# DUSTPAN distributions as limit laws for Mahonian statistics on forests

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Based on joint work with
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Papers: [BS20], [BKS20] / arXiv: 2010.12701, 1905.00975
Slides: https://www.jpswanson.org/talks/2021_UCSD.pdf

Presented at University of California, San Diego (UCSD)
Combinatorics Seminar
November 16th, 2021

# Outline

- Background: Irwin-Hall Limits
- Hook Length Formulas
- DUSTPAN Distributions
- DUSTPAN Limits
- Further Directions

# Background: Irwin-Hall Limits

**Def** Let $U[a,b]$ denote a uniform continuous random variable supported on $[a,b]$.

The $M$th Irwin-Hall distribution is

$$\mathcal{IH}_M = \underbrace{U[0,1] + \cdots + U[0,1]}_{M \text{ i.i.d. r.v.'s}}$$

**Notation** Write $X^* = \dfrac{X-\mu}{\sigma}$ for standardized r.v.

**Ex** By CLT, $\mathcal{IH}_M^* \Rightarrow \mathcal{N}(0,1)$, i.e. $\forall t \in \mathbb{R}$ $\lim_{n\to\infty} \mathbb{P}[\mathcal{IH}_M^* \leq t] = \mathbb{P}[\mathcal{N}(0,1) \leq t]$

# Background: Irwin–Hall Limits

**Rem** Our motivation in [BKS20] was to study the distribution of the major index on standard Young tableaux, generalizing earlier work on the major index of permutations and words.

$T =$

| | | | | | |
|---|---|---|---|---|---|
| 1 | 2 | 4 | 7 | 9 | 12 |
| 3 | 6 | 10 | | | |
| 5 | 8 | 11 | | | |

$\in SYT\underbrace{(6,3,3)}_{\text{an integer partition}}$ has

$\mathrm{Des}(T) = \{2, 4, 7, 9, 10\}$

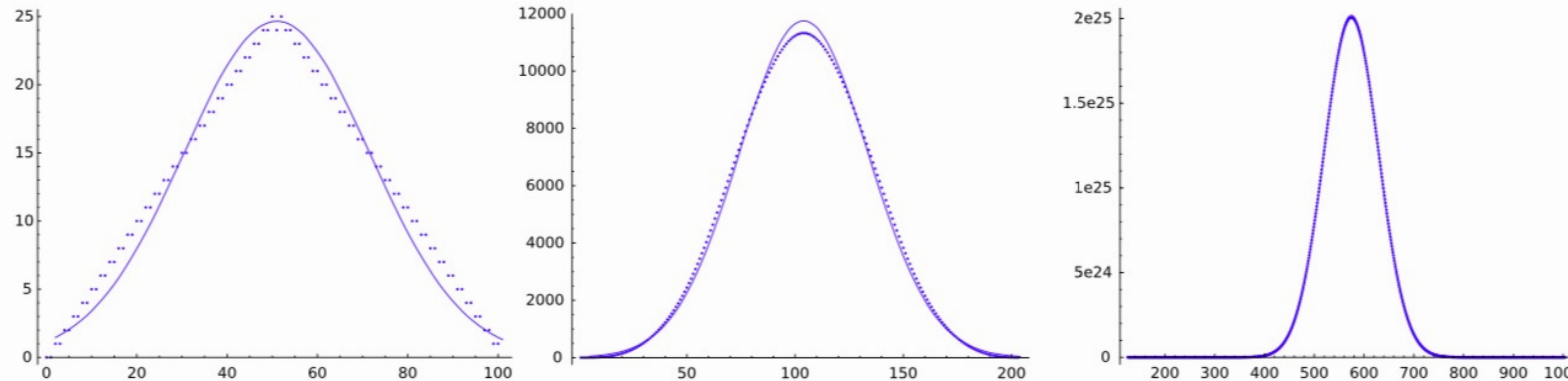
$\mathrm{des}(T) = |\mathrm{Des}(T)| = 5$

