Differential coinvariant algebras

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Outline

- Combinatorial representation theory background
- Higher coinvariant algebras
- Hilbert series identities
- Tanisaki witness relations

Coinvariant Algebras

Thm (Newton) $\mathbb{Q}[x_1,\dots,x_n]^{S_n} = \mathbb{Q}[e_1,\dots,e_n]$ where $e_i = \underbrace{\sum_{1\leq i_1<\dots<i_k\leq n} x_{i_1}\cdots x_{i_k}}_{\text{elementary symmetric polynomial}}$

Thm (Hilbert) $\langle \mathbb{Q}[x_1,\dots,x_n]^{S_n}_+ \rangle = \langle e_1,\dots,e_n \rangle$

Def The coinvariant algebra is

$$R_n = \frac{\mathbb{Q}[x_1,\dots,x_n]}{\langle e_1,\dots,e_n \rangle}$$

Ex (Borel) $R_n \cong \overbrace{H^*}^{\text{singular cohomology}}(\underbrace{Fl_n}_{\text{complete flag manifold}})$

Ex (Chevalley) $R_n \cong \mathbb{Q}S_n$

$\Rightarrow \dim R_n = n!$

Combinatorial Representation Theory

The irreducible $\mathbb{C}$-representations of a finite group G are equinumerous with the conjugacy classes.

Miracle For S_n, there is a canonical bijection!

$$\text{cycle type } \lambda \vdash n \longmapsto \text{Specht module } S^\lambda \in \mathbb{C}\text{-Rep}(S_n)$$

Thm (Lusztig-Stanley)

multiplicity of S^λ in deg. i component of R_n = # standard Young tableaux of shape λ and major index i

Combinatorial Representation Theory

Def The Frobenius characteristic is given by

$$\mathrm{Frob}(S^\lambda) = \underbrace{s_\lambda}_{\text{Schur function}}$$

Ex Lusztig-Stanley says

$$\mathrm{GrFrob}(R_n;q) = \sum_{\lambda \vdash n} \langle R_n^i, S^\lambda \rangle q^i s_\lambda$$

$$= \sum_{T \in SYT(n)} q^{\mathrm{maj}(T)} s_{sh(T)}$$

Diagonal Coinvariants

Def (Garsia-Haiman) The diagonal coinvariant algebra is

$$DR_n = \frac{\mathbb{Q}[\underline{x}_n, \underline{y}_n]}{\langle \mathbb{Q}[\underline{x}_n, \underline{y}_n]^{S_n}_{+} \rangle} \qquad \begin{pmatrix} x_i \mapsto x_{\sigma(i)} \\ y_i \mapsto y_{\sigma(i)} \end{pmatrix}$$

Thm (Haiman) $\mathrm{GrFrob}(DR_n; q,t) = \underbrace{\nabla}_{\text{a modified Macdonald eigenoperator}} e_n$

- Very hard! Proof uses Hilbert scheme of n points in $\mathbb{C}^2$

Super Diagonal Coinvariants

Def (Zabrocki) The super diagonal coinvariant algebra is

$$SDR_n = \frac{\mathbb{Q}[\underline{x}_n, \underline{y}_n, \underline{\theta}_n]}{\langle \mathbb{Q}[\underline{x}_n, \underline{y}_n, \underline{\theta}_n]^{S_n}_{+} \rangle} \qquad \begin{pmatrix} x_i \mapsto x_{\sigma(i)} \\ y_i \mapsto y_{\sigma(i)} \\ \theta_i \mapsto \theta_{\sigma(i)} \end{pmatrix}$$

where $\theta_i\theta_j = -\underline{\theta_j\theta_i}$

Conj (Zabrocki)

$$\underbrace{\boxed{\mathrm{GrFrob}(SDR_n; q,t,z) = \sum_{k=1}^{n} z^{k-1} \Delta'_{e_{n-k}} e_n}}_{\text{Delta conjecture of Haglund-Remmel-Wilson}}$$

- Probably super hard!

