

Type B q-Stirling numbers

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Based on joint work with *Bruce Sagan* (arXiv: 2205.14078)
and *Nolan Wallach* (arXiv: 2109.03407)

Slides: https://www.jpswanson.org/talks/2023_NDSU_q-Stirlings.pdf

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Outline

- Stirling variations
 - Classical type A
 - Classical q -analogue
- Type B q -Stirling numbers
- Enumerative results
- Super coinvariant algebra motivations

Overview

	Type A	Type B	Other Types
(classical)	Stirling 1730	Zaslavski '81?	Zaslavski '81?
q -analogue	Carlitz '33 Gould '61	S.-Walbach '21 S.-Sagan '22 <small>Today!</small> Bagno-Garber '22+	S.-Sagan '23+

Variations on unsigned/signed Stirling numbers of the first kind
 and unordered/ordered Stirling numbers of the second kind

Classical Stirling Numbers

Def The (type A) Stirling numbers of the second kind
are defined recursively by

$$S(n, k) = S(n-1, k-1) + kS(n-1, k)$$

$$S(0, k) = \delta_{0k}$$

Ex

$n=0:$

1 0 0 0 0 0 ...

(See A008277)

$n=1:$

0 1 0 0 0 0 ...

$n=2:$

0 1 1 0 0 0 ...

$n=3:$

0 1 3 1 0 0 ...

$n=4:$

0 1 7 6 1 0 ...

$n=5:$

0 1 15 25 10 1 ...

$$= 7 + 3 \cdot 6 = S(5, 3)$$

Classical Stirling Numbers

Def A (type A) set partition of $[n] = \{1, 2, \dots, n\}$, written $P \vdash [n]$,

is

s.t.

blocks
of P

$$P = \{B_1, \dots, B_k\}$$

$$B_1 \cup \dots \cup B_k = [n]$$

$$B_i \cap B_j = \emptyset \text{ for } i \neq j$$

$$B_i \neq \emptyset \text{ for all } i$$

- Set $S([n], k) = \{P \vdash [n] : P \text{ has } k \text{ blocks}\}$

Ex $\{\{1, 3\}, \{5, 6\}, \{2, 4\}\} \vdash [6]$

\uparrow

$13/24/56 \in S([6], 3)$

$\boxed{[\quad | \quad]}$

write in increasing
order, say

Classical Stirling Numbers

Thm $S(n,k) = \# S([n], k)$

Pf Remove n : if singleton, $k-1$ blocks, otherwise k copies with k blocks

Thm $S(n,k) = h_{n-k}(1,2,\dots,k) = \sum_{j_1+\dots+j_k=n-k} 1^{j_1} 2^{j_2} \dots k^{j_k}$

Pf $h_{n-k}(x_1, \dots, x_k) = h_{n-k}(x_1, \dots, x_{k-1}) + x_k h_{n-k-1}(x_1, \dots, x_{k-1})$ with $x_i = i$

Ex $S(5,3) = 1^2 2^0 3^0 + 1^0 2^2 3^0 + 1^0 2^0 3^2 + 1^1 2^1 3^0 + 1^1 2^0 3^1 + 1^0 2^1 3^1$
 $= 1 + 4 + 9 + 2 + 3 + 6 = 25$

Classical Stirling Numbers

Def A (type A) ordered set partition of $[n]$, written $P \models [n]$, is $P = (B_1, \dots, B_k)$ s.t. $\{B_1, \dots, B_k\} \vdash [n]$.

- Set $S^o([n], k) = \{P \models [n] \text{ where } P \text{ has } k \text{ blocks}\}$
- The (type A) ordered Stirling numbers of the second kind are

$$S^o(n, k) = k! S(n, k) \\ = \# S^o([n], k)$$

Ex $13/56/24 \in S^o([6], 3)$ and $S^o(5, 3) = 3! S(5, 3) = 6 \cdot 25 = 150$

Classical q -Stirling Numbers

Def The q -integers are defined by

$$[k] = 1 + q + \dots + q^{k-1} = \frac{1 - q^k}{1 - q}$$

Have $\lim_{q \rightarrow 1} [k] = k$.