$\mathrm{maj}(T) = 2 + 4 + 7 + 9 + 10 = 32$

# Background: Irwin-Hall Limits

**Def** Each integer partition $\lambda$ has a random variable $X_\lambda[\mathrm{maj}]$

**Ex** Distributions of $X_\lambda[\mathrm{maj}]$:




(a) $\lambda = (50, 2)$, $\mathrm{aft}(\lambda) = 2$ (b) $\lambda = (50, 3, 1)$, $\mathrm{aft}(\lambda) = 4$ (c) $\lambda = (8, 8, 7, 6, 5, 5, 5, 2, 2)$, $\mathrm{aft}(\lambda) = 39$

**Fig. 1.** Plots of $\#\{T \in \mathrm{SYT}(\lambda) : \mathrm{maj}(T) = k\}$ as a function of $k$ for three partitions $\lambda$, overlaid with scaled Gaussian approximations using the same mean and variance.

# Background: Irwin-Hall Limits

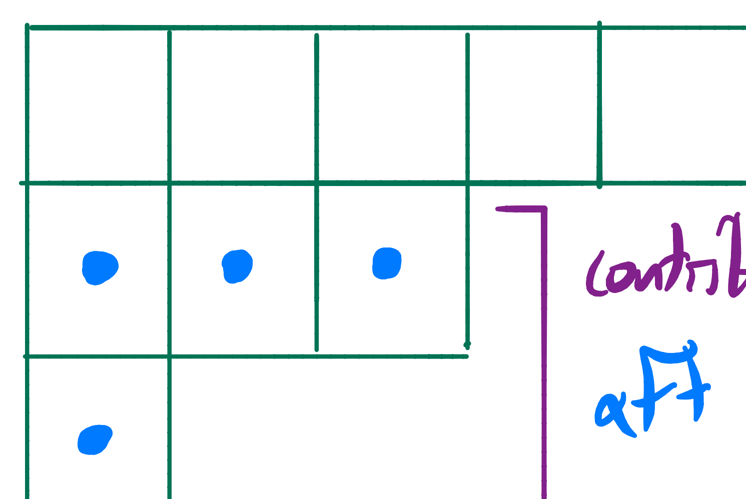
**Note** • $\chi_{(50,2)}[\mathrm{maj}]$ "looks like" $IH_2 = U[0,1] + U[0,1]$

• $\chi_{(8,8,7,6,5,5,5,2,2)}[\mathrm{maj}]$ "looks like" $N(0,1)$!

