Type B q-Stirling numbers

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Based on joint work with *Bruce Sagan* (in preparation)
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Slides: https://www.jpswanson.org/talks/2022_Waterloo_q-Stirlings.pdf

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Outline

- Stirling variations
  - Classical type A
  - Classical q-analogue
- Type B q-Stirling numbers
- Enumerative results
- Super coinvariant algebra motivations

Overview

	Type A	Type B	Other Types
Classical	Stirling 1730	Zaslavski '81?	Zaslavski '81?
q-analogue	Carlitz '33 Gould '61	S.-Wallach '21 S.-Sagan '22 Today! Bagno-Garber '22+	S.-Sagan '22+

Variations on unsigned/signed Stirling numbers of the first kind and unordered/ordered Stirling numbers of the second kind

Classical Stirling Numbers

Def The (type A) Stirling numbers of the second kind are defined recursively by

$$S(n,k) = S(n-1,k-1) + kS(n-1,k)$$

$$S(0,k) = \delta_{0k}$$

Ex

(See A008277)

$n=0$: 1 0 0 0 0 0 …

$n=1$: 0 1 0 0 0 0 …

$n=2$: 0 1 1 0 0 0 …

$n=3$: 0 1 3 1 0 0 …

$n=4$: 0 1 7 6 1 0 …

$n=5$: 0 1 15 25 10 1 …

$= 7 + 3 \cdot 6 = S(5,3)$

Classical Stirling Numbers

Def A (type A) set partition of $[n] = \{1, 2, \dots, n\}$, written $P \vdash [n]$,

is

$$P = \{\underbrace{B_1, \dots, B_k}_{\text{blocks of } P}\}$$

s.t.

$$B_1 \cup \cdots \cup B_k = [n]$$

$$B_i \cap B_j = \emptyset \text{ for } i \neq j$$

$$B_i \neq \emptyset \text{ for all } i$$

Ex $\{\{1,3\},\{5,6\},\{2\}\} \vdash [6]$

$$\Updownarrow$$

$$\underbrace{13/2/56}_{\text{write in increasing order, say}} \in S([6],3)$$

- Set $S([n],k) = \{P \vdash [n] : P \text{ has } k \text{ blocks}\}$

Classical Stirling Numbers

Thm $S(n,k) = \# S([n],k)$

Pf Remove n: if singleton, k-1 blocks, otherwise k copies with k blocks

Thm $S(n,k) = h_{n-k}(1,2,\dots,k) = \sum_{\gamma_1+\dots+\gamma_k=n-k} 1^{\gamma_1}2^{\gamma_2}\cdots k^{\gamma_k}$

Pf $h_{n-k}(x_1,\dots,x_k) = h_{n-k}(x_1,\dots,x_{k-1}) + x_k h_{n-k-1}(x_1,\dots,x_k)$ with $x_i=i$

Ex $S(5,3) = 1^2 2^0 3^0 + 1^0 2^2 3^0 + 1^0 2^0 3^2 + 1^1 2^1 3^0 + 1^1 2^0 3^1 + 1^0 2^1 3^1$

$= 1 + 4 + 9 + 2 + 3 + 6 = 25$

Classical Stirling Numbers

Def A (type A) ordered set partition of $[n]$, written $P \vDash [n]$, is $P = (B_1, \ldots, B_k)$ s.t. $\{B_1, \ldots, B_k\} \vdash [n]$.

- Set $S^o([n], k) = \{P \vDash [n]$ where P has k blocks$\}$
- The (type A) ordered Stirling numbers of the second kind are

$$S^o(n,k) = k!\,S(n,k)$$
$$= \# S^o([n],k)$$

Ex $13/56/2 \in S^o([6],3)$ and $S^o(5,3) = 3!\,S(5,3) = 6 \cdot 25 = 150$

Classical q-Stirling Numbers

Def The q-integers are defined by

$$[k] = 1+q+\cdots+q^{k-1} = \frac{1-q^k}{1-q}$$

Have $\lim_{q\to 1}[k]=k$.

