q-Stirling numbers in type B

Joshua P. Swanson[1] Bruce Sagan[2]

[1]University of Southern California, presenter [2]Michigan State University

Classical Stirling combinatorics

The *(signless) Stirling numbers of the first kind* are

$$c(n,k) = \#\{\text{permutations of } n \text{ elements with } k \text{ cycles}\}.$$

The *Stirling numbers of the second kind* are

$$S(n,k) = \#\{\text{set partitions of } [n] \text{ with } k \text{ blocks}\}.$$

The *ordered Stirling numbers of the second kind* are

$$S^o(n,k) = \#\{\text{ordered set partitions of } [n] \text{ with } k \text{ blocks}\}.$$

Classical Stirling relations

They can alternatively be defined by the relations $c(0,k) = \delta_{0,k} = S(0,k)$ for $k \in \mathbb{Z}$ and, for $n \geq 1$,

$$\begin{aligned} c(n,k) &= c(n-1,k-1) + (n-1)c(n-1,k) \\ S(n,k) &= S(n-1,k-1) + kS(n-1,k) \\ S^o(n,k) &= k!S(n,k). \end{aligned} \tag{1}$$

They also satisfy

$$\begin{aligned} c(n,k) &= e_{n-k}(1,2,\ldots,n-1) \\ S(n,k) &= h_{n-k}(1,2,\ldots,k) \\ S^o(n,k) &= e_k(1,2,\ldots,k)h_{n-k}(1,2,\ldots,k) \end{aligned} \tag{2}$$

where e and h are *elementary symmetric polynomials* and *complete homogeneous symmetric polynomials*.

They are also transition coefficients between the *falling factorial basis* and *monomial basis*:

$$t(t-1)(t-2)\cdots(t-(n-1)) = \sum_{k=0}^{n} (-1)^{n-k} c(n,k) t^k$$

$$t^n = \sum_{k=0}^{n} S(n,k) t(t-1)(t-2)\cdots(t-(k-1)).$$

Classical Whitney numbers

Let P be a finite ranked poset with minimum $\hat{0}$. Let $\mathrm{Rk}(P,k)$ denote the set of elements at rank k and let $\mu(x) = \mu(\hat{0},x)$ be the *Möbius function* of P. The *Whitney numbers of the second kind* are

$$W(P,k) = \#\mathrm{Rk}(P,k).$$

The *Whitney numbers of the first kind* are

$$w(P,k) = \sum_{x \in \mathrm{Rk}(P,k)} \mu(x).$$

Let Π_n denote the *lattice of set partitions* of $[n]$, where $\rho \leq \sigma$ when ρ refines σ. This is isomorphic to the *intersection lattice* of the type A_{n-1} *hyperplane arrangement* $\{x_i = x_j : 1 \leq i < j \leq n\}$ ordered by reverse containment. Then

$$\begin{aligned} W(\Pi_n,k) &= S(n,n-k) \\ w(\Pi_n,k) &= (-1)^k c(n,n-k). \end{aligned} \tag{3}$$

Type B permutations and set partitions

A *type B_n permutation* is a permutation π of $\pm[n]$ where $\pi(-i) = -\pi(i)$ for all i.

A *type B_n set partition* is a set partition of $\{-n,\ldots,0,\ldots,n\}$ of the form

$$S_0 \mid S_1 \mid \cdots \mid S_{2k}$$

where $0 \in S_0$, $i \in S_0 \Rightarrow -i \in S_0$, and $S_{2i} = -S_{2i-1}$ for $i \geq 1$. For example,

$$\begin{aligned} \pi &= (1,\overline{3},\overline{1},3)\ (\overline{4})(4)\ (2,\overline{5},7)(\overline{2},5,\overline{7})\ (\overline{6},6) \\ \rho &= 0\overline{1}1\overline{3}3\overline{6}6 \mid \overline{4}/4 \mid 2\overline{5}7/\overline{2}5\overline{7}. \end{aligned}$$

Here $(2,\overline{5},7)(\overline{2},5,\overline{7})$ are *paired cycles*.

For a *type B_n ordered set partition*, we fix an order of each block pair $\{S_{2i}, S_{2i-1}\}$, and also order the block pairs amongst themselves.

