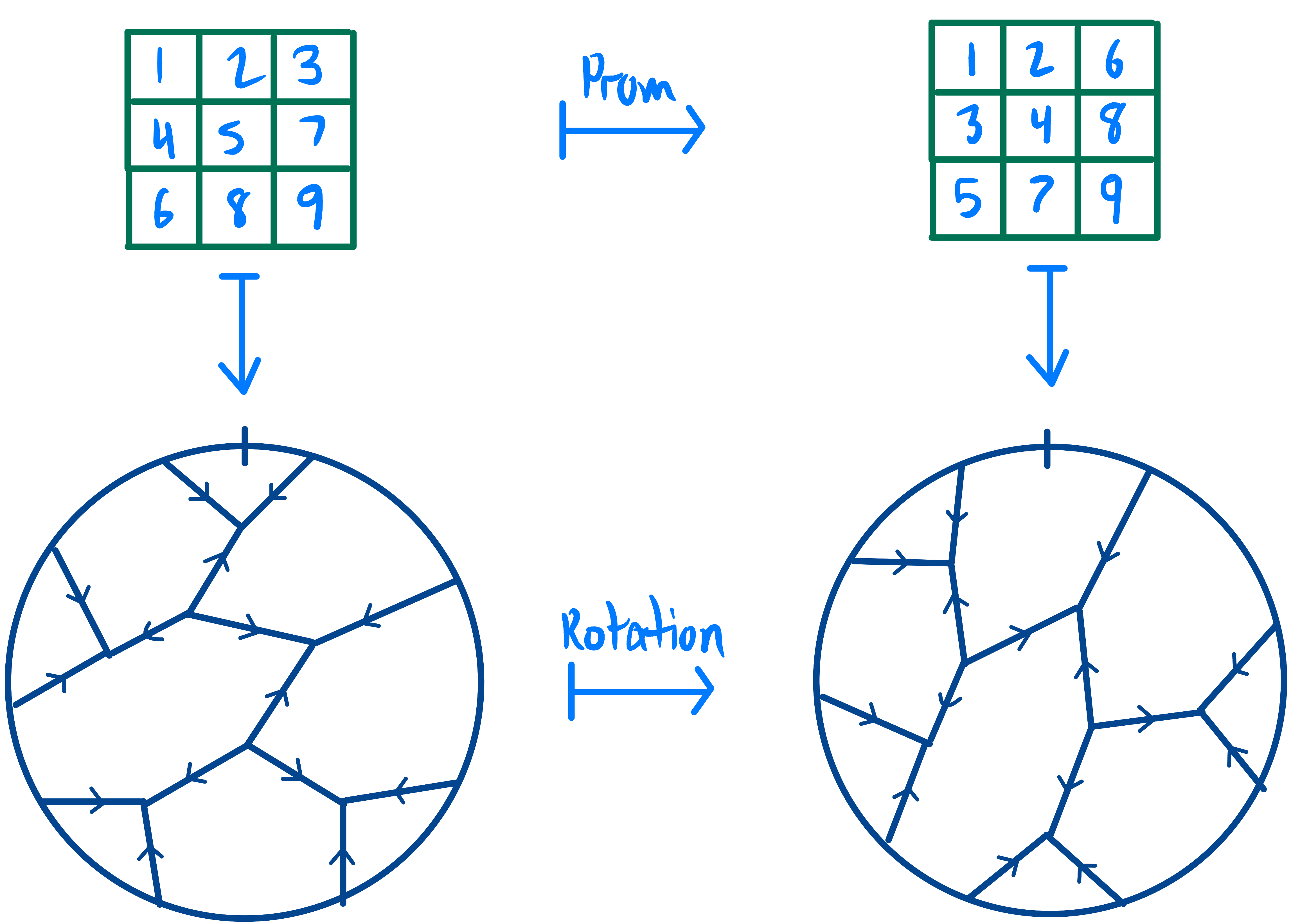
SL_3-Web basis

Thm (Petersen-Pylyavskyy-Rhoades)
The bijection from non-elliptic webs to $SYT(3\times\frac{n}{3})$ sends rotation to promotion
reflection to evacuation.

- "Hidden" dihedral action on $SYT(3\times m)$!

SL_3-Web basis

Ex



SL_3-Web basis

Applications

- Quantum link invariants
  - SL_3 link polynomials, foams, ...
- Cluster algebras
  - Cluster structures on $\mathbb{C}[x_{ij}, y_{kl}]^{SL_3}$
- Enumerative combinatorics
  - promotion, evacuation, cyclic sieving
- Dimer models
- Representation theory

The web basis problem

Problem (Khovanov–Kuperberg '96)

Give a web basis* for $\mathrm{Inv}_{SL_r}(V_1 \otimes \cdots \otimes V_n)$ for $r \geq 4$.

*with desirable properties for use in applications:

- testability
- reduction rules
- rotation invariance

The web basis problem

<u>Next</u>: our solution for $r=4$!

- Unifies $r=2,3,4$
- Many pieces work for general r (TBD!)
- Introduces new plabic graphs with multiple trip's
- Combinatorially beautiful,

  <u>e.g.</u> connects ASM's and PP's in a certain sense

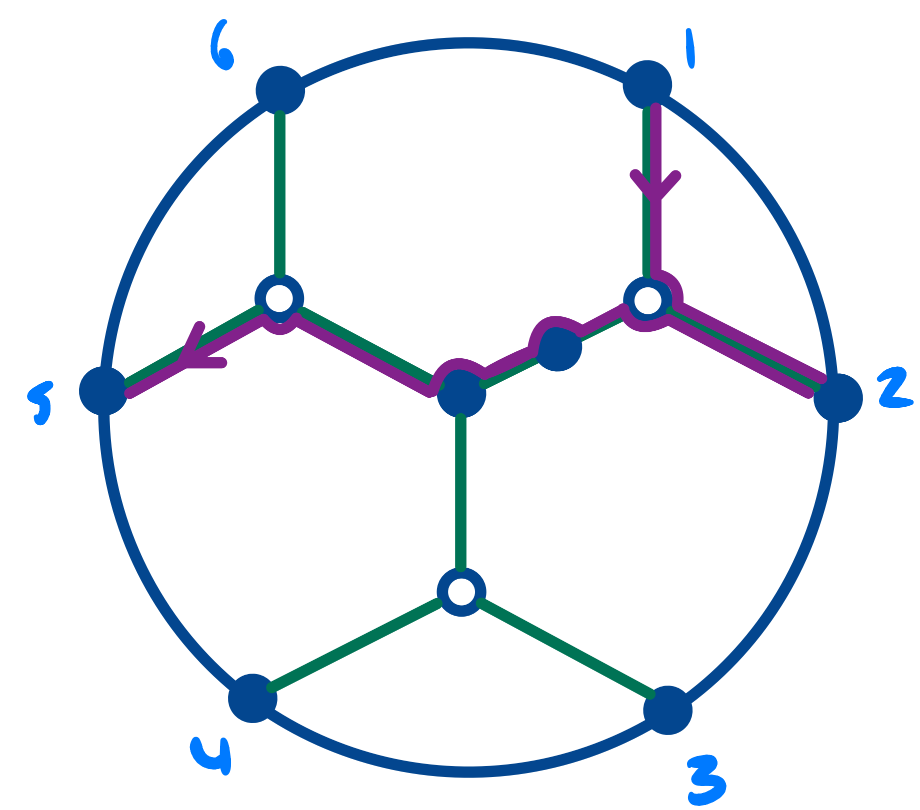
Plabic graphs

- Postnikov introduced plabic graphs to parametrize cells in $Gr_{\geq 0}(k,n)$. "planar bicolored"
- They have trip permutations:

Rules of the road

Left at ○
Right at ●

Ex

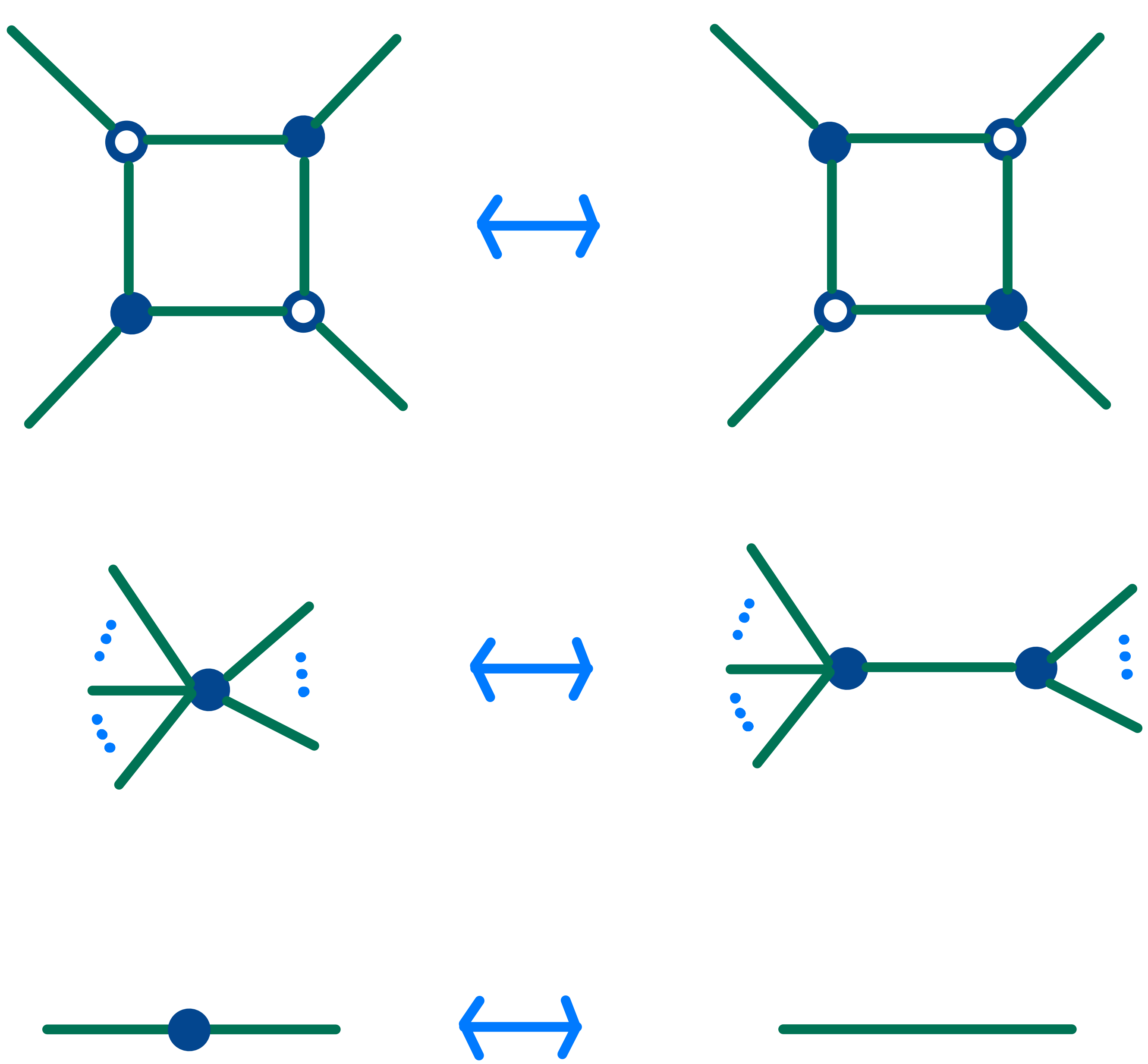


$1 \mapsto 2$
$2 \mapsto 5$
$\vdots$

Trip = (1 2 5 6 3 4)