Def (Carlitz '33; Gould '61)

The (type A) q-Stirling numbers of the second kind are defined recursively by

$$\left.\begin{array}{l} S[n,k] = S[n-1,k-1]+[k]S[n-1,k] \\ S[0,k] = \delta_{0k} \end{array}\right] \Rightarrow S[n,k]\Big|_{q=1} = S(n,k)$$

Classical q-Stirling Numbers

Ex

	$k=0$:	$k=1$:	$k=2$:	$k=3$:	$k=4$:	$k=5$:
$n=0$:	1	0	0	0	0	0 ...
$n=1$:	0	1	0	0	0	0 ...
$n=2$:	0	1	1	0	0	0 ...
$n=3$:	0	1	$2+q$	1	0	0 ...
$n=4$:	0	1	$3+3q+q^2$	$3+2q+q^2$	1	0 ...
$n=5$:	0	1	$4+6q+4q^2+q^3$	$\boxed{6+8q+7q^2+3q^3+q^4}$	$4+3q+2q^2+q^3$	1 ...

$$=(3+3q+q^2)+(1+q+q^2)(3+2q+q^2)=S[5,3]$$

Classical q-Stirling Numbers

Thm $S[n,k] = h_{n-k}([1],[2],\dots,[k]) = \sum_{\gamma_1+\cdots+\gamma_k=n-k} [1]^{\gamma_1}[2]^{\gamma_2}\cdots[k]^{\gamma_k}$

Pf Same!

Thm (Milne '82; Garsia-Remmel '86; Wachs-White '91; ...)

There are statistics $\mathrm{stat}: S([n],k) \to \mathbb{Z}_{\geq 0}$ such that

$$S[n,k] = \sum_{p \in S([n],k)} q^{\mathrm{stat}(p)}$$

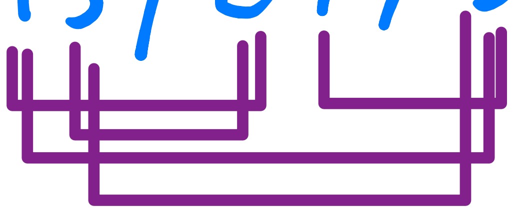
Classical q-Stirling Numbers

- For instance, set

$$\mathrm{inv}(P) = \#\{(a, B_j) : a \in B_i \text{ for some } i<j \text{ and } a > \min B_j\}$$

Then $\displaystyle\sum_{P \in S([n],k)} q^{\mathrm{inv}(P)} = S[n,k]$

Ex 145/27/36 has inv = 5



Classical q-Stirling Numbers

Def The (type A) ordered q-Stirling numbers of the second kind are

$$S^o[n,k] = [k]!\,S[n,k]$$

(The q-Factorial is $[k]! = [k][k-1]\cdots[1]$)

- Encountered in Carlitz '33 originally!
- Statistics and additional work by Steingrímson '01, Haglund-Rhoades-Shimozono '18, ...

e.g. $$\sum_{p \in S^o([n],k)} q^{\mathrm{inv}(p)} = S^o[n,k]$$

Classical Type B Stirling Numbers

Def The type B Stirling numbers of the second kind are defined recursively by

$$S_B(n,k) = S_B(n-1,k-1) + (2k+1)S_B(n-1,k)$$

where $S_B(0,k) = \delta_{0k}$

Ex

(See A039755)

n=0:	1	0	0	0	0	0	…
n=1:	1	1	0	0	0	0	…
n=2:	1	4	1	0	0	0	…
n=3:	1	13	9	1	0	0	…
n=4:	1	40	58	16	1	0	…
n=5:	1	121	330	$\textcircled{170}$	25	1	…

$= 58 + 7\cdot 16 = S_B(5,3)$

Classical Type B Stirling Numbers

Def A type B set partition of $\langle n\rangle = \{-n, -n+1, \dots, -1, 0, 1, \dots, n-1, n\}$, written $P \vdash \langle n\rangle$, is a set partition of $\langle n\rangle$ such that

$$B \in P \ni -B = \{-b : b \in B\} \in P$$

and $B \neq -B$ unless $0 \in B$.