Type B Stirling combinatorics

The *type B Stirling numbers of the first kind* are

$$c_B(n,k) = \#\{\text{type } B_n \text{ permutations with } 2k \text{ paired cycles}\}.$$

The *type B Stirling numbers of the second kind* are

$$S_B(n,k) = \#\{\text{type } B_n \text{ set partitions with } 2k+1 \text{ blocks}\}.$$

The *type B ordered Stirling numbers of the second kind* are

$$S_B^o(n,k) = \#\{\text{type } B_n \text{ ordered set partitions with } 2k+1 \text{ blocks}\}.$$

THEOREM: [3] We have

$$\begin{aligned} W(\Pi_{B_n},k) &= S_B(n,n-k) \\ w(\Pi_{B_n},k) &= (-1)^k c_B(n,n-k) \\ \#B(\rho) &= (-1)^{n-k}\mu(\rho). \end{aligned}$$

q-Stirling numbers in type B

The *q-analogue of n* is $[n] = 1 + q + \cdots + q^{n-1}$. Carlitz, Gould, and others have introduced q-analogues of the (type A) Stirling numbers. We introduce the following q-analogues in type B:

DEFINITION: [3]
The *type B q-Stirling numbers of the first kind* are $c_B[0,k] = \delta_{0,k}$ and

$$c_B[n,k] = c_B[n-1,k-1] + [2n-1]c_B[n-1,k].$$

The *type B q-Stirling numbers of the second kind* are $S_B[0,k] = \delta_{0,k}$ and

$$S_B[n,k] = S_B[n-1,k-1] + [2k+1]S_B[n-1,k].$$

The *ordered type B q-Stirling numbers of the second kind* are

$$S_B^o[n,k] = [2k]!!S_B[n,k]$$

where $[2k]!! = [2k][2k-2]\cdots[2]$

Bagno–Garber–Komatsu [1] have independently introduced some of these ideas and results, with q,r-Stirling analogues.

Type B q-Stirling relations

The following specializations follow easily from the recurrences:

THEOREM: [3] We have

$$\begin{aligned} c_B[n,k] &= e_{n-k}([1],[3],\ldots,[2n-1]) \\ S_B[n,k] &= h_{n-k}([1],[3],\ldots,[2k+1]) \\ S_B^o[n,k] &= e_k([2],[4],\ldots,[2k])h_{n-k}([1],[3],\ldots,[2k+1]) \end{aligned} \tag{4}$$

We also prove a variety of generating function identities, including

$$(t-[1])(t-[3])\cdots(t-[2n-1]) = \sum_{k=0}^{n} (-1)^{n-k} c_B[n,k] t^k$$

$$t^n = \sum_{k=0}^{n} S_B[n,k](t-[1])(t-[3])\cdots(t-[2k+1])$$

$$\sum_{n \geq 0} S_B^o[n,k]\frac{x^n}{[n]!} = \frac{1}{q^{k^2}} \sum_{i=0}^{k} (-1)^{k-i} q^{2\binom{k-i}{2}} \begin{bmatrix} k \\ i \end{bmatrix}_{q^2} \exp_q([2i+1]x),$$

We have statistics for which $c_B[n,k]$, $S_B[n,k]$, $S_B^o[n,k]$ are generating functions. For $S_B^o[n,k]$:

$$\mathrm{inv}_B(\rho) = \#\{(s,S_j) : s \in S_i, i < j, s \geq \min|S_j| \tag{5}$$

Algebraic Considerations

Chan–Rhoades [2] introduced *type B generalized coinvariant algebras* $R_{n,k}^{(2)}$.

THEOREM: [3] We have $\mathrm{rev}_q \operatorname{Hilb}(R_{n,k}^{(2)}; q) = S_B^o[n,k]$.

We also have conjectured bases for the *super coinvariant algebras* in types A and B which correspond to $S^o[n,k]$ and $S_B^o[n,k]$. Related to these conjectures, we have proven a conjecture of Swanson–Wallach [5]: $\sum_{k=0}^{n}(-1)^k S_B^o[n,k] = 1$.

Future work

Many conjectures and further avenues of exploration are listed in [3]. For example:

CONJECTURE: [3] For each k,n, the coefficients of $c[n,k]$ and $S[n,k]$ are log-concave (hence unimodal), and the coefficients of $c_B[n,k]$ and $S_B[n,k]$ are *parity log-concave* (hence *parity unimodal*).

We are also working to generalize to complex reflection groups $G(r,p,n)$ [4].

References

[1] Eli Bagno, David Garber, and Takao Komatsu. A q,r-analogue for the Stirling numbers of the second kind of type B. In preparation.

[2] Kin Tung Jonathan Chan and Brendon Rhoades. Generalized coinvariant algebras for wreath products. *Adv. in Appl. Math.*, 120:102060, 61, 2020.

[3] Bruce Sagan and Joshua P. Swanson. q-Stirling numbers in type B. Submitted. arXiv: 2205.14078.

[4] Bruce Sagan and Joshua P. Swanson. Stirling numbers for complex reflection groups. In preparation.

[5] Joshua P. Swanson and Nolan R. Wallach. Harmonic differential forms for pseudo-reflection groups II. Bi-degree bounds. Submitted. arXiv:2109.03407, 2022.