Plabic graphs

- Moves preserve Trip:



Thm (Postnikov)

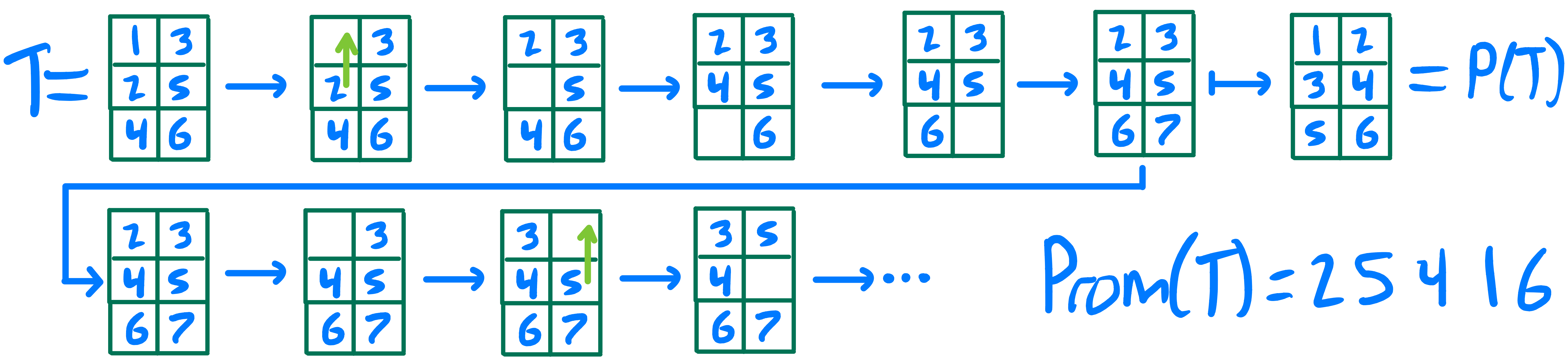
Two reduced plabic graphs have the same Trip if and only if they are connected by a sequence of moves.

Promotion permutations

Obs (Hopkins–Rubey) 3-row basis webs are reduced plabic graphs. What are their Trip's?

Def The **promotion permutation** of $T \in SYT(3 \times \frac{n}{3})$ tracks what enters the top row when computing $P^n(T)$:

Ex



$\mathrm{Prom}(T) = 2\,5\,4\,1\,6\,3$

Promotion and trip permutations

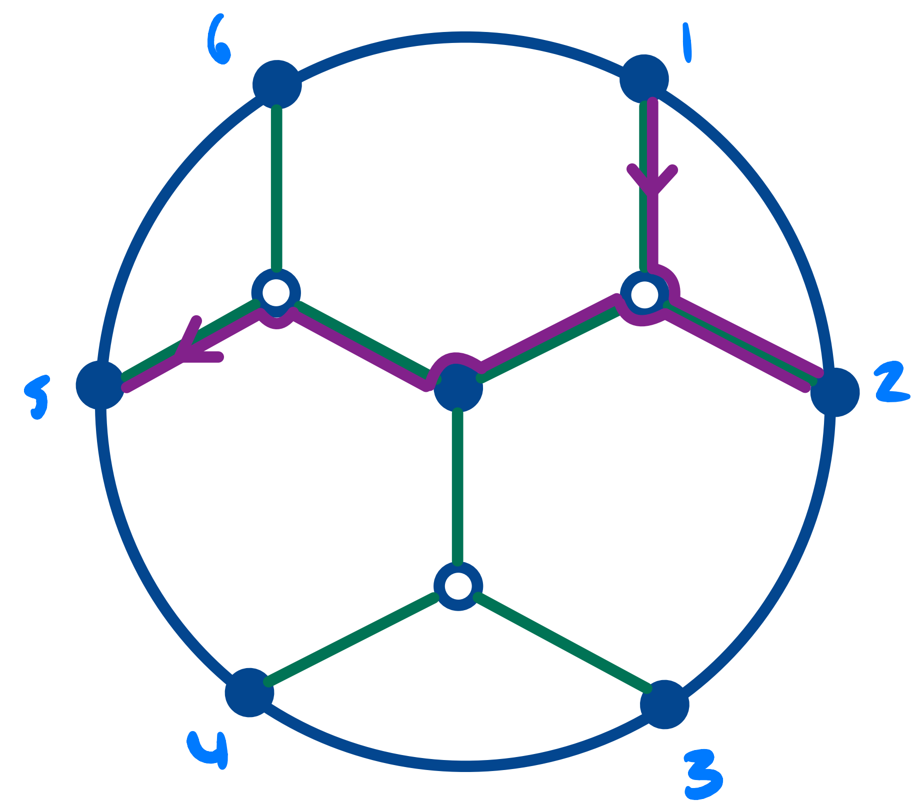
Thm (Hopkins–Rubey) The bijection between SL_3 basis webs and $SYT(3 \times \frac{n}{3})$ sends Trip to Prom.

Ex

$T =$

1	3
2	5
4	6

$\longleftrightarrow$



$= G$

$$\mathrm{Prom}(T) = 254163 = (125634) = \mathrm{Trip}(G)$$

Promotion matrices

- $\text{prom}: SYT(3\times\frac{n}{3}) \to S_n$ is injective, but

  $\text{prom}: SYT(4\times\frac{n}{4}) \to S_n$ is <u>not</u>.

- <u>Fix:</u> let prom_i record which number enters row i

  Thm $T \in SYT(r\times\frac{n}{r})$ is uniquely determined by the sequence of promotion permutations $\text{prom}_1, \ldots, \text{prom}_{r-1}$.

  Moreover, $\text{prom}_i^{-1} = \text{prom}_{r-i}$.

Multiple trips

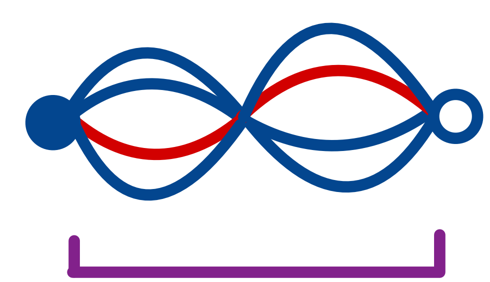
- Given a plabic graph (or similar), let

  trip_i: take the ith left at ○

  ith right at ●

- <u>Idea:</u> if r-valent, $\text{trip}_i^{-1} = \text{trip}_{r-i}$.

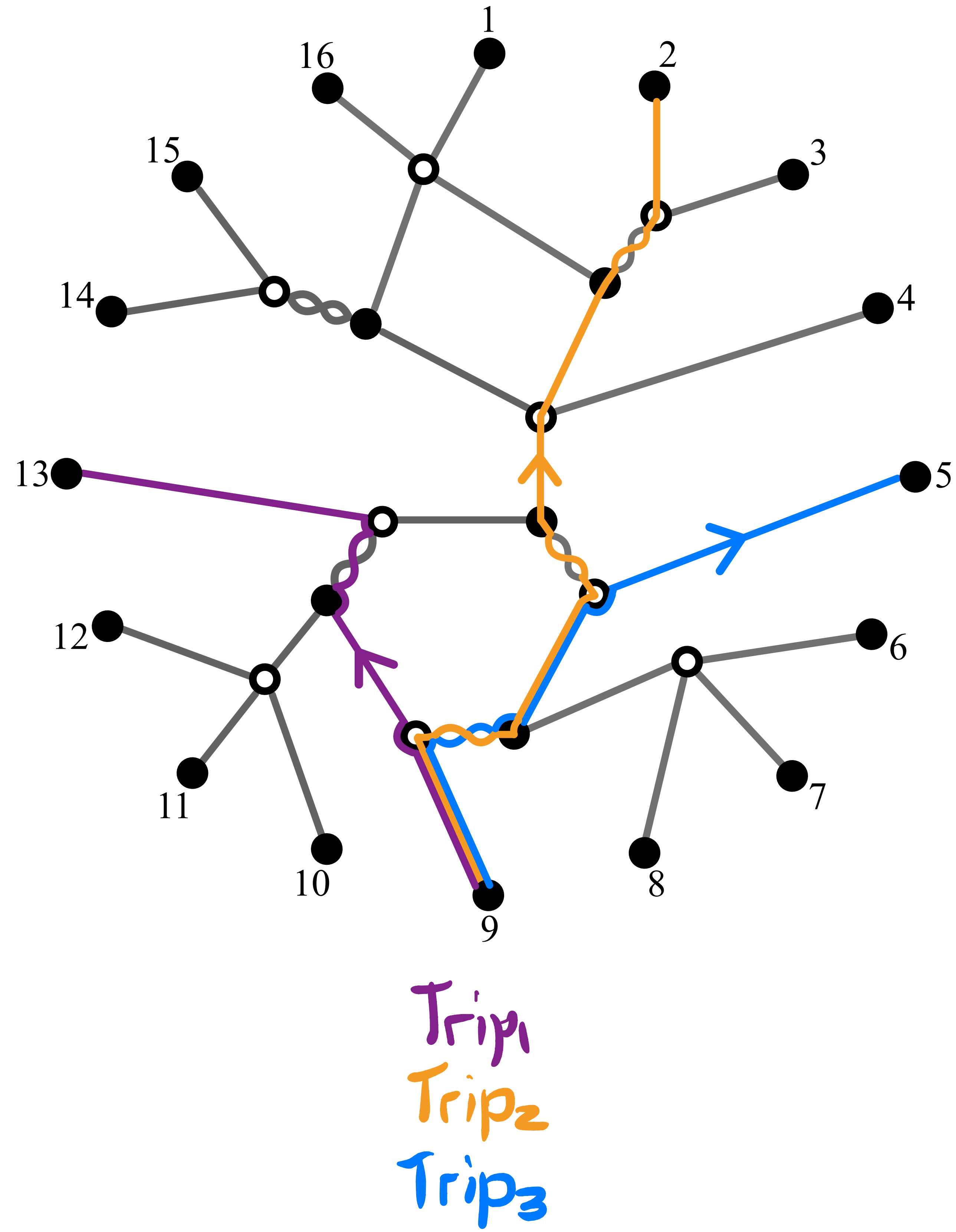
- An hourglass is an edge



4-hourglass

Multiple trips

Ex



Main theorem

Thm $\{[W] \mid W$ is a top fully reduced hourglass plabic graph$\}$ is an SL_4 web basis. Furthermore, we have a bijection

$$SYT(4 \times \tfrac{n}{4}) \to TFRHPG(n)$$

$$T \mapsto G$$

$$\mathrm{prom}_i(T) = \mathrm{trip}_i(G) \qquad (i = 1, \ldots, r-1)$$

→

Main theorem

While $[W]$ is a collection of webs, they represent the same tensor invariant.