**Def** Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \vdash n$, let

$$\mathrm{aft}(\lambda) = n - \max\{\lambda_1, k\}.$$

**Ex** $\lambda = (5,3,1) \Rightarrow \mathrm{aft}(\lambda) = 4$



contribute to aft

**Rem** On FindStat as St001214; on OEIS as A338621

# Background: Irwin-Hall Limits

**Thm** (Billey-Konvalinka-S. [BKS20, Thm. 1.7])

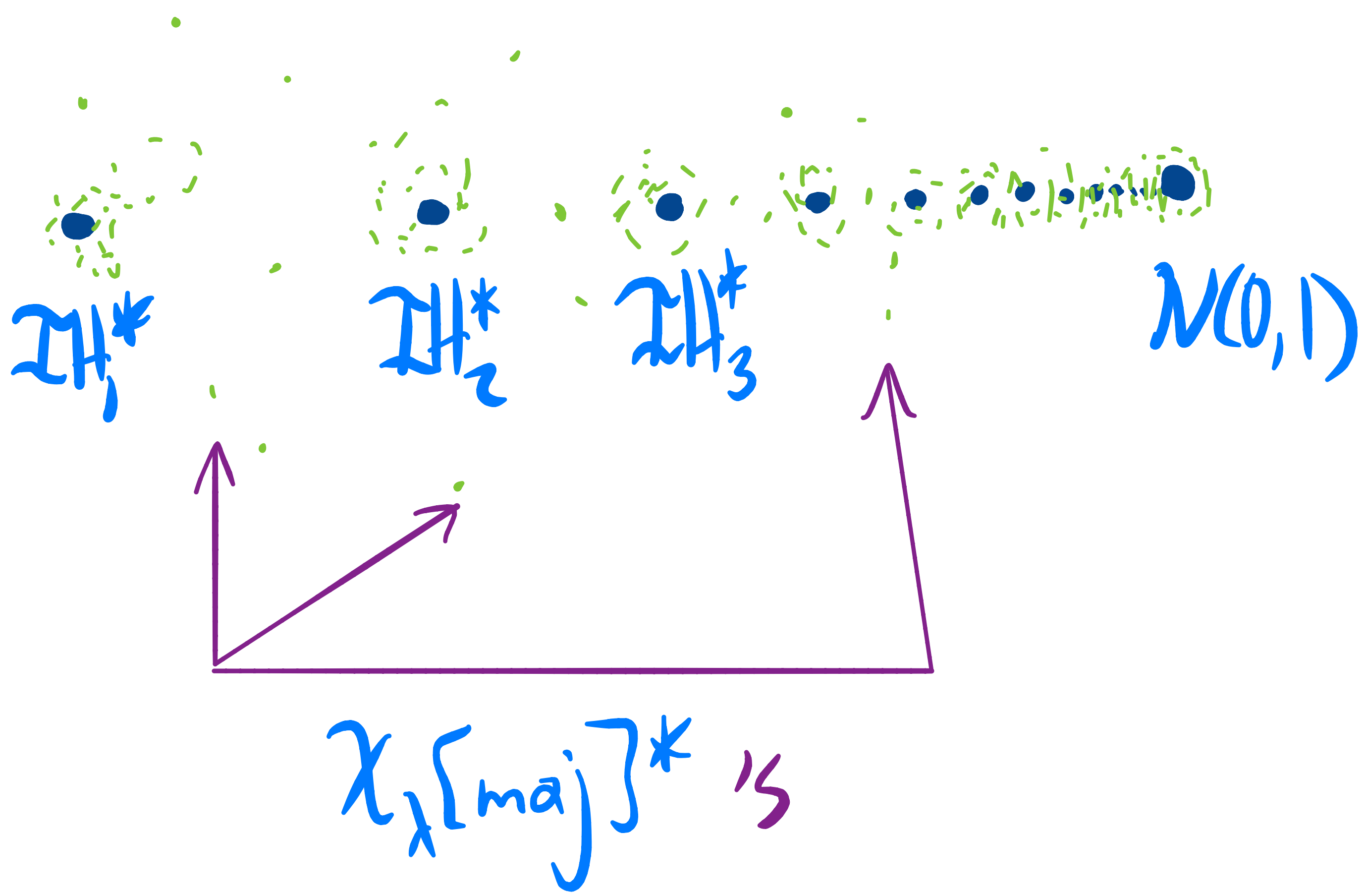
Suppose $\lambda^{(1)}, \lambda^{(2)}, \ldots$ is a sequence of partitions. Then $\mathcal{X}_{\lambda^{(1)}}[\mathrm{maj}]^*, \mathcal{X}_{\lambda^{(2)}}[\mathrm{maj}]^*, \ldots$ converges in distribution if and only if

- (i) $\mathrm{aft}(\lambda^{(N)}) \to \infty$; or
- (ii) $|\lambda^{(N)}| \to \infty$ and $\mathrm{aft}(\lambda^{(N)}) \to M < \infty$; or
- (iii) the distribution of $\mathcal{X}_{\lambda^{(N)}}[\mathrm{maj}]^*$ is eventually constant.

The limit law is $\mathcal{N}$ in case (i), $\mathcal{IH}_M^*$ in case (ii), and discrete in case (iii).

# Background: Irwin-Hall Limits

**Idea** The metric space of SYT distributions under (say) the Lévy metric:




$$M_{SYT} = \{\chi_\lambda[\mathrm{maj}]^*\}$$

$$M_{IH} = \{IH_M^*\}$$

$$\Rightarrow \overline{M_{IH}} = M_{IH} \sqcup \{N(0,1)\}$$

**Cor** $\overline{M_{SYT}} = M_{SYT} \sqcup \underbrace{\overline{M_{IH}}}_{\text{the set of limit points}}$

# Hook Length Formulas

**Rem** Proof in [BKS20, Thm. 1.7] relies on Stanley's q-hook length formula:

$$\sum_{T \in SYT(\lambda)} q^{\mathrm{maj}(T)} = q^{r(\lambda)} \frac{[n]_q!}{\prod_{c \in \lambda} [h_c]_q}$$