Differential Coinvariants

Def (t=0 specialization) The super coinvariant algebra is

$$SR_n = \frac{\mathbb{Q}[\underline{x}_n, \underline{\theta}_n]}{\langle \mathbb{Q}[\underline{x}_n, \underline{\theta}_n]^{S_n}_{+} \rangle}$$

(think $\theta_i = \underline{dx_i}$)

Conj ($t=0$, $s_\lambda \mapsto \dim S^\lambda$ specialization)

$$\mathrm{Hilb}(SR_n; q, z) = \sum_{k=0}^{n-1} \underbrace{[n-k]_q!\, \mathrm{Stir}_q(n, n-k)}_{} z^k$$

q-counts ordered set partitions of $[n]$ with $n-k$ blocks

Differential Alternants

Thm (Wallach-S.) $(t=0,\ \lambda=(1^n))$

$$\mathrm{Hilb}(SR_n^{sgn};q,z)=\prod_{j=1}^{n-1}(z+i)$$

Basis: $\{d_{i_1}\cdots d_{i_k}\Delta_n : 1\leq i_1<\cdots\leq i_k\leq n-1\}$

where $d_i=\sum_{j=1}^{n}\frac{\partial^i}{\partial x_j}\theta_i$ is a generalized exterior derivative

$\Delta_n=\prod_{1\leq i<j\leq n}(x_j-x_i)$ is the Vandermonde determinant

Pseudo-Reflection Semi-Invariants

Thm Generalizes uniformly to all pseudo-reflection groups G:

$$\mathrm{Hilb}(SR_G^{\det}; q, z) = \prod_{i=1}^{\mathrm{rank}(G)} \Big(z + \underset{\substack{\uparrow \\ \text{co-exponents of } G}}{q^{e_i^*}}\Big)$$

- Includes Weyl groups: type A, B/C, D

  Shephard-Todd groups: $G(m,1,n) = C_m \text{ wr } S_n$

  $G(m,p,n)$

Exterior Differentiation

Thm (Wallach-S.) Let SR_G^k = θ-degree k component. Then

$$0 \to K \to R_G \xrightarrow{d} SR_G^1 \xrightarrow{d} \cdots \xrightarrow{d} SR_G^r \xrightarrow{d} 0$$

is exact.

Cor $\mathrm{Hilb}(SR_G; q, -q) = 1$

Cor $t=0, z=-q$ specialization holds

- Proof uses an algebraic Hodge decomposition and Laplacians

Exterior Differentiation

- Have $\underbrace{\chi}_{\text{graded Euler characteristic}}(H(SR_G^*, \underline{d_i}); q) = \mathrm{Hilb}(SR_G; q, q^{-e_i^*}) - 1$

- If e_i^* are distinct (typical), $\chi(H(SR_G^*, d_i); q)$ determine $\mathrm{Hilb}(SR_G; q, z)$

> **Q** Is there a complex homotopic to (SR_n^*, d_i) with $\chi(H(SR_n^*, d_i); q)$ consistent with q-Stirling formula?

Harmonics

- Have positive-definite form on $\mathbb{Q}[\underline{x}_n]$

$$\langle f, g\rangle = (\partial_g f)(0)$$

Def The S_n-harmonics are

$$H_n = \langle e_1, \ldots, e_n\rangle^{\perp} = \{f \in \mathbb{Q}[\underline{x}_n] : \partial_{e_i} f = 0\}$$

- $H_n \xrightarrow{\sim} R_n$ as graded S_n-modules

Harmonic Differential Forms

- Generalizes to S_n-harmonic differential forms

$$SH_n \xrightarrow{\sim} SR_n \quad \text{as bigraded } S_n\text{-modules}$$

Thm (Wallach-S.)

$$SH_n^{sgn} = \mathbb{Q}\{d_{i_1}\cdots d_{i_k}\Delta_n : 1 \le i_1 < \cdots \le i_k \le n-1\}$$

Conj (Wallach-S.)

$$SH_n = \mathbb{Q}[\partial_{x_1}, \ldots, \partial_{x_n}]\{\underline{d_I\Delta_n} : I \subset [n-1]\}$$

- Call $g\cdot f = \partial_g f$ the flip action

A "succinct" formula

- Combining numerous results and conjectures suggests

$$\mathrm{Gr\text{-}Frob}(SR_n; q, z) = \sum_{\mu \vdash n} z^{n-\ell(\mu)} q^{\sum_{i=1}^{\ell(\mu)} (i-1)(\mu_i - 1)} \binom{\ell(\mu)}{m_1(\mu), \ldots, m_n(\mu)}_q \underbrace{\omega Q'_\mu(\underline{x}; q)}_{\text{dual Hall-Littlewood symmetric function}}$$

__where__ $\mathrm{Gr\text{-}Frob}(R_\mu; q) = q^{b(\mu)} Q'_\mu(\underline{x}; q^{-1})$

__where__ $R_\mu = \mathbb{Q}[\underline{x}_n]/I_\mu \quad \left(\cong H^*(\underbrace{\bar{X}_\mu}_{\text{Springer Fiber}})\right)$

__where__ I_μ is a Tanisaki ideal

A potential Filtration

Q Is there a total order $I_1 < I_2 < \cdots$ on $2^{[n-1]}$ and a bijection

$$\Phi_n : \{I \subset [n-1]\} \to \{\overbrace{\alpha \vDash n}^{\text{weak compositions}}\}$$

where the composition factor

$$\sum_{j \leq m} SH_{I_j} \Big/ \sum_{j < m} SH_{I_j}$$

is annihilated precisely by $\mathcal{I}_{\Phi_n(I_m)}$?