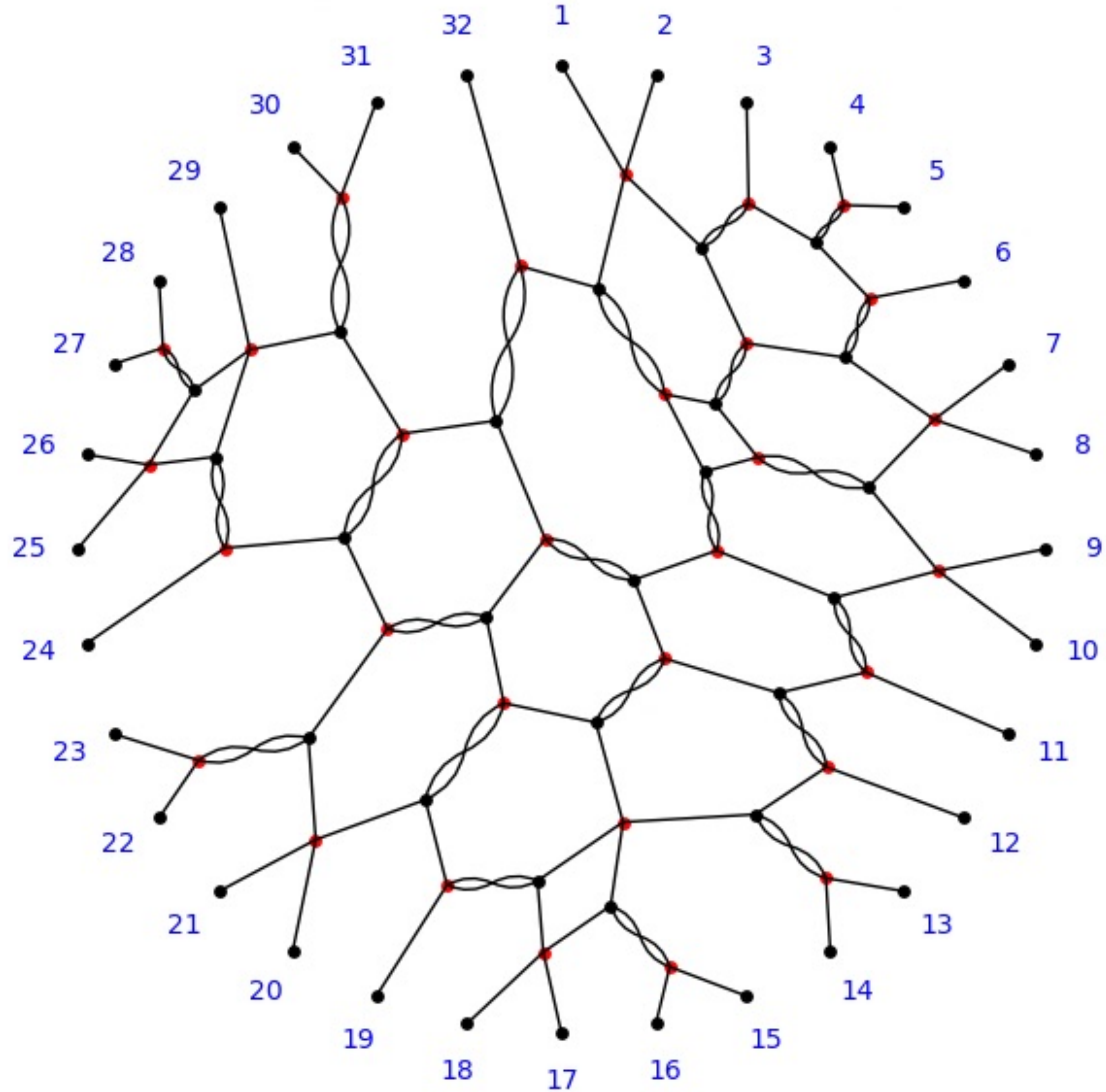
Hence we have a rotation-invariant basis of tensor invariants, encoded by webs!

4-row webs (new!)

Ex

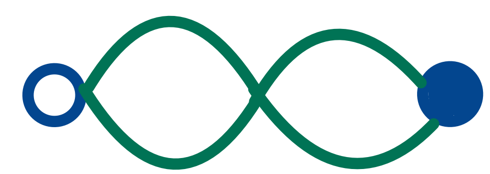
1	3	4	7	8	17	19	23
2	5	6	9	14	18	21	24
10	12	13	15	16	25	26	28
11	20	22	27	29	30	31	32

⟼



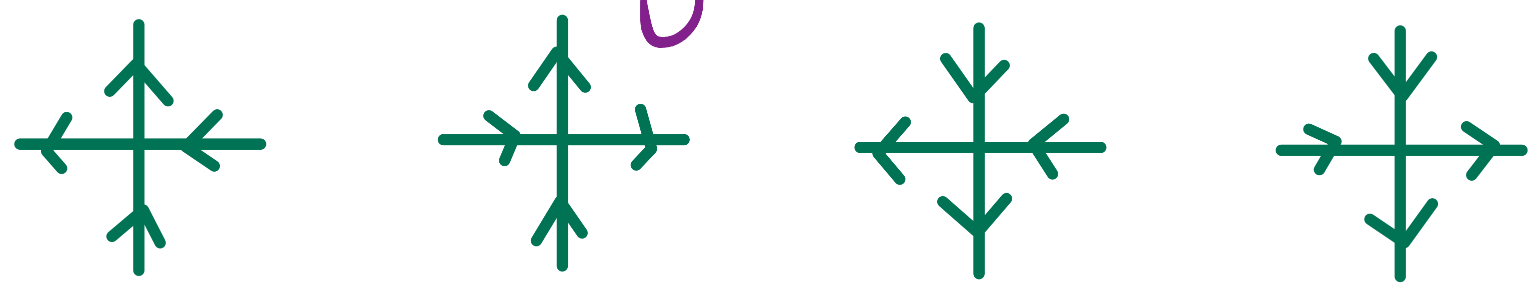
4-row webs (new!)

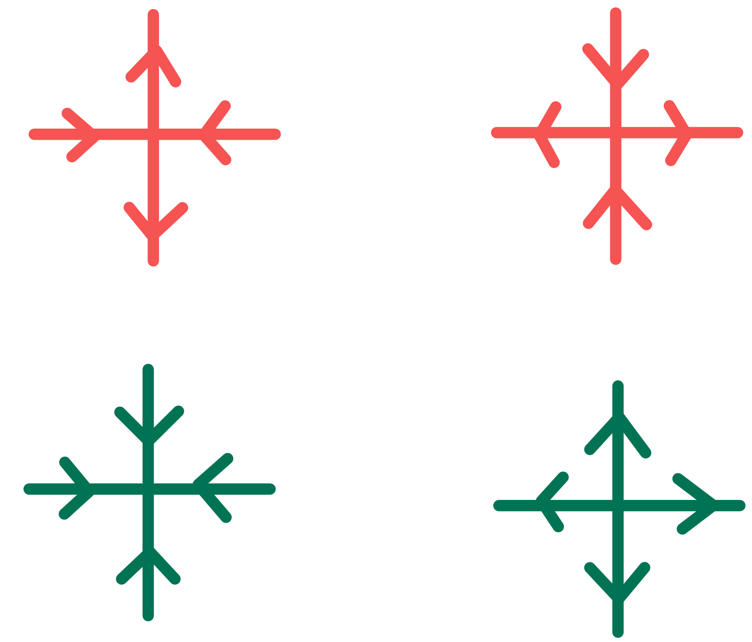
Def A 4-row web is...

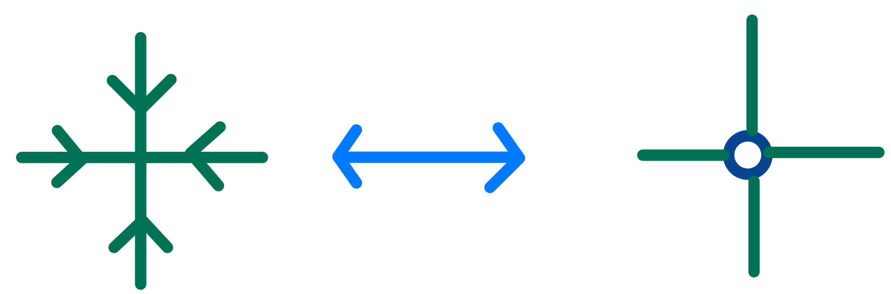
- Planar-embedded graph in disk, allowing hourglass edges 
- 4-valent interior vertices, univalent boundary
- Bipartite, marked "initial" outer vertex