ratio is key to cumulant formula! then method of moments

**Q** What other combinatorial statistics arise as quotients of q-integers? (Cyclotomic generating functions)

**Recall** $[n]_q = \frac{1-q^n}{1-q} = 1+q+\cdots+q^{n-1}$

# Hook Length Formulas

**Thm** The rank on semistandard tableaux of shape $\lambda$ and entries $\leq m$ is

$$\sum_{T \in SSYT_{\leq m}(\lambda)} q^{\mathrm{rank}(T)} = \underbrace{q^{r(\lambda)} \prod_{u \in \lambda} \frac{[m+c_u]_q}{[h_u]_q}}_{\text{Stanley's } q\text{-hook-content formula}} = \underbrace{q^{r(\lambda)} \prod_{1 \leq i < j \leq m} \frac{[\lambda_i - \lambda_j + j - i]_q}{[j-i]_q}}_{q\text{-Weyl dimension formula (type A)}} .$$

**Thm** The size on plane partitions in an $a \times b \times c$ box is

$$\sum_{P \in PP(a \times b \times c)} q^{\mathrm{size}(P)} = \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{[i+j+k-1]_q}{[i+j+k-2]_q} \quad \text{MacMahon}$$

# Hook Length Formulas

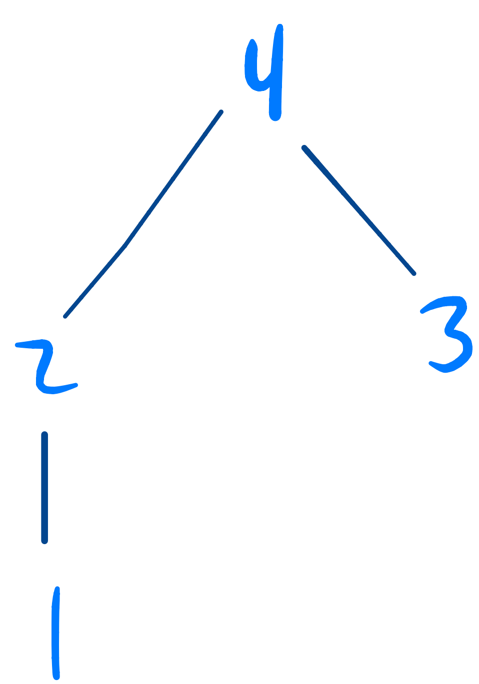
**Def** Let $P$ be a forest viewed as a poset with roots as maximal elements. Fix an order-preserving bijection $w: P \to [n]$.

Let $\mathcal{L}(P,w) = \{$linear extensions of $P$, viewed as permutations of $[n]$ via $w\}$.

The inversion number of $\pi \in S_n$ is $\mathrm{inv}(\pi) = \#\{(i,j) : 1 \le i < j \le n,\ \pi(i) > \pi(j)\}$.

**Ex**

$(P,w) =$




$\mathcal{L}(P,w) = \{1234, 1324, 3124\} \Rightarrow$ inv's $= \{0,1,2\}$

# Hook Length Formulas

**Thm** (Stanley, Björner-Wachs)

$$\sum_{\pi \in \mathcal{L}(P,w)} q^{\mathrm{inv}(\pi)} = \frac{[n]_q!}{\prod_{u \in P} [h_u]_q}.$$

**Q** What does the metric space of forest distributions

$$M_{\text{Forest}} = \{ \chi_P^*[\mathrm{maj}] \}$$

look like?

# DUSTPAN Distributions

**Def** The rank of $P$ is the length of a maximal chain.

**Thm** ("Generic" case) Given a sequence of forests $P$,

$$\chi_P[\mathrm{inv}]^* \Rightarrow \mathcal{N}(0,1)$$

if

$$|P| \to \infty \quad \text{and} \quad \limsup \frac{\mathrm{rank}(P)}{|P|} < 1.$$

**Idea**

$\mathcal{N}(0,1)$

# DUSTPAN Distributions

Let $n = |P|$ and $r = \mathrm{rank}(P)$.

**Q** Consider the set of rooted, unlabeled forests with $n$ vertices, sampled uniformly at random. What is the expected value of the rank $r$? How does $r$ compare to $n$ as $n \to \infty$?

**Rem** Brantin-Flajolet proved $\mathbb{E}[r] \sim C\sqrt{n}$ for binary trees.

Typically $\mathbb{E}[r] \sim D\sqrt{n}$ for ordered/labeled variations.