Ex $\{\{4\}, \{1,-3\}, \{-2,0,2\}, \{-4\}, \{-1,3\}\}$

$$\updownarrow$$

$$\bar{2}\,0\,2 \,/\, \begin{smallmatrix}\bar{1}3\\ \textcircled{1}\bar{3}\end{smallmatrix} \,/\, \begin{smallmatrix}\bar{4}\\ \textcircled{4}\end{smallmatrix} \in S_B(\langle 4\rangle, 2)$$

increasing minima, say, after negative pair

Why? Naturally indexes type B_n hyperplane arrangement intersection lattice

- Set $S_B(\langle n\rangle, k) = \{P \vdash \langle n\rangle : P \text{ has } 2k+1 \text{ blocks}\}$

Classical Type B Stirling Numbers

Thm $S_B(n,k) = \# S_B(\langle n \rangle, k)$

Pf Remove ±n: if singleton, 2k-1 blocks, otherwise 2k+1 copies with 2k+1 blocks

Thm $S_B(n,k) = h_{n-k}(1,3,\dots,2k+1) = \sum_{\gamma_0+\cdots+\gamma_k=n-k} 1^{\gamma_0} 3^{\gamma_1} \cdots (2k+1)^{\gamma_k}$

Pf Same!

Classical Type B Stirling Numbers

Def A type B ordered set partition of $\langle n\rangle$, written $P \vDash \langle n\rangle$, is $P=(B_0, B_1, \dots, B_{2k})$ s.t. $\{B_0, \dots, B_k\} \vdash \langle n\rangle$ s.t. $0 \in B_0$ and $B_{2i} = -B_{2i-1}$ for $i \geq 1$

- Set $S_B^o(\langle n\rangle, k) = \{P \vDash \langle n\rangle$ where P has $2k+1$ blocks$\}$
- The type B ordered Stirling numbers of the second kind are

$$S_B^o(n,k) = 2^k k!\, S_B(n,k)$$
$$= \# S_B^o([n], k)$$

Ex $\bar{2}02 / \begin{smallmatrix}\bar{4}\\4\end{smallmatrix} / \begin{smallmatrix}1\bar{3}\\\bar{1}3\end{smallmatrix} \in S_B^o(\langle 4\rangle, 2)$ and $S_B^o(5,3) = 8 \cdot 3! \cdot 170 = 8160$

New Type B q-Stirling Numbers

Def (S.-Wallach '21; Bagno-Garber '22+)

The type B q-Stirling numbers of the second kind are defined recursively by

$$\left.\begin{aligned} S_B[n,k] &= S_B[n-1,k-1] + [2k+1]\,S_B[n-1,k] \\ S_B[0,k] &= \delta_{0k} \end{aligned}\right] \Rightarrow S_B[n,k]\big|_{q=1} = S_B(n,k)$$

Why? Type B super coinvariant algebra Hilbert series data; later

New Type B q-Stirling Numbers

Ex

	k=0:	k=1:	k=2:	k=3:	
n=0:	1	0	0	0	...
n=1:	1	1	0	0	...
n=2:	1	$2+q+q^2$	1	0	...
n=3:	1	$3+3q+4q^2+2q^3+q^4$	$\boxed{3+2q+2q^2+q^3+q^4}$	1	...

$$= (2+q+q^2) + (1+q+q^2+q^3+q^4)\cdot 1 = S_B[3,2]$$

New Type B q-Stirling Numbers

Thm $S_B[n,k] = h_{n-k}([1],[3],\dots,[2k+1]) = \sum_{\gamma_0+\dots+\gamma_k=n-k} [1]^{\gamma_0}[3]^{\gamma_1}\cdots[2k+1]^{\gamma_k}$

Pf Same!

Thm (S.-Sagan '22; Bagno-Garber '22+)

There are statistics $\mathrm{stat}: S([n],k) \to \mathbb{Z}_{\geq 0}$ such that

$$S_B[n,k] = \sum_{p\in S_B(\langle n\rangle,k)} q^{\mathrm{stat}(p)}$$

New Type B q-Stirling Numbers

- Our inversion-style type B set partition statistic is...