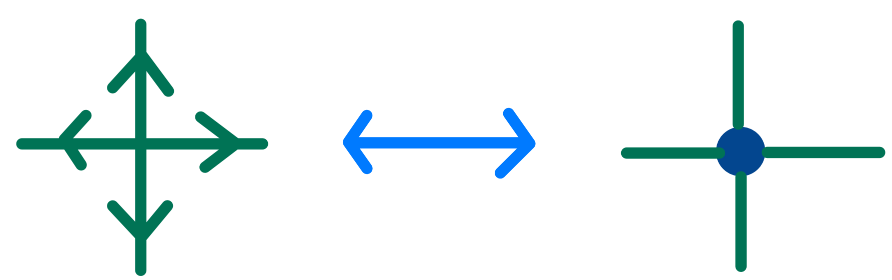
6-vertex model

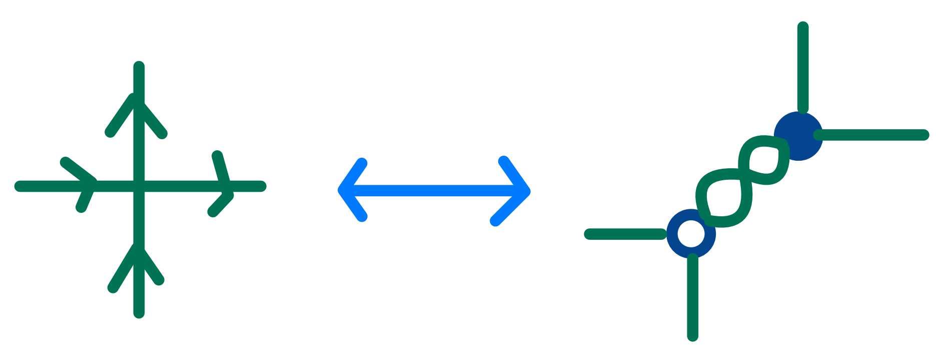
Alternate encoding:





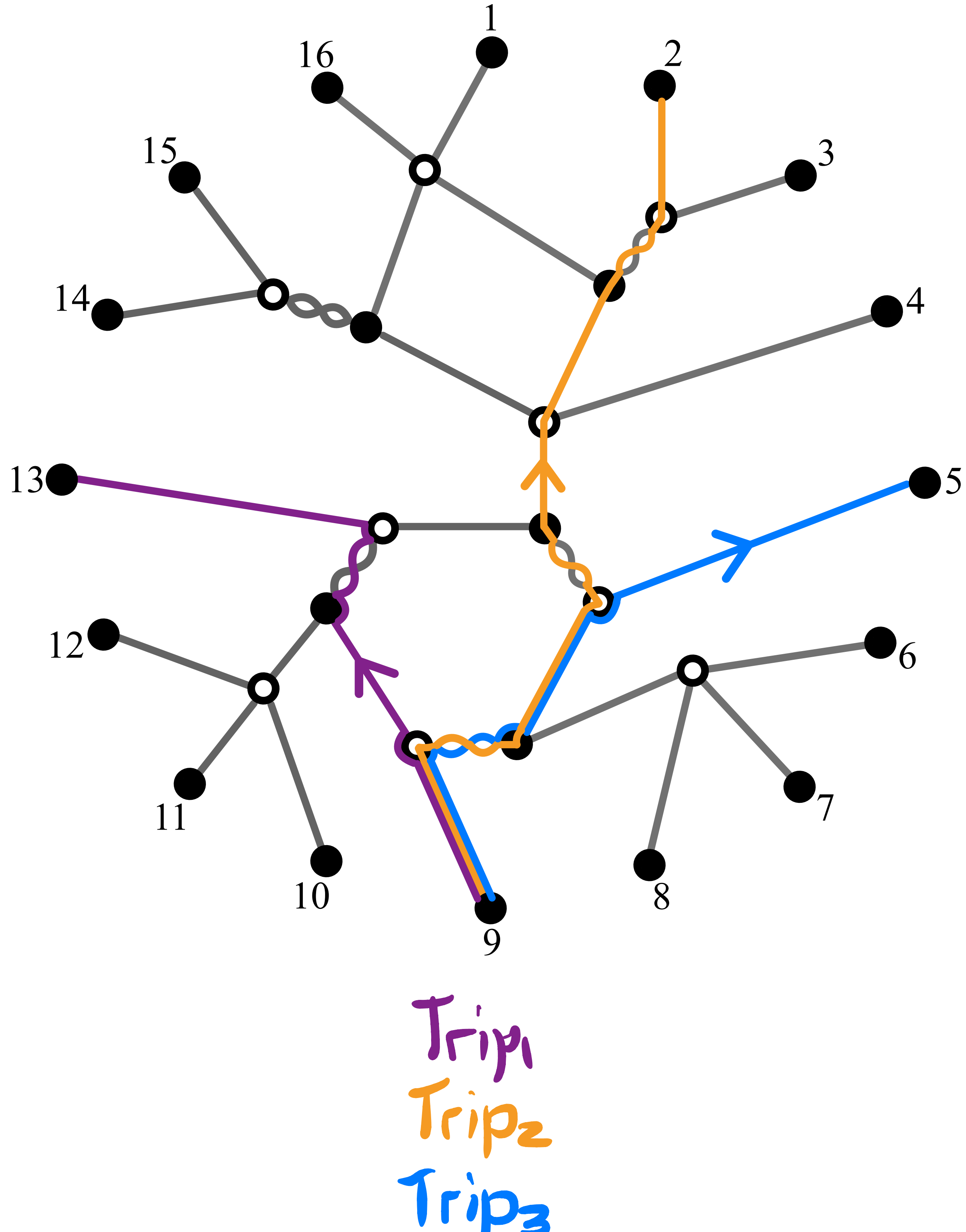

(Sink)


(Source)


(Transmitting)

6-vertex model

Ex



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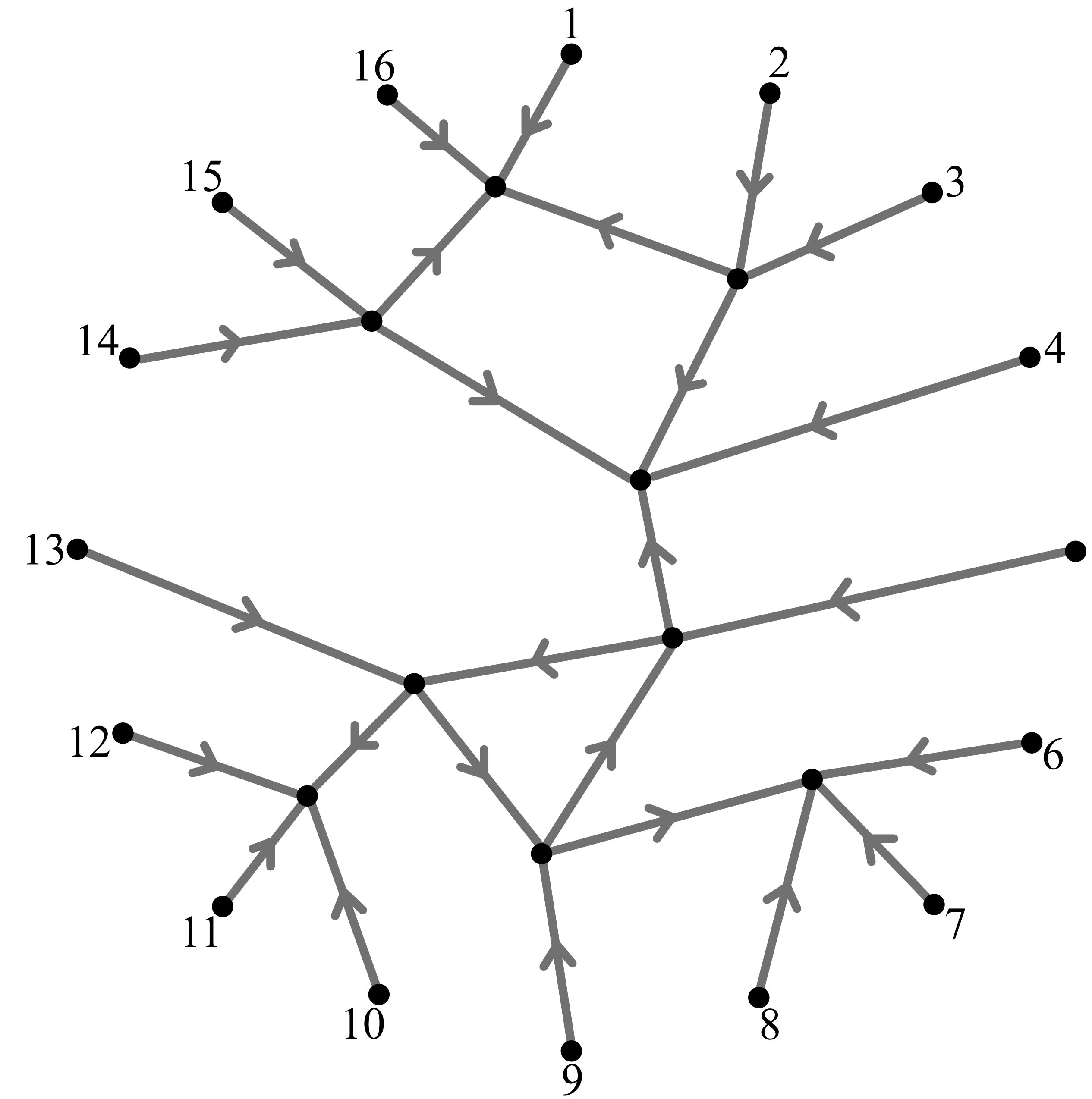
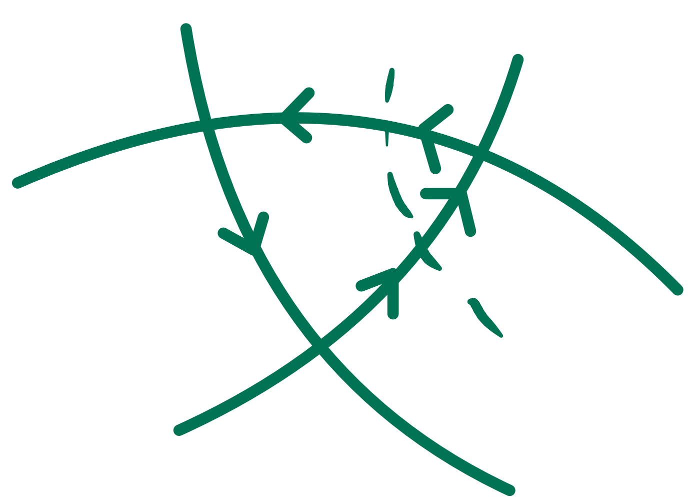


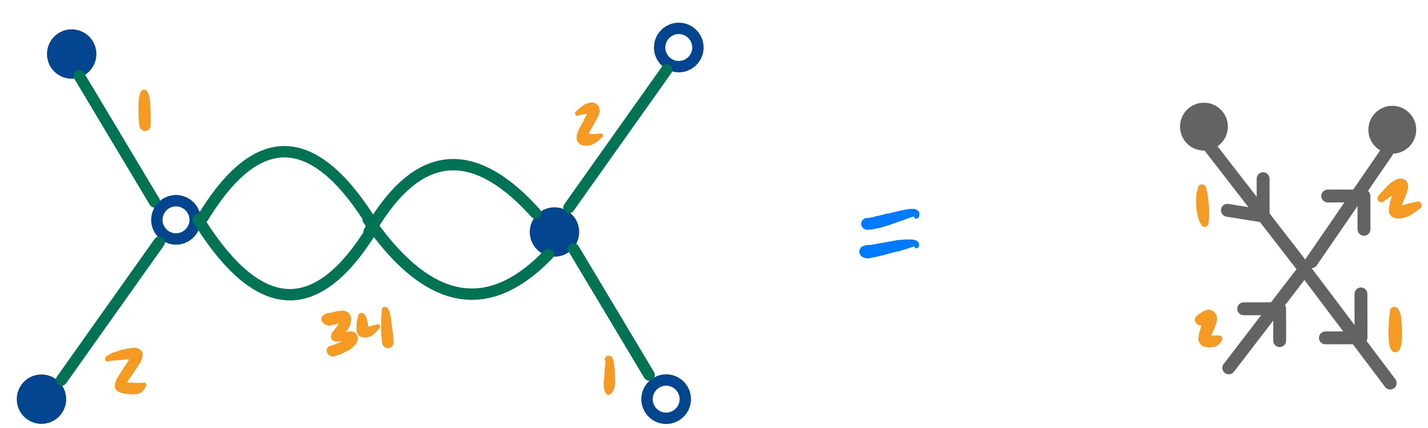
Top fully reduced

Def A 4-row basis web is top fully reduced if all triangles* in the 6-vertex configuration are oriented counterclockwise.