(Certainly $\limsup \frac{r}{n} < 1$ is "typical"!)

# DUSTPAN Distributions

**Rem** The "degenerate" case $n \sim r$ hides a world of behavior!!

**Def** Let $l_2 = \{(t_1, t_2, \ldots) : t_i \geq 0, \sum_{i=1}^{\infty} t_i^2 < \infty\}$ and

$\tilde{l}_2 = \{(t_1, t_2, \ldots) : t_1 \geq t_2 \geq \cdots \geq 0, \sum_{i=1}^{\infty} t_i^2 < \infty\}$

be sequence spaces.

Let $t \in \tilde{l}_2$. The corresponding generalized uniform sum random variable is

$$S_t = \sum_i U\left[-\frac{t_i}{2}, \frac{t_i}{2}\right]$$

converges iff $\|t\|_2^2 = \sum_i t_i^2 < \infty$

# DUSTPAN Distributions

**Def** A DUSTPAN Distribution is a distribution associated to a uniform sum for t plus a normal distribution

$$S_t + \mathcal{N}(0, \sigma^2) \qquad \text{where } t \in \tilde{\ell}_2,\ \sigma \in \mathbb{R}_{\geq 0}.$$

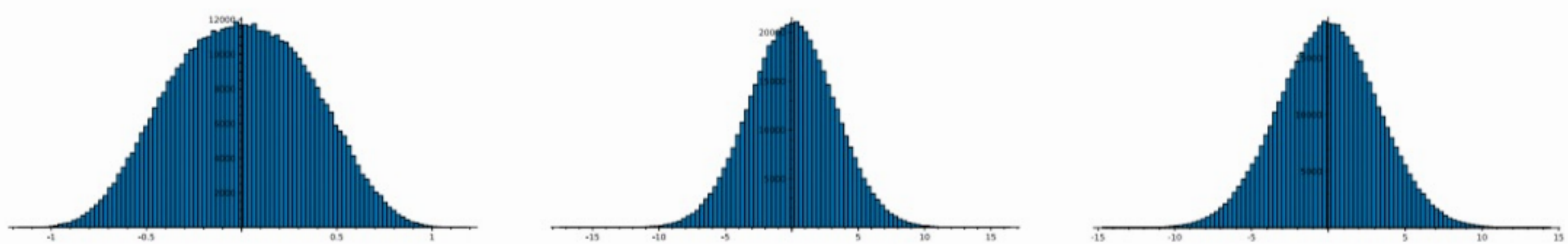


FIGURE 5. Histograms of the distributions $\mathcal{S}_{\mathbf{t}}$, $\mathcal{N}(0,\sigma)$, and $\mathcal{S}_{\mathbf{t}} + \mathcal{N}(0,\sigma)$ with $\mathbf{t} = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8})$ and $\sigma \approx 3.22$.

# DUSTPAN Distributions

**Def** The metric space of standardized DUSTPAN distributions is

$$M_{DUST} = \left\{ S_t + N(0,\sigma^2) : \underbrace{\frac{\|t\|_2^2}{12} + \sigma^2}_{\text{variance}} = 1 \right\}.$$

The standardized DUSTPAN parameter space is

$$P_{DUST} = \{ t \in \tilde{\ell}_2 : \|t\|_2^2 \leq 12 \} .$$

metric space under pointwise convergence

# DUSTPAN Distributions

**Thm** The map

$$\Phi : P_{DUST} \to M_{DUST}$$

where

$$\Phi(t) = S_t + \mathcal{N}(0, \sigma^2) \quad \text{where } \sigma = \sqrt{1 - \|t\|_2^2 / 12}$$

is a homeomorphism.