Def (S.-Sagan '22)

For $P \vdash \langle n \rangle$, set

$$\mathrm{inv}(P) = \#\{(a, B_j) : a \in B_i \text{ for some } i < j \text{ and } a \geq \min|B_j|\}$$

- Then $\displaystyle\sum_{P \in S_B(\langle n \rangle, k)} q^{\mathrm{inv}(P)} = S_B[n,k]$.

Ex $\bar{2}0\bar{2} \Big/ \begin{bmatrix} \bar{1}3 \\ 1\bar{3} \end{bmatrix} \Big/ \genfrac{}{}{0pt}{}{\bar{4}}{4}$ has $\mathrm{inv} = 3$

New Type B q-Stirling Numbers

Def The type B ordered q-Stirling numbers of the second kind are

$$S^{0}_{B}[n,k] = [2k]!!\, S_{B}[n,k]$$

$$= [2]^{k}[k]_{q^{2}}!\, S_{B}[n,k]$$

where the q-double factorial is

$$[n]!! = \begin{cases} [n][n-2]\cdots[2] & \text{if } n \geq 2 \text{ is even} \\ [n][n-2]\cdots[1] & \text{if } n \geq 1 \text{ is odd} \\ 1 & \text{if } n=0 \text{ or } n=-1 \end{cases}$$

- In Fact, $\sum_{P \in S^{0}_{B}(\langle n \rangle, k)} q^{inv(P)} = S^{0}_{B}[n,k]$! We have a maj version too.

Enumerative Results

Thm (Classical)
$$x^n = \sum_{k=0}^{n} S(n,k)\, x(x-1)\cdots(x-k+1)$$

(Bala '15, Bagno-Biagioli-Garber '19)
$$x^n = \sum_{k=0}^{n} S_B(n,k)(x-1)(x-3)\cdots(x-2k+1)$$

(S.-Sagan '22)
$$x^n = \sum_{k=0}^{n} S_B[n,k](x-[1])(x-[3])\cdots(x-[2k-1])$$

Enumerative Results

Thm (Classical)

$$\sum_{n\geq 0} S(n,k)\,x^n = \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}$$

(Bagno-Garber '22+)

$$\sum_{n\geq 0} S_B(n,k)\,x^n = \frac{x^k}{(1-x)(1-3x)\cdots(1-(2k+1)x)}$$

(S.-Sagan '22)

$$\sum_{n\geq 0} S_B[n,k]\,x^n = \frac{x^k}{(1-[1]x)(1-[3]x)\cdots(1-[2k+1]x)}$$

Enumerative Results

Thm (Classical) $\sum_{n\geq 0} S^0(n,k)\frac{x^n}{n!} = (e^x-1)^k$

(S.-Sagan '22)

$$\sum_{n\geq 0} S^0[n,k]\frac{x^n}{[n]!} = \sum_{i=0}^{k} \frac{(-1)^{k-i}}{q^{\binom{i}{2}+i(k-i)}} \begin{bmatrix} k \\ i \end{bmatrix} \exp_q([i]x)$$

where $\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}$ and $\exp_q(x) = \sum_{n\geq 0} \frac{x^n}{[n]!}$

- Proof uses q-differential equations, $D_q F(x) = \frac{F(x)-F(qx)}{x-qx}$

Enumerative Results

Thm (?)

$$\sum_{n\geq 0} S_B^0(n,k)\frac{x^n}{n!} = e^x\left(e^{2x}-1\right)^k$$

(S.-Sagan '22)

$$\sum_{n\geq 0} S_B^0[n,k]\frac{x^n}{[n]!} = \sum_{i=0}^{k} \frac{(-1)^{k-i}}{q^{i^2+(2i+1)(k-i)}} \begin{bmatrix} k \\ i \end{bmatrix}_{q^2} \exp_q([2i+1]x)$$

- Have additional generating function identities and first kind versions, e.g. $\sum_{n,k\geq 0} c[n,k]\frac{x^n}{[n]!}y^k = \exp_q[-y\log_q[1-x]]$