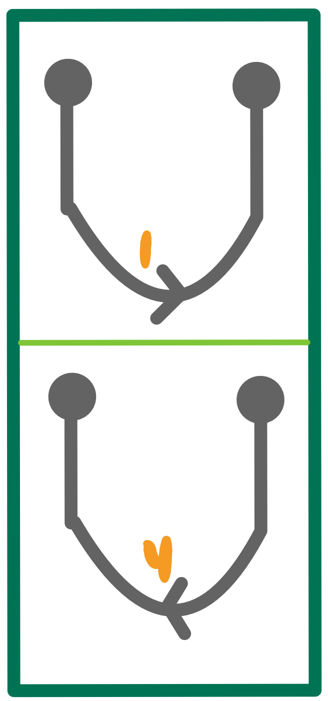
*even "big" triangles

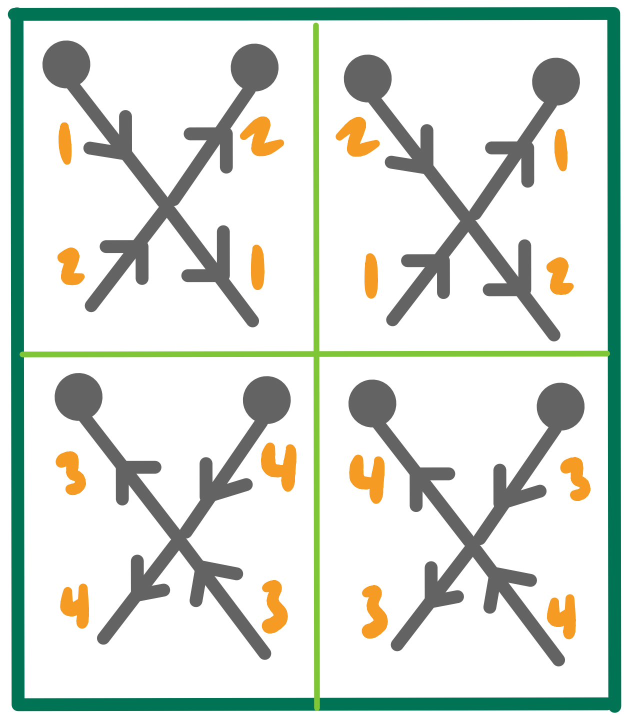


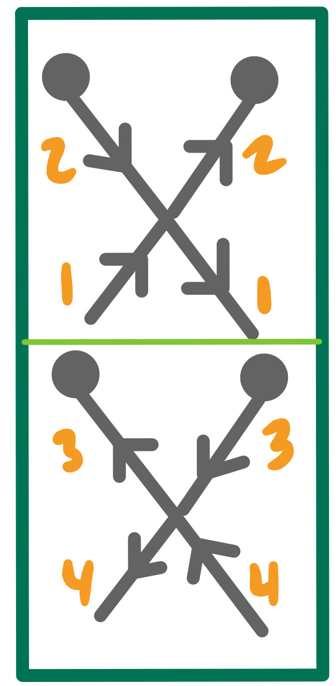
Some sample 4-row growth rules:

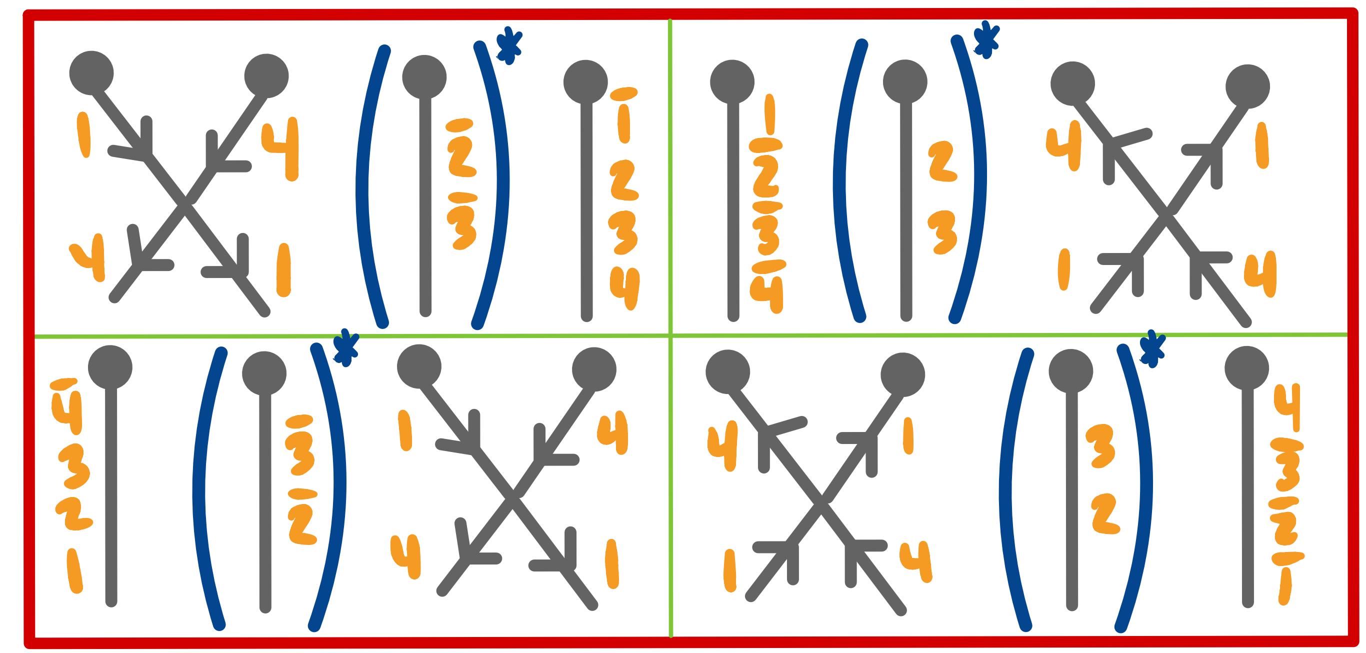


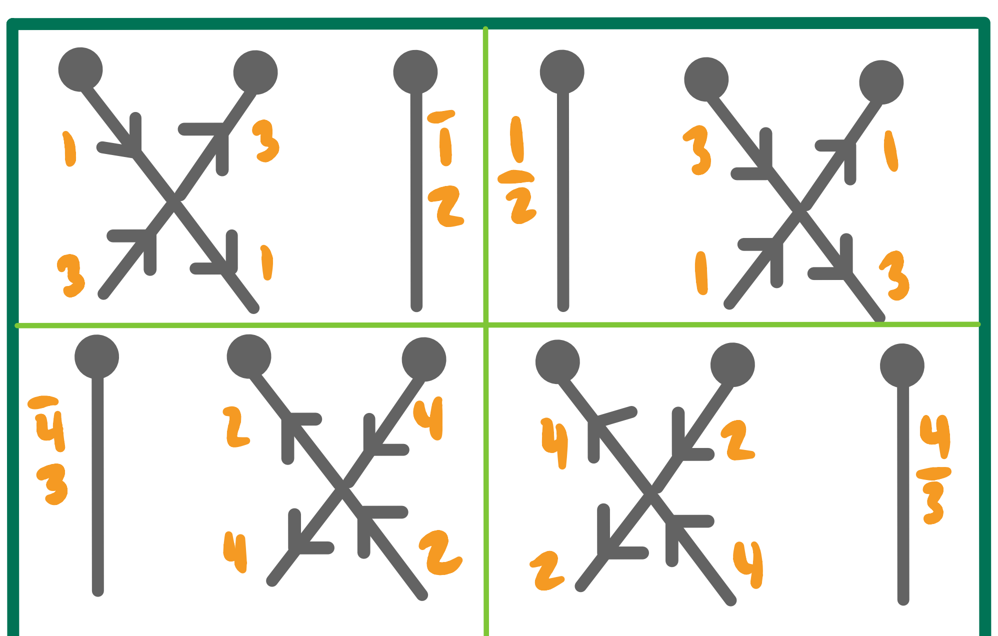
4-row growth rules

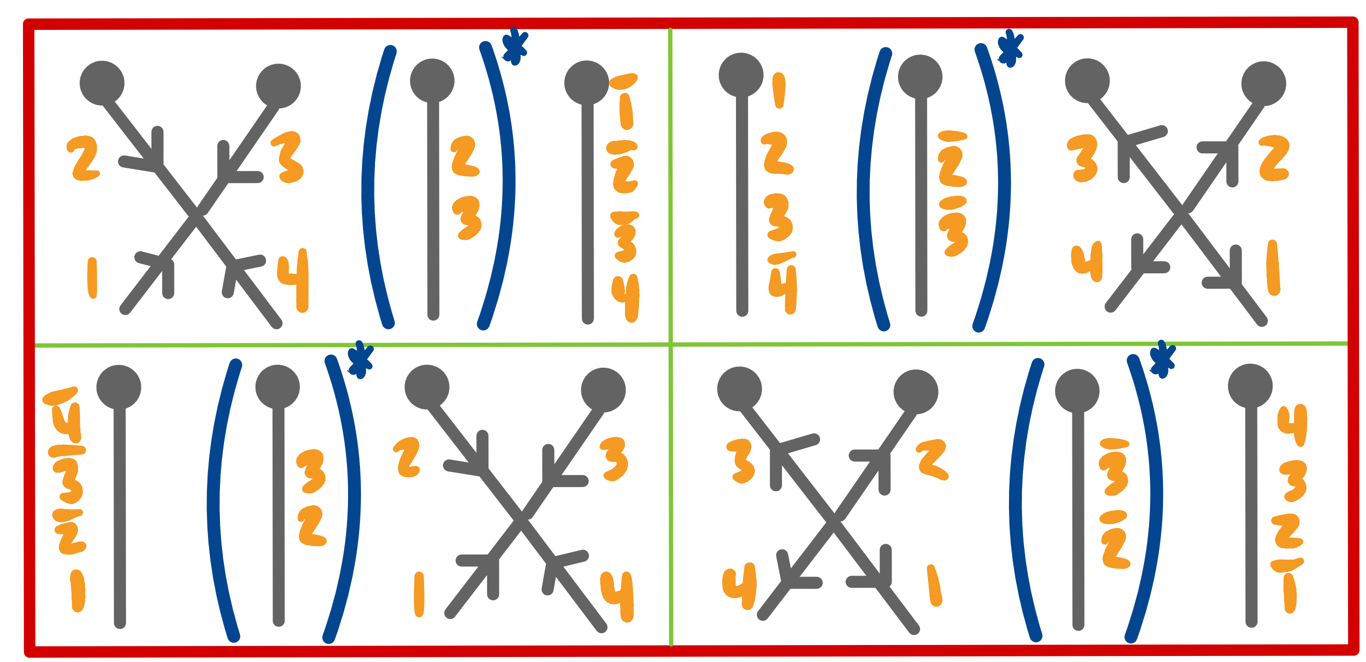




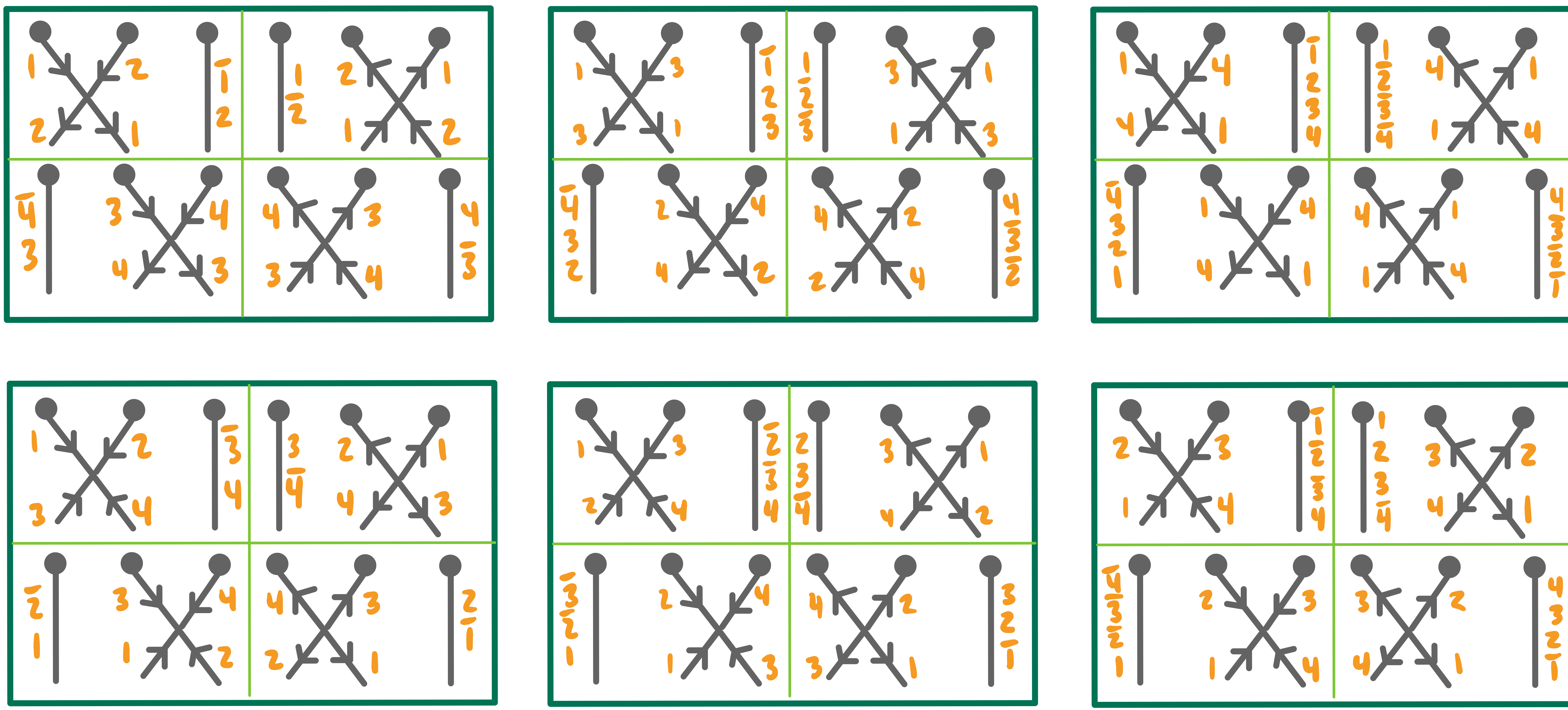








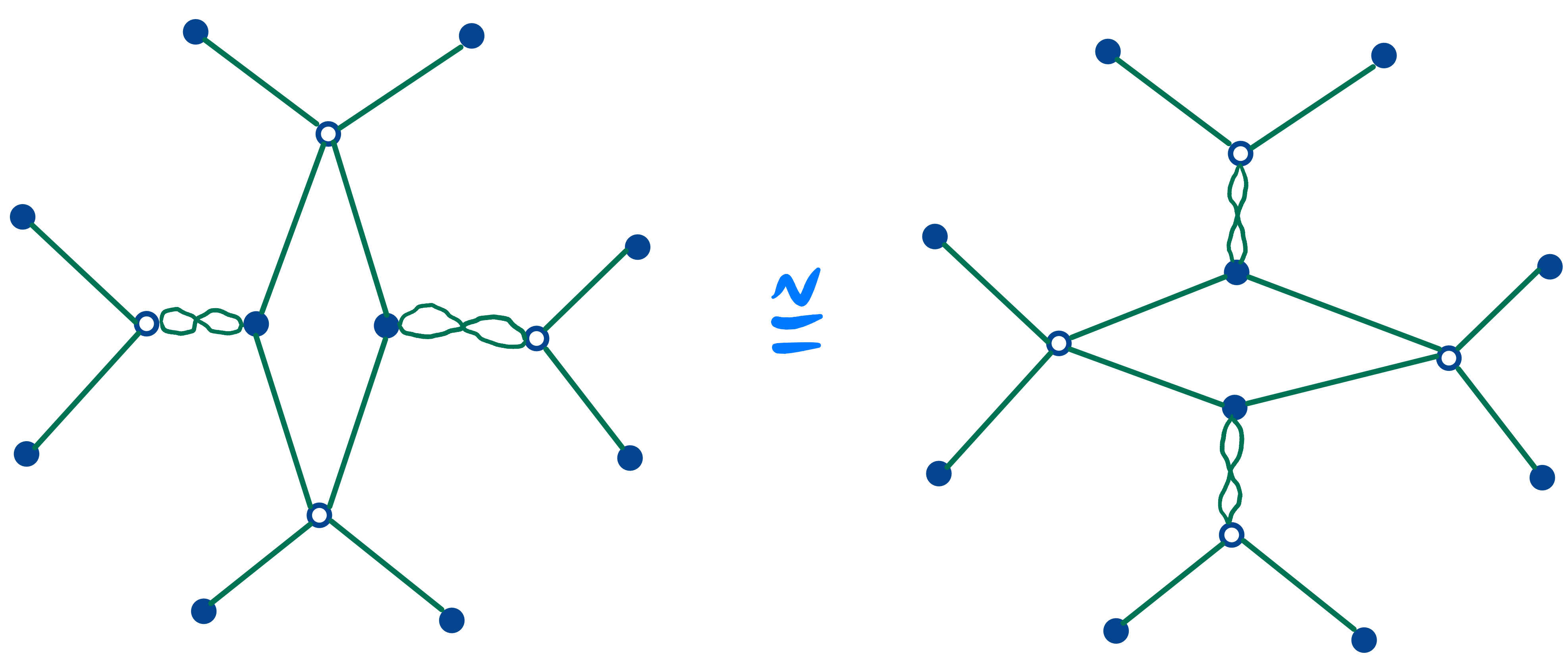
4-row growth rules



4-row moves

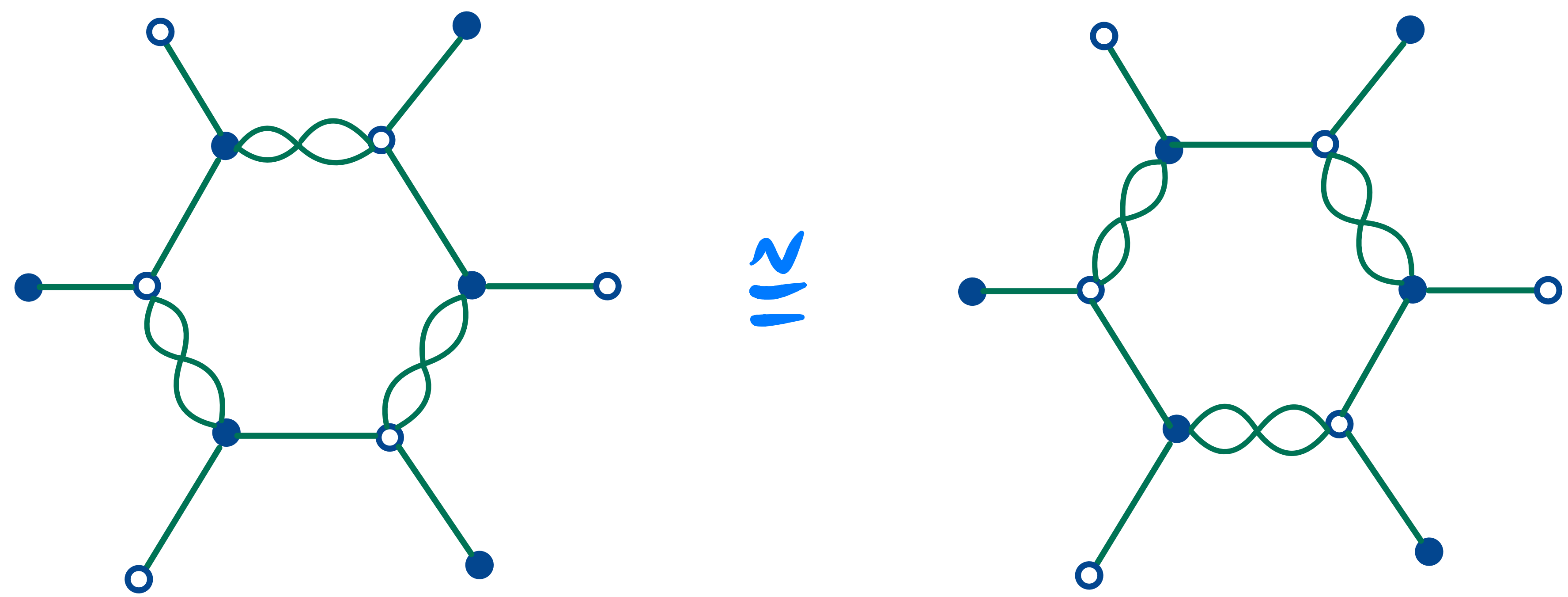
Thm Flip equivalence essentially arises from two moves:

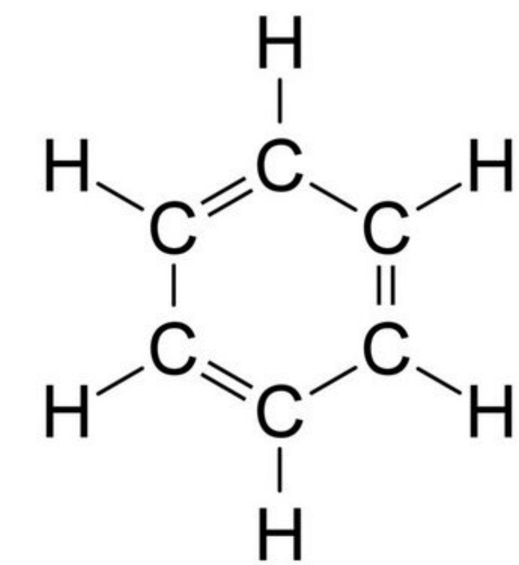
1. Square moves: e.g.



4-row moves

2. Benzene moves:

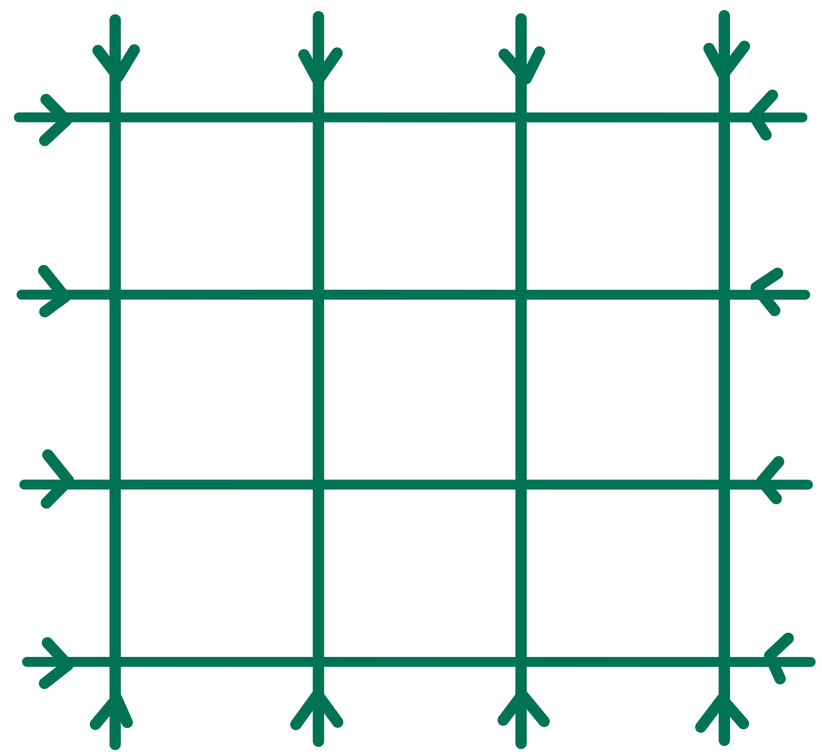




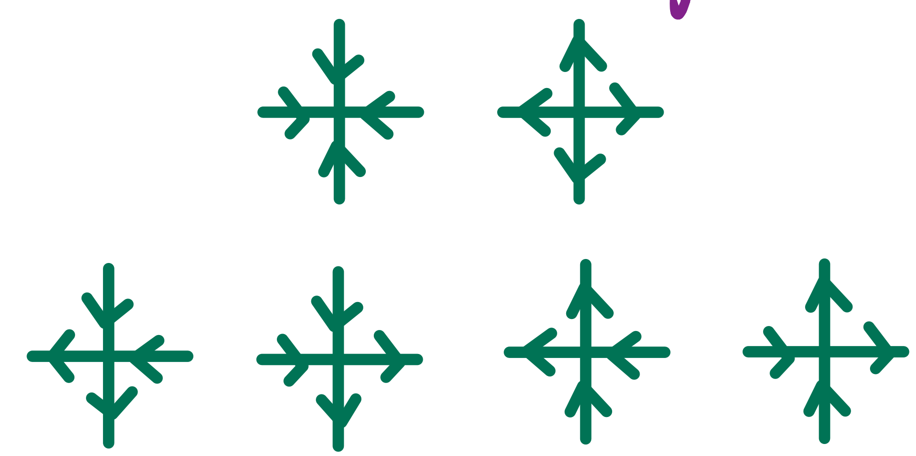
(Actual benzene)

ASM case

Thm The tableau with lattice word $1^n 2^n 3^n 4^n$ has webs



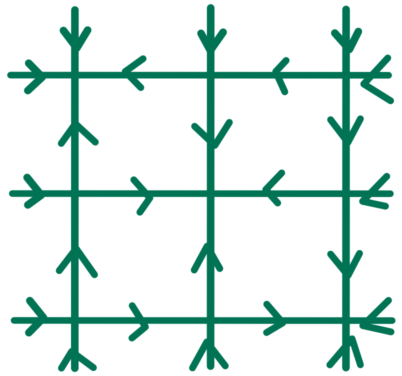
Completed in all ways using...



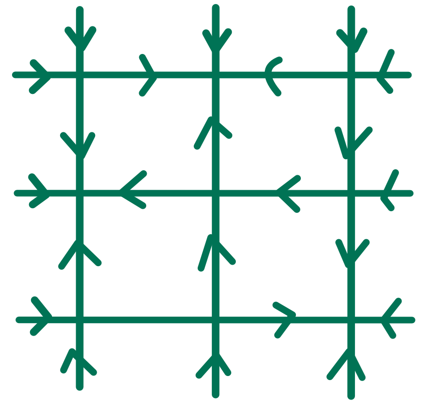
Note These are naturally alternating sign matrices!

ASM case

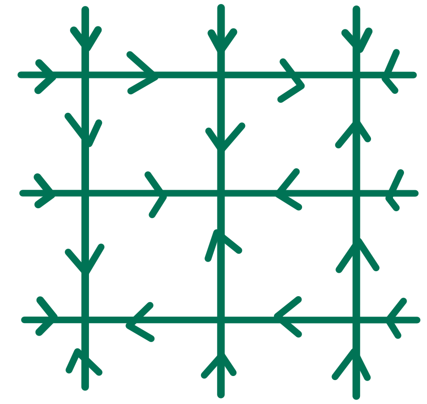
Ex



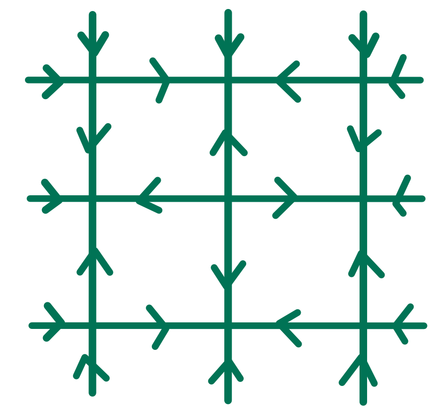
$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$$



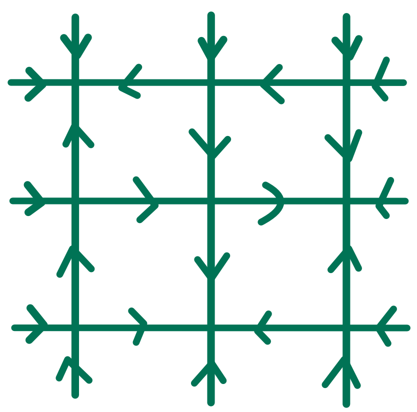
$$\begin{matrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{matrix}$$



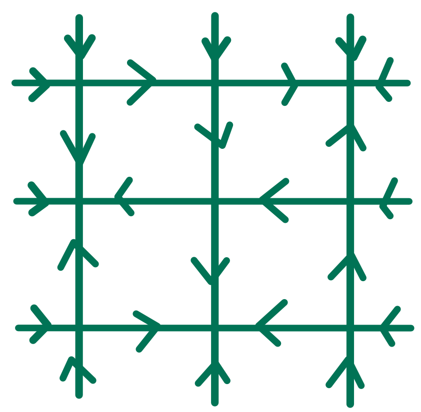
$$\begin{matrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{matrix}$$



