Super coinvariants beyond type A

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Outline

I. Super coinvariant algebras of complex reflection groups

II. Semi-invariants, degree bounds, and operator conjectures

III. Coxeter-like complexes

Super Coinvariant Algebras

- Superspace is $\underbrace{\mathbb{Q}[x_1, \dots, x_n, \theta_1, \dots, \theta_n]}_{Sym(x_1, \dots, x_n) \otimes \Lambda(\theta_1, \dots, \theta_n)}$ where $\theta_i \theta_j = -\theta_j \theta_i$ anti-commute (and $x_i \theta_j = \theta_j x_i$, $x_i x_j = x_j x_i$)

- S_n acts diagonally: $\sigma(x_i) = x_{\sigma(i)}$, $\sigma(\theta_i) = \theta_{\sigma(i)}$

- Think of θ variables as differential forms $\theta_i = dx_i$,

$$\theta_i \theta_j = dx_i \wedge dx_j$$

Super Coinvariant Algebras

- The exterior derivative is

$$d = \sum_{i=1}^{n} \partial_{x_i} dx_i \in \mathrm{End}(\mathbb{Q}[\underline{x}_n, d\underline{x}_n])$$

<u>Thm</u> (Solomon)

$$\langle \mathbb{Q}[\underline{x}_n, d\underline{x}_n]^{S_n}_{+} \rangle = \langle e_1, \ldots, e_n, de_1, \ldots, de_n \rangle$$

Super Coinvariant Algebras

Def The <u>super coinvariant algebra</u> of S_n is

$$\underline{SR_n} = \mathbb{Q}[\underline{x}_n, \underline{\theta}_n] / \langle \mathbb{Q}[\underline{x}_n, \underline{\theta}_n]_+^{S_n} \rangle.$$

Thm (Conjectured by Zabrocki '19 and Fields Group; proved by Wilson-Rhoades '23+)

$$\mathrm{Hilb}(SR_n; q, z) = \sum_{k=1}^{n} [k]! \underbrace{S[n,k]}_{q\text{-Stirlings of the second kind}} z^{n-k}$$

Complex Reflection Groups

Def A complex reflection group is a finite subgroup of $GL(\mathbb{C}^n)$ generated by pseudoreflections g s.t. $\operatorname{codim}(\operatorname{fix}(g))=1$ and $\exists m$ s.t. $g^m=1$.

Thm (Shephard-Todd '54)

The irreducible complex reflection groups fall into one infinite family plus 34 exceptional groups.

Complex Reflection Groups

Ex S_n = permutation matrices $(n \geq 1)$

B_n = signed permutation matrices

D_n = signed permutation matrices with evenly many -1's

$G(m,1,n)$ = pseudopermutation matrices with non-zero entries $\zeta \in \mathbb{C}$ s.t. $\zeta^m = 1$ $(m \geq 1)$

$G(m,p,n)$ = index p subgroup of $G(m,1,n)$ s.t. $\left(\prod \zeta_i\right)^{\frac{m}{p}} = 1$ $(p \mid m)$

Note $S_n = G(1,1,n)$ $B_n = G(2,1,n)$ $D_n = G(2,2,n)$

G-Super Coinvariant Algebras

Def The super coinvariant algebra of G is

$$SR_G = \underbrace{\mathbb{C}[\underline{x}_n, \underline{\theta}_n]}_{\substack{\text{natural } G\text{-action} \\ \text{on } Sym((\mathbb{C}^n)^*) \otimes Ext((\mathbb{C}^n)^*)] \text{ bi-graded}}} / \langle \mathbb{C}[\underline{x}_n, \underline{\theta}_n]^G_+ \rangle .$$

Q What is the bigraded irreducible decomposition of SR_G?

Exactness Works!

Thm (Wallach–S. '21)

The exterior derivative complex of SR_G,

$$0 \to \mathbb{C} \to \underbrace{SR_G^0}_{=R_G} \xrightarrow{d} \underbrace{SR_G^1}_{\theta\text{-degree } 1} \xrightarrow{d} \cdots \xrightarrow{d} SR_G^r \to 0,$$

is exact.

Cor $\mathrm{Hilb}(SR_G; q, -q) = \sum_{k=0}^{r} (-q)^{r-k}\, \mathrm{Hilb}(SR_G^k; q) = 1$

G-semi-invariants

Def Let $\chi : G \to \mathbb{C}^*$ be a 1-dimensional character of G.
The χ-semi-invariants of a G-module V are

$$V^\chi = \{ v \in V : g \cdot v = \chi(g) v \}.$$

Ex
- $\chi(g) = 1, \forall g \in G$ recovers the G-invariants V^G
- $\chi(\sigma) = \mathrm{sgn}(\sigma)$ in type A with $V = \mathbb{C}[\underline{x}_n]$ recovers the alternating polynomials
- $\chi(g) = \det_{\mathbb{C}^n}(g)$ or $\chi(g) = \det_{\mathbb{C}^n}(g)^j$ for fixed j

G-semi-invariants

Fact For all G and χ, there is a Vandermondian Δ_χ s.t.

$$\mathbb{C}[\underline{x}_n]^\chi = \mathbb{C}[\underline{x}_n]^G \Delta_\chi .$$

- Let $\Delta_G = \Delta_{\det(\mathbb{C}^n)}$. We have $\Delta_\chi \mid \Delta_G$.