**Cor** $M_{DUST}$ is closed under convergence in distribution!

# DUSTPAN Distributions

DUSTPAN distributions are natural in the following sense:

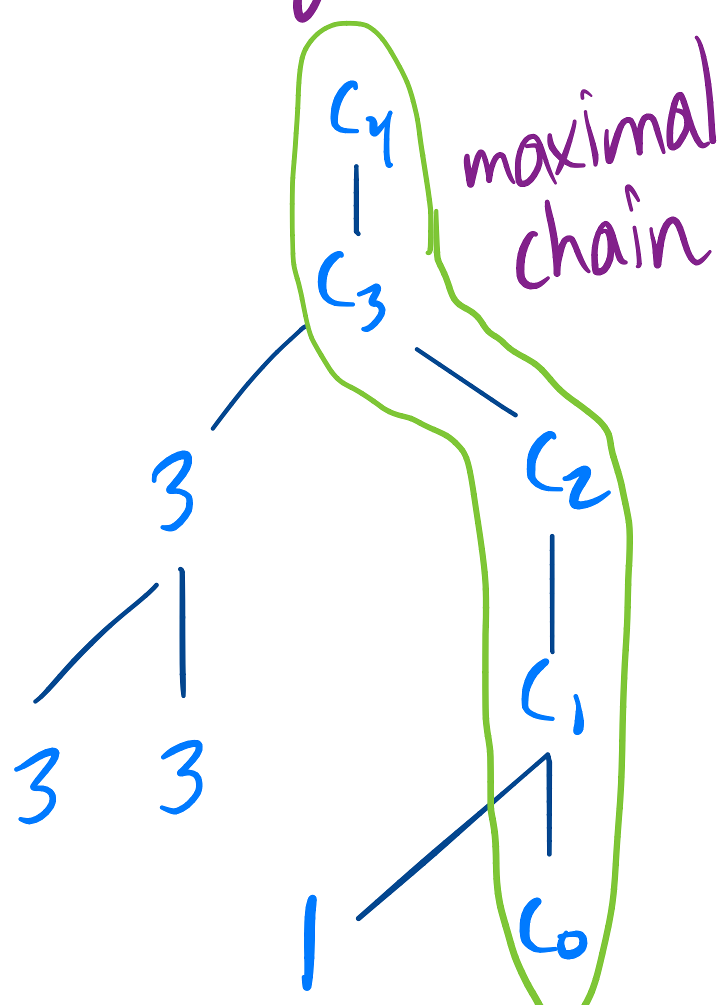
**Cor** The standardized DUSTPAN distributions $M_{DUST}$ are the smallest family which:

i. Consists of variance 1 real-valued distributions
ii. Contains $\mathcal{U}[0,1]^*$
iii. Is closed under standardized independent sums
iv. Is closed under convergence in distribution

# DUSTPAN Limits

**Def** A rooted tree is *standardized* if its root has at least two children.

**Def** The *elevation multiset* of $P$ relative to a fixed maximal chain lists the heights at which subtrees are attached to the chain:




$$e = \{1, 3, 3, 3\} = (3, 3, 3, 1, 0, 0, \ldots)$$

**Rem** Normalize via

$$\hat{e} = \frac{\sqrt{12}}{\|e\|_2} e$$

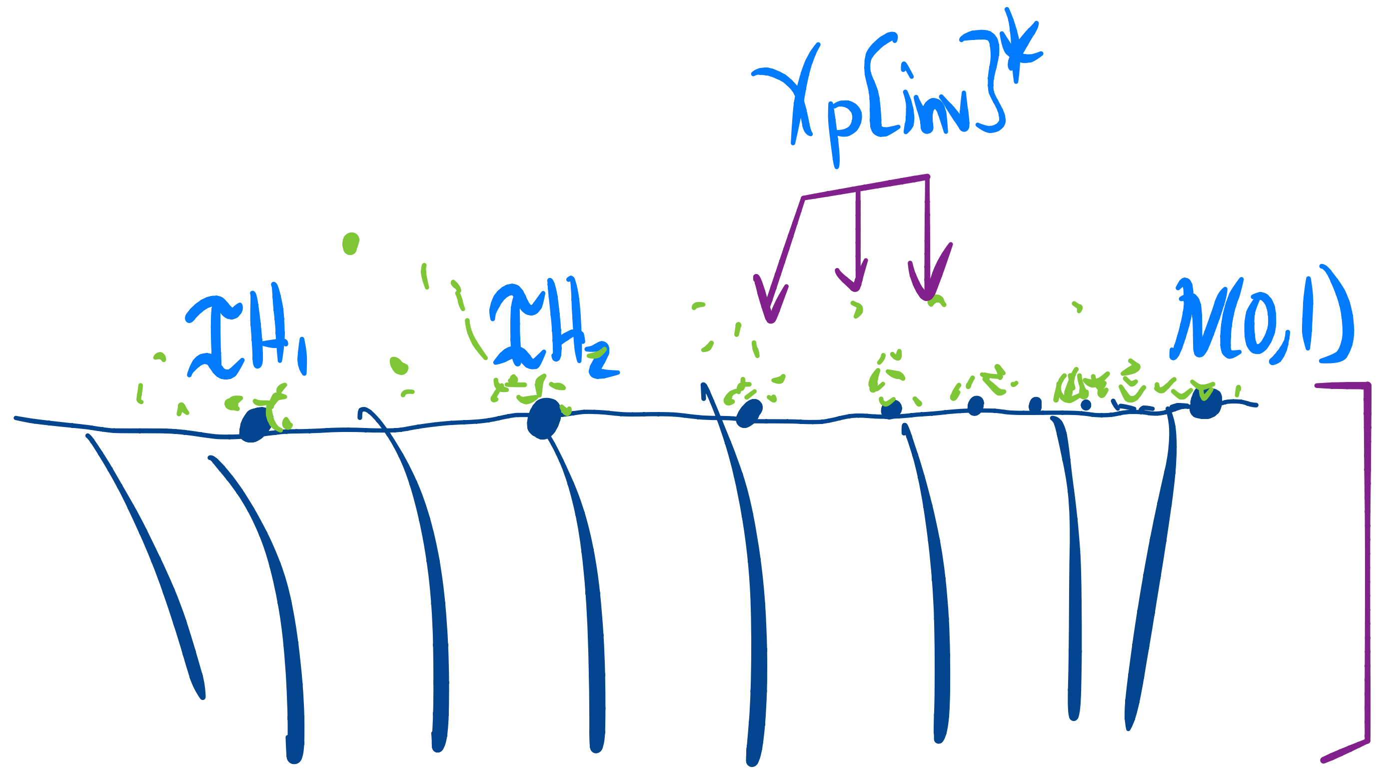
# DUSTPAN Limits

**Thm** [BS20, Thm. 1.13] Let $P$ be an infinite sequence of standardized trees with $n-r=o(n^{1/2})$. Then $X_P^*[\mathrm{inv}]$ converges in distribution if and only if the elevation multisets $\hat{e}$ associated to $P$ converge pointwise to some $t \in \tilde{\ell}_2$.

In that case, the limit distribution is $S_t + \mathcal{N}(0,\sigma^2)$ where $\frac{\|t\|_2^2}{12} + \sigma^2 = 1$.

# DUSTPAN Limits

Idea




$M_{DUST}$ is a large collection of limit points!

Cor: For any fixed $\varepsilon > 0$, let $\varepsilon TREE$ be the set of standardized trees for which $n - r < n^{\frac{1}{2} - \varepsilon}$. Let $M_{\varepsilon TREE} = \{\mathcal{X}_p^* : p \in \varepsilon TREE\} \subset M_{Forest}$. Then

$$\overline{M_{\varepsilon TREE}} = M_{\varepsilon TREE} \cup \underbrace{M_{DUST}}_{\text{all limit points!}}.$$