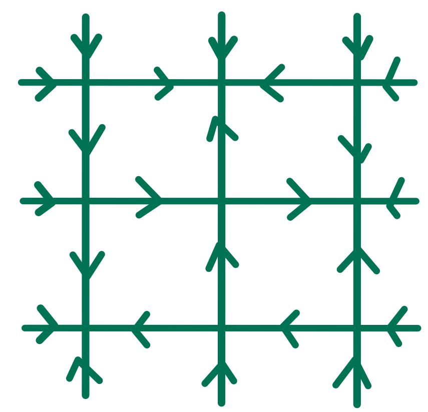
$$\begin{matrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{matrix}$$



$$\begin{matrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{matrix}$$



$$\begin{matrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{matrix}$$



$$\begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{matrix}$$

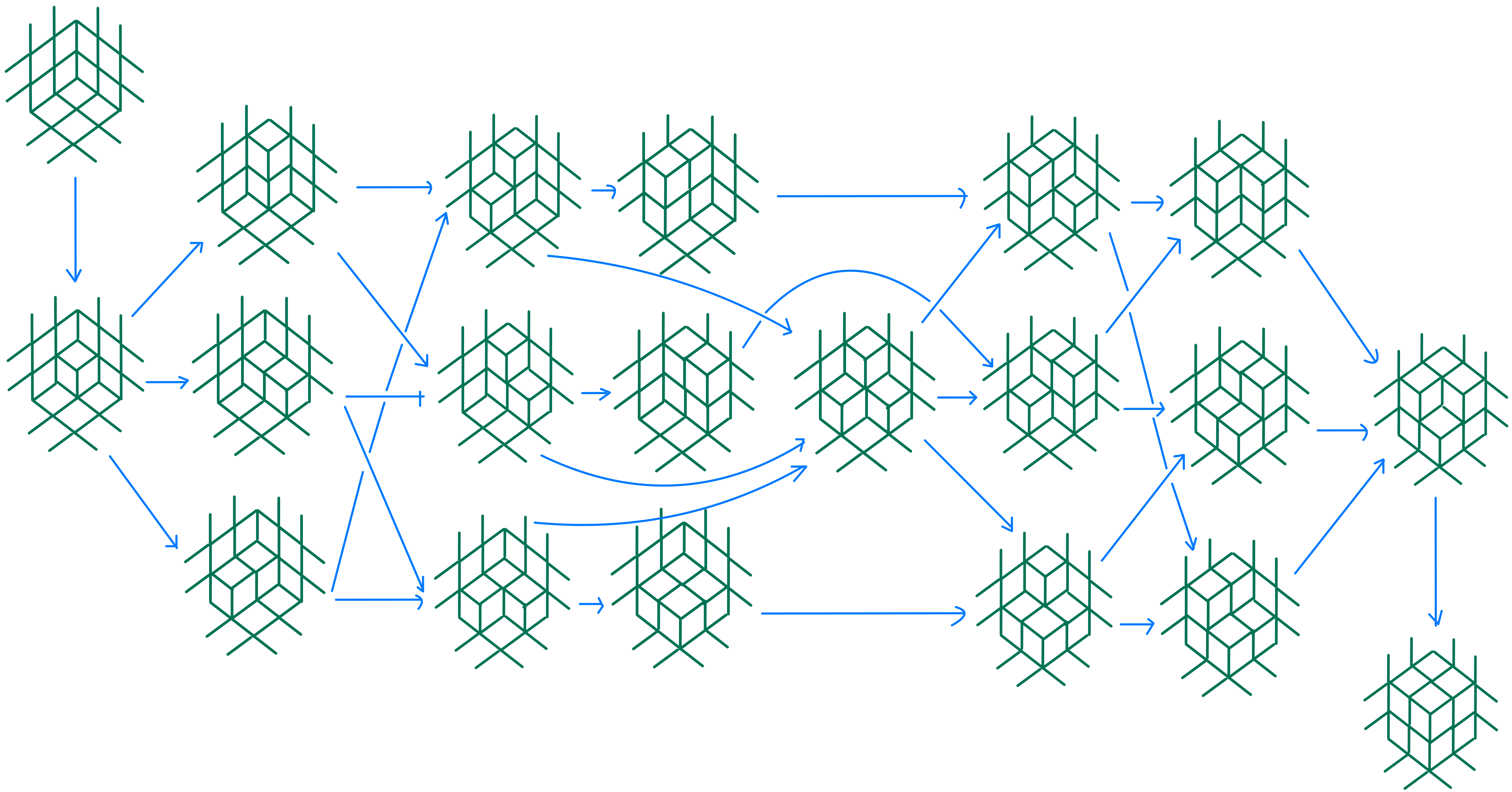
Plane partition case

Thm The tableau with lattice word $1^a\bar{4}^b 2^c \bar{1}^{a-c} \bar{2}^c 4^b \bar{1}^c$ $(a \geq c)$ has equivalence class

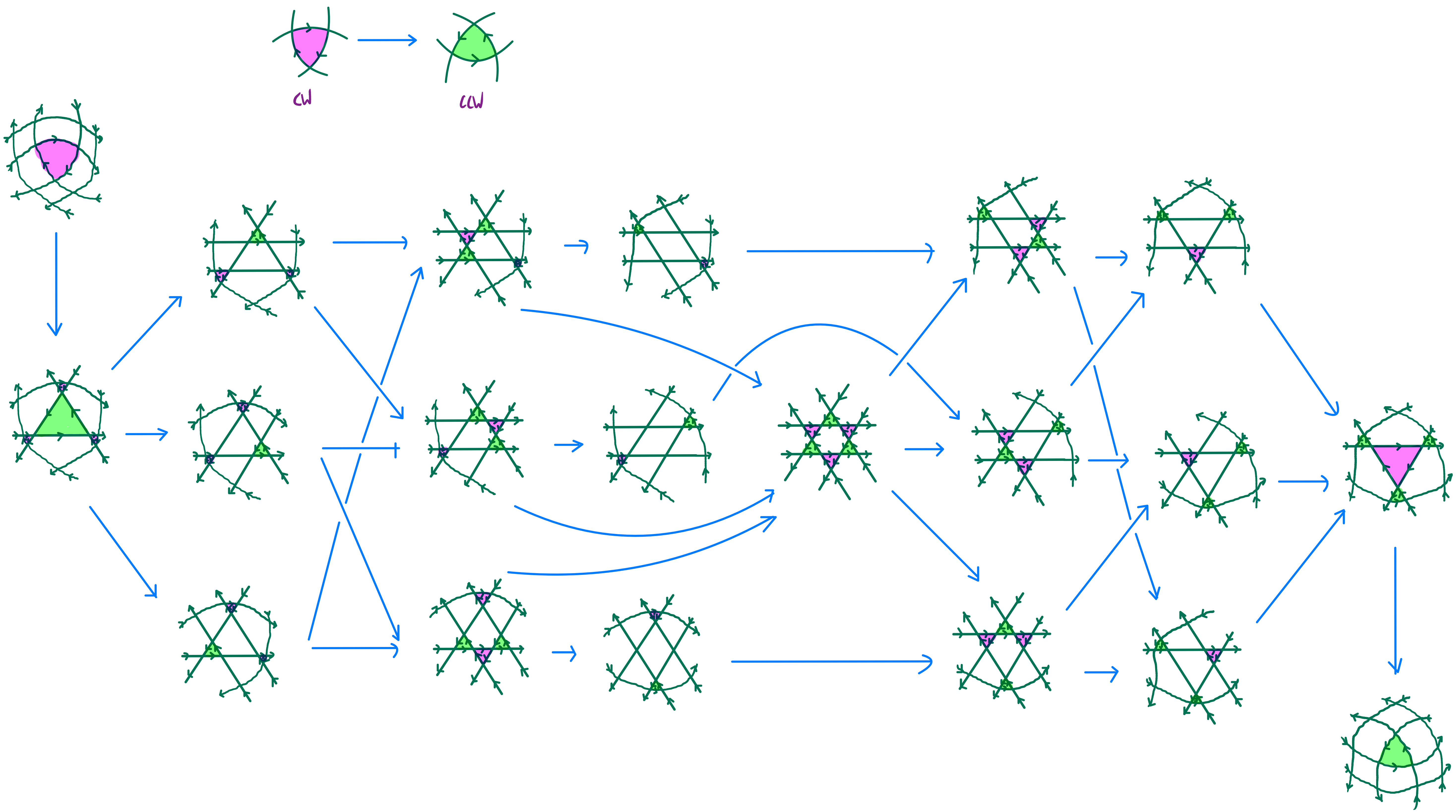
$$\left\{\begin{array}{c}\text{plane partitions} \\ \text{in the } a \times b \times c \text{ box}\end{array}\right\}$$

Plane partition case

Ex



Plane partition case



Fraser webs

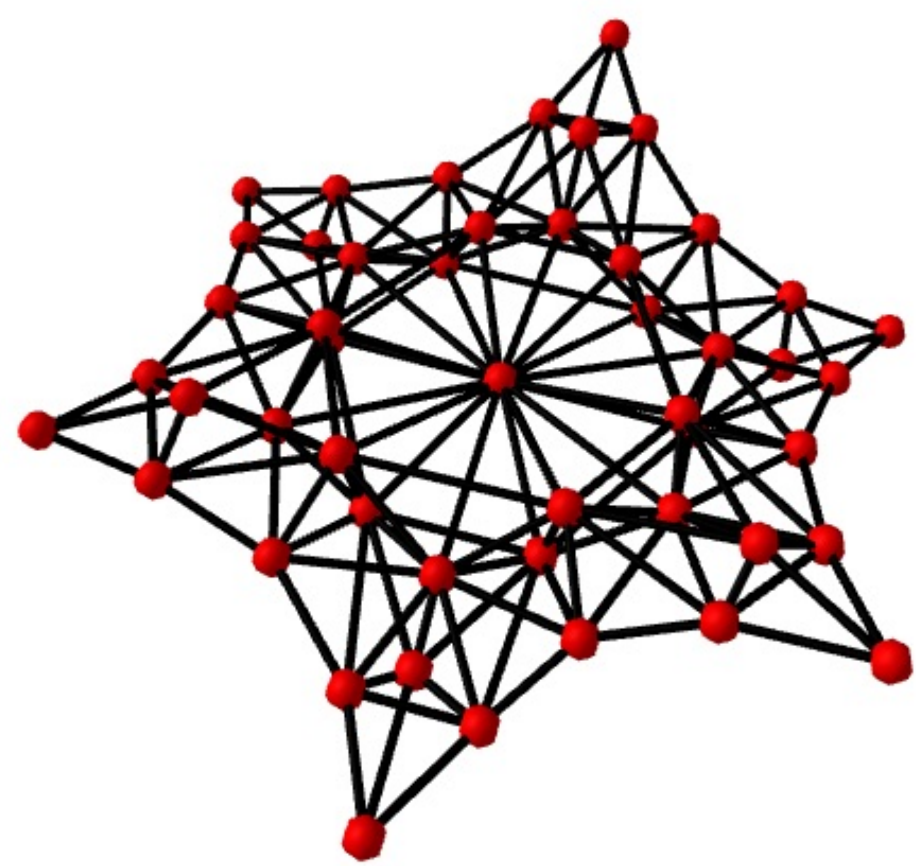
- Chris Fraser introduced a 2-column web basis.
- We showed Trip=Prom (and "fully reduced" — no bad double crossings)
- Hence the hourglass plabic graph framework unites the Tamari, ASM, and PP lattices!

Future Work

- Understand Prom.
  - Characterize image
  - Figure out duality
- General SL_r case
  - Mostly know how to do $r=5$
- Strong Knuth equivalence behind growth rules
- ASM/PP connections
- Code w/ GUI Front-end

Future Work

- Affine building geometry and "pockets"
  - Suggested by Kuperberg based on [Fontaine–Kamnitzer–Kuperberg]



THANKS!