Def (Carlitz '33; Gould '61)

The (type A) q -Stirling numbers of the second kind
are defined recursively by

$$S[n, k] = S[n-1, k-1] + [k]S[n-1, k]$$

$$S[0, k] = \delta_{0k}$$

$$\Rightarrow S[n, k] \Big|_{q=1} = S(n, k)$$

Classical q -Stirling Numbers

Ex

	$k=0:$	$k=1:$	$k=2:$	$k=3:$	$k=4:$	$k=5:$
$n=0:$	1	0	0	0	0	$0 \dots$
$n=1:$	0	1	0	0	0	$0 \dots$
$n=2:$	0	1	1	0	0	$0 \dots$
$n=3:$	0	1	$2+q$	1	0	$0 \dots$
$n=4:$	0	1	$3+3q+q^2$	$3+2q+q^2$	1	$0 \dots$
$n=5:$	0	1	$4+6q+4q^2+q^3$	$6+8q+7q^2+3q^3+q^4$	$4+3q+2q^2+q^3$	$1 \dots$

$$-(3+3q+q^2) + (1+q+q^2)(3+2q+q^2) = S[5,3]$$

Classical q -Stirling Numbers

Thm $S[n, k] = h_{n-k}([1], [2], \dots, [k]) = \sum_{\gamma_1 + \dots + \gamma_k = n-k} [1]^{\gamma_1} [2]^{\gamma_2} \dots [k]^{\gamma_k}$

Pf Same!

Thm (Milne '82; Garsia-Remmel '86; Wachs-White '91; ...)

There are statistics $\text{stat}: S[n, k] \rightarrow \mathbb{Z}_{\geq 0}$ such that

$$S[n, k] = \sum_{p \in S[n, k]} q^{\text{stat}(p)}$$

Classical q-Stirling Numbers

- For instance, set

$\text{inv}(P) = \#\{(a, B_j) : a \in B; \text{ for some } k_j \text{ and } a > \min B_j\}$

Then $\sum_{P \in S[n,k]} q^{\text{inv}(P)} = S[n,k]$

Ex] $145/27/36$ has $\text{inv} = 5$



Classical q -Stirling Numbers

Def The (type A) ordered q -Stirling numbers of the second kind are

$$S^o[n, k] = [k]! S[n, k]$$

(The q -Factorial is
 $[k]! = [k][k-1]\dots[1]$)

- Encountered in Carlitz '33 originally!
- Statistics and additional work by Steingrímsson '01,
Haglund-Rhoades-Shimozono '18, ...

e.g. $\sum_{P \in S^o[n, k]} q^{\text{inv}(P)} = S^o[n, k]$

Classical Type B Stirling Numbers

Def

The Type B Stirling numbers of the second kind
are defined recursively by

$$S_B(n, k) = S_B(n-1, k-1) + (2k+1) S_B(n-1, k)$$

where $S_B(0, k) = \delta_{0k}$

Ex

$n=0:$

1 0 0 0 0 0 ...

(See A039755)

$n=1:$

1 1 0 0 0 0 ...

$n=2:$

1 4 1 0 0 0 ...

$n=3:$

1 13 9 1 0 0 ...

$$= 58 + 7 \cdot 16 = S_B(5, 3)$$

$n=4:$

1 40 58 16 1 0 ...

$n=5:$

1 121 330 170 25 1 ...

Classical Type B Stirling Numbers

Def A type B set partition of $\langle_n\rangle = \{-n, -n+1, \dots, -1, 0, 1, \dots, n-1, n\}$,

written $P \vdash \langle_n\rangle$, is a set partition of $\langle_n\rangle$ such that

$$B \in P \ni -B = \{-b : b \in B\} \in P$$

and $B \neq -B$ unless $0 \in B$.

• Set $S_B(\langle_n\rangle, k) = \{P \vdash \langle_n\rangle : P \text{ has } 2k+1 \text{ blocks}\}$

Ex $\{\{\{4\}, \{1, -3\}, \{-2, 0, 2\}, \{-4\}, \{-1, 3\}\}\}$
 $\bar{z}02/\bar{1}\bar{3}/\bar{4} \in S_B(\langle_4\rangle, 2)$
increasing minima, say

Why? Naturally indexes type B_n hyperplane arrangement intersection lattice

(classical) Type B Stirling Numbers

Thm $S_B(n, k) = \# S_B(\langle n \rangle, k)$

Pf Remove $\pm n$: if singleton, $2k-1$ blocks, otherwise $2k+1$ copies with $2k+1$ blocks

Thm $S_B(n, k) = h_{n-k}(1, 3, \dots, 2k+1) = \sum_{\gamma_0 + \dots + \gamma_k = n-k} 1^{\gamma_0} 3^{\gamma_1} \dots (2k+1)^{\gamma_k}$

Pf Same!