Thm (Wallach–S. '21) Suppose $(\mathbb{C}^n)^G = 0$.

Then

$$\mathrm{Hilb}(SR_G^{\det}; q, z) = q^\alpha \prod_{i=1}^{\wedge} (z + q^{e_i^*})$$

- Comes with an explicit basis built from Δ_G "top-down"
- Holds in more generality than stated here.

Degree Bounds

Thm (Wallach–S. '23+) For $G(m,1,n)$,

$$\underbrace{SR^{i,k}_{G(m,1,n)}}_{\substack{x\text{-degree } i \\ \theta\text{-degree } k}} \neq 0 \iff i + k + m\binom{k}{2} \leq m\binom{n}{2} + (m-1)n$$

Thm (Wallach–S. '23+) For $G(m,p,n)$ with $p \neq m$ or $p = 1$,

$$\bigoplus_{i+k=l} SR^{i,k}_{G(m,p,n)} \neq 0 \iff 0 \leq l \leq m\binom{n}{2} + n\left(\frac{m}{p} - 1\right)$$

- Conjectured to hold for <u>all</u> G, where RHS is $\deg \Delta_G$.

G-Super Operator Conjecture

- There is a natural notion of G-super harmonics SH_G. These are canonical coset representatives of SR_G. See Wallach-S. '21 for details.

- The SH_G are preserved by partial derivatives and generalized exterior derivatives $d_1, \ldots, d_r$ where d_i lowers x-degree by e_i^* and raises θ-degree by 1.

G-Super Operator Conjecture

Conj (Wallach–S. '23+) Let $G = G(m,1,n)$.

Then

$$SH_G = \text{Span}\{\partial_f d_{i_1} \cdots d_{i_k} \Delta_G : f \in \mathbb{C}[\underline{x}_n],\ 1 \le i_1 < \cdots < i_k \le r\}$$

Note
- Proven for $G = S_n$ by Wilson–Rhoades '23+
- Proof technique likely works in general.
- True for rank 2 reflection groups and H_3.
- <u>Fails</u> for F_4, D_4, D_5, and all $G(m,p,n)$ aside from $G(m,1,n)$ (or <u>maybe</u> D_n, $n \ge 6$)

Coxeter Complexes

<u>Conj</u> (Fields Group)

Forgetting the x-grading,

$SR_n \cong$ the type A Coxeter complex

up to a small twist

as S_n-modules.

- Recall that every <u>reflection</u> group G has a Coxeter complex.

Coxeter Complexes

Q True for all such G?

A
- Sadly, no. Fails for (at least) F_4
- Conjectured to hold for A_n, B_n, possibly D_n

Note See Sagan-S. for combinatorics inspired by this conjecture: type B q-Stirlings and beyond

Beyond Coxeter Complexes

- Coxeter complexes are generalized by the Milnor fiber complex of a Shephard group.
- Coxeter complexes are topologically spheres, and Milnor fiber complexes are wedges of spheres.
- However, $\mathrm{Hilb}(SR_G; q, -q) = 1$ suggests any complex associated to G is a sphere, even for $G(m,1,n)$!

Beyond Coxeter Complexes

- Haglund-Rhoades-Shimozono '18 introduced generalized coinvariant algebras $R_{n,k}$, which conjecturally correspond to SR_n^{n-k}. They are provably governed by the combinatorics of the Coxeter complex.
- Chan-Rhoades '20 introduced $G(m,1,n)$-generalized coinvariant algebras $R_{n,k}^{(m)}$. They are provably governed by related combinatorics.

Beyond Coxeter Complexes

- In Sagan–S., we let

$$S^{\circ}_{CR}[m,n,k] = [k][2k]\cdots[mk]\,h_{n-k}([1],[m+1],\ldots,[km+1]).$$

- We show

$$\mathrm{Hilb}(R^{(m)}_{n,k};q) = \mathrm{rev}_q(S^{\circ}_{CR}[m,n,k])$$

and

$$\sum_k (-q^{m-1})^{n-k} S^{\circ}_{CR}[m,n,k] = [m-1]^n \quad (m>1).$$

- This is consistent with a wedge of $m-1$ spheres.

Beyond Coxeter Complexes

- In Sagan–S., we also let
$$\bar{S}^{\circ}[m,n,k] = [k][2k]\cdots[mk]\,h_{n-k}([m-1],[2m-1],\ldots,[(k+1)m-1]).$$
- We conjecture
$$\mathrm{Hilb}(SR^{n-k}_{G(m,1,n)};q) = \bar{S}^{\circ}[m,n,k]$$
and we show, for $m>1$,
$$\sum_k (-q)^{n-k}\bar{S}^{\circ}[m,n,k] = 1.$$
- This is consistent with a topological sphere.

Beyond Coxeter Complexes

- This suggests a bifurcation, even for $G(m,1,n)$:

Super coinvariants	Generalized coinvariants
Super $G(m,1,n)$-ordered set partition	Chan-Rhoades ordered set partition
Simplicial spheres	Wedges of spheres
(Ongoing research!)	Coxeter/Milnor Fiber complexes

agree for $m \leq 2$, disagree for $m \geq 3$!

Sorry I couldn't be in Georgia!