Enumerative Results

Thm (Conjectured in S.-Wallach '21, proved in S.-Sagan '22)

$$\sum_{k=0}^{n} (-q)^{n-k} S^{o}[n,k] = 1$$

and

$$\sum_{k=0}^{n} (-q)^{n-k} S^{o}_{B}[n,k] = 1$$

- Algebraic and combinatorial proofs
- Conjectured description of $\sum_{k=0}^{n} (-q^2)^{n-k} S^{o}[n,k] - 1$ and more

Coinvariant Algebras

Thm (Newton) $\mathbb{Q}[x_1,\ldots,x_n]^{S_n} = \mathbb{Q}[e_1,\ldots,e_n]$ where $e_i = \underbrace{\sum_{1\le i_1<\cdots<i_k\le n} x_{i_1}\cdots x_{i_k}}_{\text{elementary symmetric polynomial}}$

and $\sigma(x_i) = x_{\sigma(i)}$

Thm (Hilbert) $\langle \mathbb{Q}[x_1,\ldots,x_n]_+^{S_n} \rangle = \langle e_1,\ldots,e_n \rangle$

Def The coinvariant algebra of S_n is

$$R_n = \frac{\mathbb{Q}[x_1,\ldots,x_n]}{\langle e_1,\ldots,e_n \rangle}$$

Coinvariant Algebras

Thm (Borel) $R_n \cong \overbrace{H^*}^{\text{singular cohomology}}(\underbrace{Fl_n}_{\text{complete flag manifold}})$

Thm (Chevalley) $R_n \cong \mathbb{Q}S_n$

$\Rightarrow \dim R_n = n!$

Thm (Artin) $\{x_1^{\alpha_1} \cdots x_n^{\alpha_n} : 0 \leq \alpha_i < i\}$ descends to a basis for R_n

Cor $$\mathrm{Hilb}(R_n; q) = \sum_{d \geq 0} \dim(R_n)_d \cdot q^d = \underbrace{1 \cdot (1+q) \cdot (1+q+q^2) \cdots \underbrace{(1+q+\cdots+q^{n-1})}_{[n]_q}}_{[n]_q!}$$

Super Coinvariant Algebras

- Superspace is $\underbrace{\mathbb{Q}[x_1,-,x_n,\theta_1,-,\theta_n]}_{\mathrm{Sym}(x_1,-,x_n)\otimes\Lambda(\theta_1,-,\theta_n)}$ where $\theta_i\theta_j=-\theta_j\theta_i$ anti-commute (and $x_i\theta_j=\theta_jx_i$, $x_ix_j=x_jx_i$)

- S_n acts diagonally: $\sigma(x_i)=x_{\sigma(i)}$, $\sigma(\theta_i)=\theta_{\sigma(i)}$

Super Coinvariant Algebras

Def The super coinvariant algebra of S_n is

$$SR_n = \mathbb{Q}[\underline{x}_n, \underline{\theta}_n] / \langle \mathbb{Q}[\underline{x}_n, \underline{\theta}_n]^{S_n}_{+} \rangle$$

Conj (Zabrocki '19) $\mathrm{Hilb}(SR_n; q, z) = \sum_{k=0}^{n} S^{\circ}[n,k]\, z^{n-k}$

Super Coinvariant Algebras

- Can define SR_G for any pseudoreflection group G.

  Conj (Wallach-S. '21)