(Classical) Type B Stirling Numbers

Def A type B ordered set partition of $\langle n \rangle$, written $P \models \langle n \rangle$, is $P = (B_0, B_1, \dots, B_{2k})$ s.t. $\{B_0, \dots, B_k\} \vdash \langle n \rangle$ s.t. $0 \in B_0$ and $B_{2i} = -B_{2i-1}$ for $i \geq 1$

- Set $S_B^o(\langle n \rangle, k) = \{P \models \langle n \rangle \text{ where } P \text{ has } 2k+1 \text{ blocks}\}$
- The type B ordered Stirling numbers of the second kind are

$$\begin{aligned} S_B^o(n, k) &= 2^k k! S_B(n, k) \\ &= \# S_B^o([n], k) \end{aligned}$$

Ex $\bar{1}02/\bar{4}/\bar{1}\bar{3} \in S_B^o(\langle 4 \rangle, 2)$ and $S_B^o(5, 3) = 8 \cdot 3! \cdot 170 = 8160$

New Type B q -Stirling Numbers

Def (S.-Wallach '21; Bagno-Garber '22+)

The type B \underline{q} -Stirling numbers of the second kind
are defined recursively by

$$\left. \begin{aligned} S_B[n, k] &= S_B[n-1, k-1] + [zk+1]S_B[n-1, k] \\ S_B[0, k] &= \delta_{0k} \end{aligned} \right] \Rightarrow S_B[n, k] \Big|_{q=1} = S_B[n, k]$$

Why? Type B super coinvariant algebra Hilbert series data;
later

New Type B q -Stirling Numbers

Ex

	$k=0:$	$k=1:$	$k=2:$	$k=3:$	\dots
$n=0:$	1	0	0	0	\dots
$n=1:$	1	1	0	0	\dots
$n=2:$	1	$2+q+q^2$	1	0	\dots
$n=3:$	1	$3+3q+4q^2+2q^3+q^4$	$3+2q+2q^2+q^3+q^4$	1	\dots

$$= (2+q+q^2) + (1+q+q^2+q^3+q^4) \cdot 1 = S_B(3,2)$$

New Type B q-Stirling Numbers

Thm $S_B[n, k] = h_{n-k}([1], [3], \dots, [2k+1]) = \sum_{\gamma_0 + \dots + \gamma_k = n-k} [1]^{\gamma_0} [3]^{\gamma_1} \dots [2k+1]^{\gamma_k}$

Pf Same!

Thm (S.-Sagan '22; Bagno-Gorber '22+)

There are statistics $\text{stat}: S[\{n\}, k] \rightarrow \mathbb{Z}_{\geq 0}$ such that

$$S_B[n, k] = \sum_{p \in S_B(\langle n \rangle, k)} q^{\text{stat}(p)}$$

New Type B q -Stirling Numbers

- Our inversion-style type B set partition statistic is...

Def (S.-Sagan '22)

For $P \vdash \langle n \rangle$, set

$$\text{inv}(P) = \#\{(a, B_j) : a \in B_i \text{ for some } i \leq j \text{ and } a \geq \min|B_j|\}$$

Then $\sum_{P \in S_B(\langle n \rangle, k)} q^{\text{inv}(P)} = S_B[n, k].$

Ex $\bar{2}0\bar{2}/\bar{1}\bar{3}/\bar{4}$ has $\text{inv} = 3$

New Type B q -Stirling Numbers

Def] The type B ordered q -Stirling numbers of the second kind are

$$\begin{aligned} S_B^o[n,k] &= [2k]!! \cdot S_B[n,k] \\ &= [2]^k [k]_{q^2}! \cdot S_B[n,k] \end{aligned}$$

where the q -double factorial is

$$[n]!! = \begin{cases} [n][n-2]\dots[2] & \text{if } n \geq 2 \text{ is even} \\ [n][n-2]\dots[1] & \text{if } n \geq 1 \text{ is odd} \\ 1 & \text{if } n=0 \text{ or } n=-1 \end{cases}$$

- In Fact, $\sum_{P \in S_B^o(n,k)} q^{\text{inv}(P)} = S_B^o[n,k]!$ We have a maj version too.