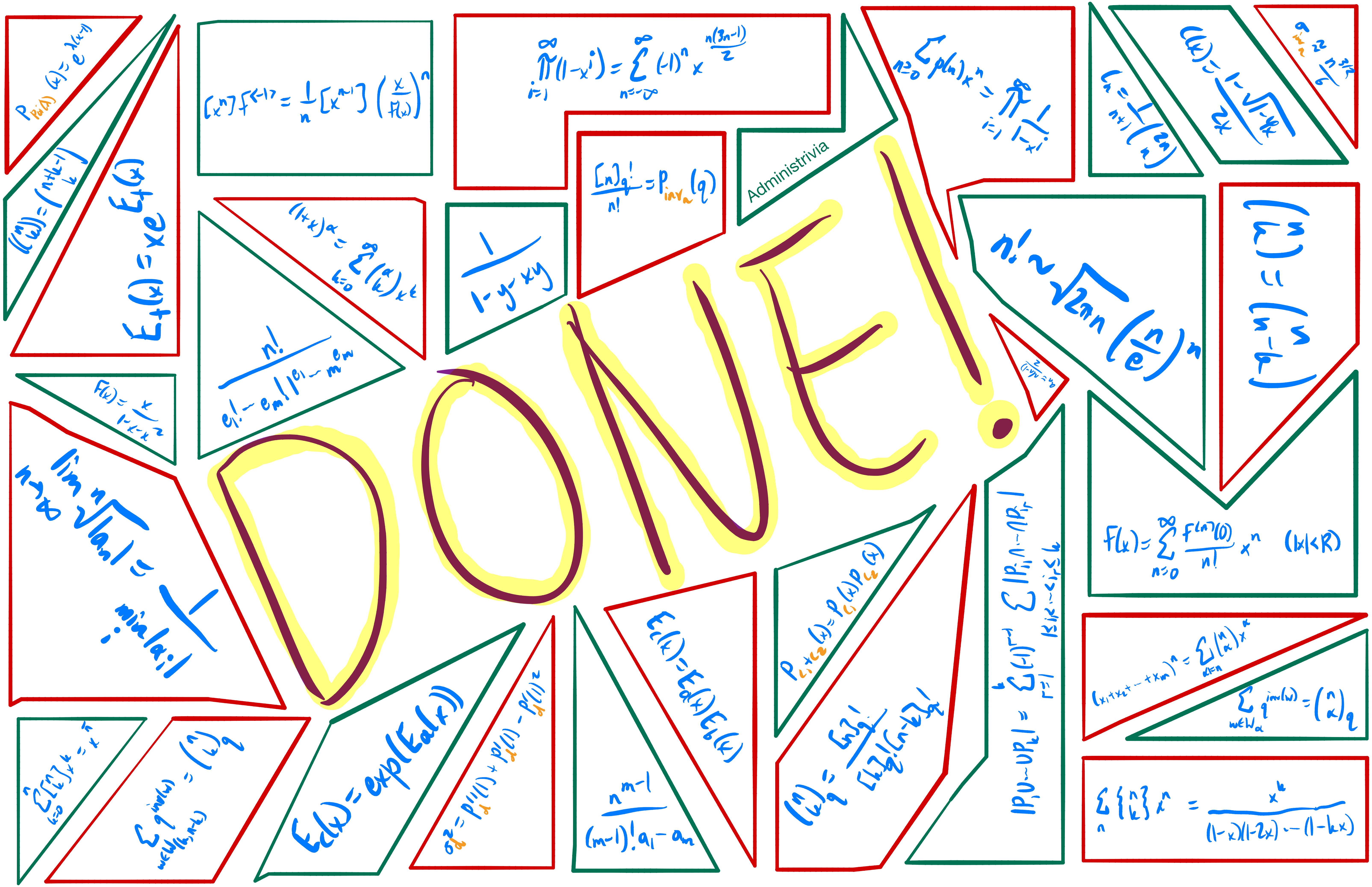
# Further Directions

**Rem** Have related results for...

- rank on $SSYT_{S_m}(\lambda)$: complicated subset of $M_{DUSTPAN}$
- size on $PP_{a\times b\times c}$: only $M_{IH}$

**Q** • What about between $n-r=\omega(n^{1/12})$ and $n-r=o(n^{1/2})$ regimes for $M_{Forest}$?

- What about $M_{CGF}$??
- More applications of DUSTPAN's?


DONE!
Administrivia
$\prod_{i=1}^{\infty}(1-x^i) = \sum_{n=-\infty}^{\infty}(-1)^n x^{\frac{n(3n-1)}{2}}$
$\sum_{n=0} p(n)x^n = \prod_{i=1} \frac{1}{1-x^i}$
$\frac{1}{1-y-xy}$
$\frac{[n]_q!}{n!} = P_{inv_n}(q)$
$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$
$F(x) = \sum_{n=0}^{\infty} \frac{F^{(n)}(0)}{n!}x^n \quad (|x|<R)$
$\frac{n!}{e_1!\cdots e_m! 1^{e_1}\cdots m^{e_m}}$
$\frac{n^{m-1}}{(m-1)!a_1\cdots a_m}$
$E_\ell(x) = \exp(E_d(x))$
$\sum_n \left\{ {n \atop k} \right\} x^n = \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}$