  $$\mathrm{Hilb}(SR_{B_n}; q, z) = \sum_{k=0}^{n} S^{0}_{B}[n,k]\, z^{n-k}$$

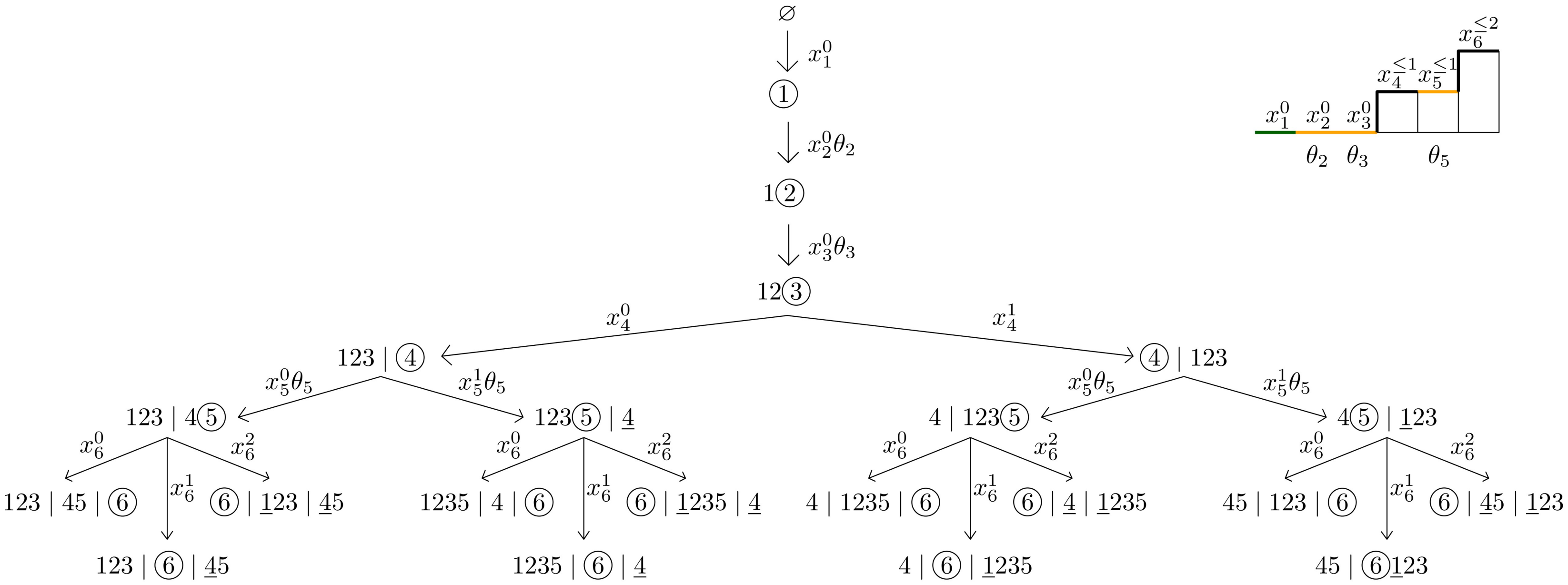
- Alternating sum identities arise from...

  Thm (Wallach-S.) $\mathrm{Hilb}(SR_G; q, -q) = 1$

- Have $\mathrm{Hilb}(SR_G; q, -q^{e_i^*}) - 1 = \chi(SR_G, d_i)$ for "generalized exterior derivatives" d_i

Super Coinvariant Algebras

- Conjectured "Artin"-style basis in types A and B
  - naturally motivates inv on $P \vDash [n]$ and $P \vDash \langle n \rangle$!



$\varnothing$
x_1^0
①
$x_2^0\theta_2$
1②
$x_3^0\theta_3$
12③
x_4^0
x_4^1
123 | ④
④ | 123
$x_5^0\theta_5$
$x_5^1\theta_5$
$x_5^0\theta_5$
$x_5^1\theta_5$
123 | 4⑤
123⑤ | $\underline{4}$
4 | 123⑤
4⑤ | $\underline{1}23$
x_6^0
x_6^1
x_6^2
123 | 45 | ⑥
123 | ⑥ | $\underline{4}5$
⑥ | $\underline{1}23$ | $\underline{4}5$
1235 | 4 | ⑥
1235 | ⑥ | $\underline{4}$
⑥ | $\underline{1}235$ | $\underline{4}$
4 | 1235 | ⑥
4 | ⑥ | $\underline{1}235$
⑥ | $\underline{4}$ | $\underline{1}235$
45 | 123 | ⑥
45 | ⑥$\underline{1}23$
⑥ | $\underline{4}5$ | $\underline{1}23$
x_1^0
x_2^0
x_3^0
$x_4^{\leq 1}$
$x_5^{\leq 1}$
$x_6^{\leq 2}$
θ_2
θ_3
θ_5

THANKS!