Enumerative Results

Thm (classical) $x^n = \sum_{k=0}^n S(n,k) x(x-1)\dots(x-k+1)$

(Bala '15, Bagno-Biagioli-Garter '19)

$$x^n = \sum_{k=0}^n S_B(n,k) (x-1)(x-3)\dots(x-2k+1)$$

(S.-Sagan '22)

$$x^n = \sum_{k=0}^n S_B[n,k] (x-[1])(x-[3])\dots(x-[2k-1])$$

Enumerative Results

Thm (classical)

$$\sum_{n=0}^{\infty} S(n,k) x^n = \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}$$

(Bagnoli-Garter '22+)

$$\sum_{n=0}^{\infty} S_B(n,k) x^n = \frac{x^k}{(1-x)(1-3x)\cdots(1-(2k+1)x)}$$

(S.-Sagan '22)

$$\sum_{n=0}^{\infty} S_B[n,k] x^n = \frac{x^k}{(1-[1]x)(1-[3]x)\cdots(1-[2k+1]x)}$$

Enumerative Results

Thm (classical) $\sum_{n=0}^{\infty} S^o(n,k) \frac{x^n}{n!} = (e^x - 1)^k$

(See S.-Sagan '22)

$$\sum_{n=0}^{\infty} S^o[n,k] \frac{x^n}{[n]!} = \sum_{i=0}^k \frac{(-1)^{k-i}}{q^{(i)+i(k-i)}} \begin{bmatrix} k \\ i \end{bmatrix} \exp_q([i]x)$$

where $\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}$ and $\exp_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]!}$

- Proof uses q -differential equations, $D_q f(x) = \frac{f(x) - f(qx)}{x - qx}$

Enumerative Results

Thm (?)

$$\sum_{n \geq 0} S_B^o(n, k) \frac{x^n}{[n]!} = e^x (e^{2x} - 1)^k$$

(S.-Sagan '22)

$$\sum_{n \geq 0} S_B^o(n, k) \frac{x^n}{[n]!} = \sum_{i=0}^k \frac{(-1)^{k-i}}{q^{i^2 + (2i+1)(k-i)}} \begin{bmatrix} k \\ i \end{bmatrix}_{q^2} \exp_q([(2i+1)x])$$

- Have additional generating function identities and

first kind versions, e.g.

$$\sum_{n, k \geq 0} c(n, k) \frac{x^n}{[n]!} y^k = \exp_q[-y \log_q[1-x]]$$

Enumerative Results

Thm (Conjectured in S.-Wallach '21, proved in S.-Sagan '22)

$$\sum_{k=0}^n (-q)^{n-k} S^o[n,k] = 1$$

and $\sum_{k=0}^n (-q)^{n-k} S_B^o[n,k] = 1$

- Algebraic and combinatorial proofs
- Conjectured description of $\sum_{k=0}^n (-q^2)^{n-k} S^o[n,k] - 1$ and more

coinvariant Algebras

Thm (Newton) $\mathbb{Q}[x_1, \dots, x_n]^{S_n} = \mathbb{Q}[e_1, \dots, e_n]$ where $e_k = \sum_{\substack{1 \leq i_1 < \dots < i_k \leq n}} x_{i_1} \cdots x_{i_k}$
and $\sigma(x_i) = x_{\sigma(i)}$

elementary symmetric
polynomial

Thm (Hilbert) $\langle \mathbb{Q}[x_1, \dots, x_n]_+^{S_n} \rangle = \langle e_1, \dots, e_n \rangle$

Def The (coinvariant) algebra of S_n is

$$R_n = \frac{\mathbb{Q}[x_1, \dots, x_n]}{\langle e_1, \dots, e_n \rangle}$$

Coinvariant Algebras

singular cohomology

$$\boxed{\text{Thm}} \quad (\text{Borel}) \quad R_n \cong H^*(\text{Fl}_n)$$

complete flag manifold

$$\boxed{\text{Thm}} \quad (\text{Chevalley}) \quad R_n \cong \mathbb{Q}S_n$$

$\Rightarrow \dim R_n = n!$

$$\boxed{\text{Thm}} \quad (\text{Artin}) \quad \left\{ x_1^{\alpha_1} \cdots x_n^{\alpha_n} : 0 \leq \alpha_i \leq i \right\} \text{ descends to a basis for } R_n$$

$$\boxed{\text{Cof}} \quad \text{Hilb}(R_n; q) = \sum_{d=0} \dim(R_n)_d \cdot q^d = 1 \cdot (1+q) \cdot (1+q+q^2) \cdots \underbrace{(1+q+\cdots+q^{n-1})}_{[n]_q!}$$

$$[n]_q!$$

Super (co)invariant Algebras

- Superspace is $\boxed{\mathbb{Q}[x_1, \dots, x_n, \theta_1, \dots, \theta_n]}$ where $\theta_i \theta_j = -\theta_j \theta_i$ anti-commute
 $\text{Sym}(x_1, \dots, x_n) \otimes \Lambda(\theta_1, \dots, \theta_n)$
(and $x_i \theta_j = \theta_j x_i$, $x_i x_j = x_j x_i$)
- S_n acts diagonally: $\sigma(x_i) = x_{\sigma(i)}$, $\sigma(\theta_i) = \theta_{\sigma(i)}$