A potential Filtration

- If so, expect

$$(*) \qquad \sum_{I\subset[n]} z^{|I|} q^{\binom{n}{2}-\mathrm{sum}(I)} = \sum_{\alpha\vDash n} z^{n-\ell(\alpha)} q^{2b(\alpha)-\binom{\ell(\alpha)}{2}+\mathrm{coinv}(\alpha)}$$

Thm (S.) Have an explicit bijection Φ_n proving $(*)$.

- This approach would give an explicit (though complicated) basis using the Garsia–Procesi monomial basis for R_μ

Tanisaki Witness Relations

- The Tanisaki ideals I_μ are generated by certain $e_i(S)$ for $S \subset \{x_1, \dots, x_n\}$

- Call the $\mathbb{Q}$-linear relations between $\partial_{e_i(\underline{x}_m)} d_I \Delta_n$ Tanisaki witness relations

Ex $\partial_{e_2(\underline{2})} d_2 \Delta_3 = 0$ $\Rightarrow$ $\mathcal{I}_{(2,1)}$ annihilates $SH_{\{2\}} < SH_3'$

$(SH_{\{1\}} + SH_{\{2\}}) / SH_{\{2\}}$

$\partial_{e_2(\underline{2})} d_1 \Delta_3 = \partial_{e_1(\underline{2})} d_2 \Delta_3$

Generic Pieri Rule

Thm (S.) Let $I=\{i_1<\cdots<i_k\}\subset[n-1]$. Then

$$\sum(-1)^d\,\partial_{e_{n-k-d}(\underline{n-1})}\,d_{j_1\cdots j_k}\Delta_n=0$$

where the sum is over all subsets $J=\{j_1<\cdots<j_k\}\subset[n-1]$ where

$$1\le i_1\le j_1<i_2\le j_2<\cdots<i_k\le j_k<n$$

and

$$d=(j_1-i_1)+\cdots+(j_k-i_k).$$

Generic Pieri Rule

Cor $\{n-1\} < \{n-2\} < \cdots < \{1\}$ gives a filtration of SH'_n by $SH_{\{i\}}$'s where the composition factors are annihilated precisely by the Tanisaki ideals $\mathcal{I}_{(2,1^{n-2})}$. Furthermore,

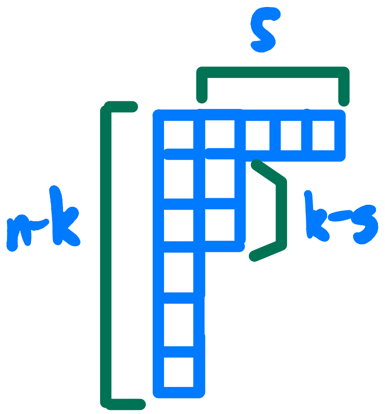
$$\mathrm{GrFrob}(SH'_n;q) = [n-1]_q \, \omega Q'_{(2,1^{n-2})}(\underline{x};q)$$

- These relations are "almost all" in a certain asymptotic sense (k fixed, $n \to \infty$)

Extreme hook relations

Thm (S.) Suppose $I=\{i_1<\cdots<i_k\}\subset[n-1]$ is such that for some $1\le s\le k$,

$$\left.\begin{array}{r} i_1,\dots,i_{k-s+1}\le n-k \\ i_{k-s+2}=n-s+1 \\ i_{k-s+3}=n-s+2 \\ \vdots \\ i_k=n-1. \end{array}\right] \Leftrightarrow \Phi_n(I)=\alpha\vDash n \text{ with } \begin{array}{l} \ell(\alpha)=n-k \\ \bar{\alpha}=(s,1^{k-s}) \end{array}$$



Pick $0\le u\le s$. Then

$$\sum(-1)^d\Delta_s(j_{k-s+1},\dots,j_k)\binom{d+u}{u}\partial_{e_{n-s+d}(\underline{n-s+u})}d_J\Delta_n=0$$

summed over $J=\{j_1<\cdots<j_k\}\subset[n-1]$ for which

$j_1=i_1,\dots,j_{k-s}=i_{k-s}$ and

$d=\mathrm{sum}(J)-\mathrm{sum}(I)\ge\underline{0}$.

Extreme hook relations

Q What are all the Tanisaki witness relations?
Is there a geometric/algebraic/topological interpretation for them and their coefficients?

THANKS!