Super Coinvariant Algebras

Def The super coinvariant algebra of S_n is

$$SR_n = \langle Q[x_n, \theta_n] / \langle Q[x_n, \theta_n]_+^{S_n} \rangle \rangle$$

Conj (Zabrocki '19) $\text{Hilb}(SR_n; q, z) = \sum_{k=0}^n S^{\circ}[n, k] z^{n-k}$

Then (Rhoades-Wilson '23+) Hilbert series is true!

Super (coinvariant) Algebras

- Can define SR_G for any pseudoreflection group G .

Conj (Walbach-S. '21)

$$\text{Hilb}(SR_{B_n}; q, z) = \sum_{k=0}^n S_B^{\circ}[n, k] z^{n-k}$$

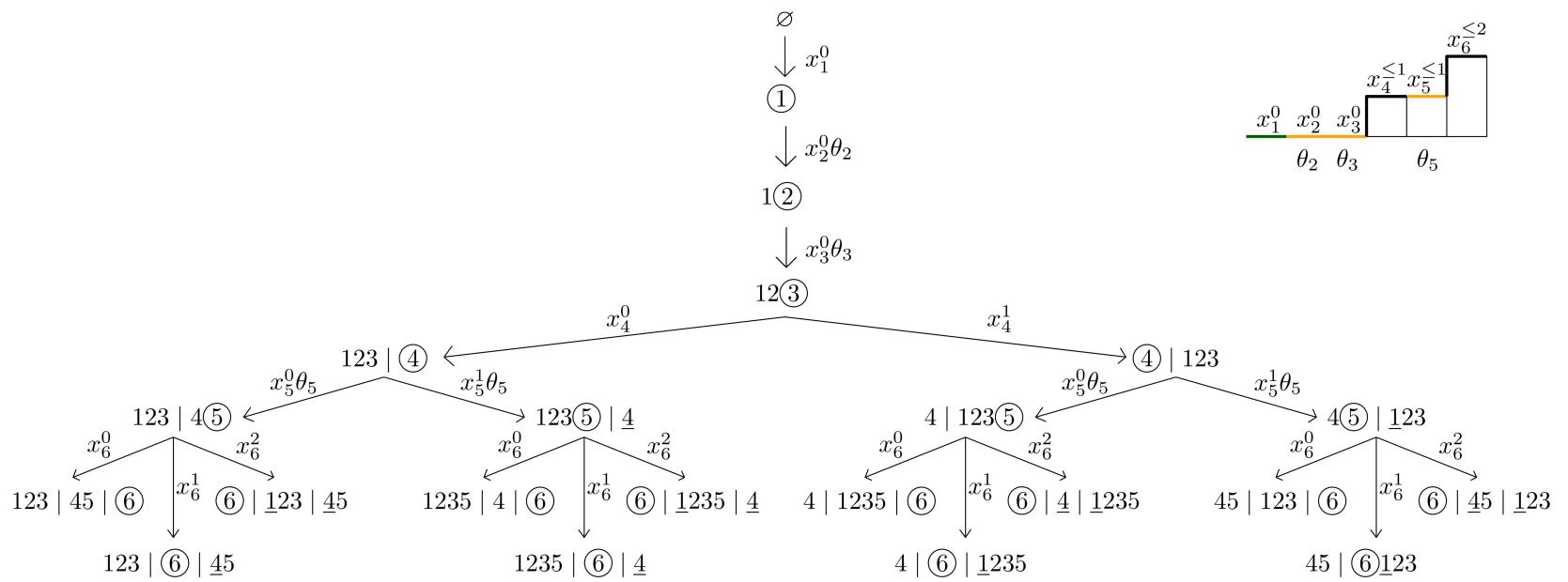
- Alternating sum identities arise from...

Thm (Walbach-S.) $\text{Hilb}(SR_G; q, -q) = 1$

- Have $\text{Hilb}(SR_G; q, -q^{e_i^*}) - 1 = \chi(SR_G, d_i)$ for "generalized exterior derivatives" d_i

Super Coinvariant Algebras

- Conjectured "Artin"-style basis in types A and B
 - naturally motivates inv on $P \models [n]$ and $P \models \langle n \rangle$!



THANKS!