Combinatorics of harmonic polynomial differential forms

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Based partly on joint work with *Nolan Wallach*
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Slides: https://www.jpswanson.org/talks/2021_UCLA_harmonics.pdf

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Outline

- Coinvariant algebras / harmonics
- Super coinvariant algebras/super harmonics
- A potential Filtration
- Tanisaki witness relations

Coinvariant Algebras

Thm (Newton) $\mathbb{Q}[x_1,\dots,x_n]^{S_n} = \mathbb{Q}[e_1,\dots,e_n]$ where $e_i = \underbrace{\sum_{1\le i_1<\dots<i_k\le n} x_{i_1}\cdots x_{i_k}}_{\text{elementary symmetric polynomial}}$

and $\sigma(x_i) = x_{\sigma(i)}$

Thm (Hilbert) $\langle \mathbb{Q}[x_1,\dots,x_n]^{S_n}_{+} \rangle = \langle e_1,\dots,e_n \rangle$

Def The coinvariant algebra of S_n is

$$R_n = \frac{\mathbb{Q}[x_1,\dots,x_n]}{\langle e_1,\dots,e_n \rangle}$$

Coinvariant Algebras

Thm (Borel) $R_n \cong \overset{\text{singular cohomology}}{\overbrace{H}^*}(\underbrace{Fl_n}_{\text{complete flag manifold}})$

Thm ((Chevalley) $R_n \cong \mathbb{Q}S_n$

$\Rightarrow \dim R_n = n!$

Thm (Artin) $\{x_1^{\alpha_1} \cdots x_n^{\alpha_n} : 0 \leq \alpha_i < i\}$ descends to a basis for R_n

Cor $$\mathrm{Hilb}(R_n; q) = \sum_{d \geq 0} \dim(R_n)_d \cdot q^d = \underbrace{1 \cdot (1+q) \cdot (1+q+q^2) \cdots \underbrace{(1+q+\cdots+q^{n-1})}_{[n]_q}}_{[n]_q!}$$

Harmonic Polynomials

- If $g=\sum_\alpha c_\alpha x_1^{\alpha_1}\cdots x_n^{\alpha_n}$, set

$$\partial_g=\sum_\alpha c_\alpha \partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n} \in \mathrm{End}(\mathbb{Q}[x_1,\dots,x_n])$$

- The form $\langle f,g\rangle=(\partial_g f)(0)$ is symmetric, positive-definite, and S_n-invariant

Def The S_n-harmonic polynomials are

$$H_n=\langle \mathbb{Q}[x_1,\dots,x_n]_+^{S_n}\rangle^\perp=\{f\in\mathbb{Q}[x_1,\dots,x_n] : \partial_{e_i}f=0\ \forall i=1,\dots,n\}$$

Lem $H_n\xrightarrow{\sim}R_n$ is a graded S_n-module isomorphism

Harmonic Polynomials

Ex Let $\Delta_n = \prod_{1\le i<j\le n}(x_j - x_i)$ = the Vandermonde determinant.

This is an alternant: $\sigma\cdot\Delta_n = \mathrm{sgn}(\sigma)\Delta_n$.

If $g \in \mathbb{Q}[x_1,\dots,x_n]^{S_n}_{+}$, then $\partial_g \Delta_n$ is a lower-degree alternant, so 0!

$$\Rightarrow \partial_{e_i}\Delta_n = 0 \quad \forall i\ge 1 \quad \Rightarrow \underline{\Delta_n \in \mathcal{H}_n}$$

Thm (Steinberg) $\mathcal{H}_n = \mathbb{Q}[\partial_{x_1},\dots,\partial_{x_n}]\Delta_n$

Super Coinvariant Algebras

- Superspace is $\underbrace{\mathbb{Q}[x_1,\dots,x_n,\theta_1,\dots,\theta_n]}_{\mathrm{Sym}(x_1,\dots,x_n)\otimes\Lambda(\theta_1,\dots,\theta_n)}$ where $\theta_i\theta_j=-\theta_j\theta_i$ anti-commute (and $x_i\theta_j=\theta_jx_i$, $x_ix_j=x_jx_i$)

- S_n acts diagonally: $\sigma(x_i)=x_{\sigma(i)}$, $\sigma(\theta_i)=\theta_{\sigma(i)}$

Super Coinvariant Algebras

Def The super coinvariant algebra of S_n is

$$SR_n = \mathbb{Q}[\underline{x}_n, \underline{\theta}_n] / \langle \mathbb{Q}[\underline{x}_n, \underline{\theta}_n]^{S_n}_{+} \rangle$$

Conj (Zabrocki)

$$\mathrm{Hilb}(SR_n; q, z) = \sum_{k=0}^{n-1} \underbrace{[n-k]_q! \, \mathrm{Stir}_q(n, n-k)}_{\text{q-counts ordered set partitions of } [n] \text{ with } n-k \text{ blocks}} z^k$$

Thm (Wallach-S.) $\mathrm{Hilb}(SR_n; q, -q) = 1$

Aside Type B version: $\mathrm{Stir}^B_q(n,k)$

ongoing: Sagan-Sulzgruber-S.

Harmonic polynomial differential forms

- Think of θ variables as differential forms $\underline{\theta_i = dx_i}$, $\theta_i \theta_j = dx_i \wedge dx_j$

- The exterior derivative is

$$d = \sum_{i=1}^{n} \partial_{x_i} dx_i \in \mathrm{End}(\mathbb{Q}[\underline{x}_n, d\underline{x}_n])$$

$\underline{\boxed{\text{Thm}}}$ (Solomon) $\langle \mathbb{Q}[\underline{x}_n, d\underline{x}_n]^{S_n}_{+} \rangle = \langle e_1, \ldots, e_n, de_1, \ldots, de_n \rangle$

Harmonic polynomial differential forms

- The interior product is

$$\partial_{\theta_i}\theta_{i_1}\cdots\theta_{i_k}=\begin{cases}(-1)^{\ell-1}\theta_{i_1}\cdots\hat{\theta}_{i_\ell}\cdots\theta_{i_k} & \text{if } i=i_\ell\\ 0 & \text{otherwise}\end{cases}$$

(and is x-linear)

- If $g=\sum_{\alpha,I}c_{\alpha,I}\,x_1^{\alpha_1}\cdots x_n^{\alpha_n}\,\theta_{i_1}\cdots\theta_{i_k}$, set

$$\partial_g=\sum_{\alpha,I}c_{\alpha,I}\,\partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n}\,\partial_{\theta_{\underline{i_k}}}\cdots\partial_{\theta_{\underline{i_1}}}\in\mathrm{End}(\mathbb{Q}[\underline{x}_n,\underline{\theta}_n])$$

Harmonic polynomial differential forms

- The form $\langle f,g \rangle$ = constant term of $\partial_g f$

  is symmetric, positive-definite, and S_n-invariant

Def The S_n-super harmonics are

$$SH_n = \langle \mathbb{Q}[\underline{x}_n, \underline{\theta}_n]_+^{S_n} \rangle^{\perp}$$

$$= \{ f \in \mathbb{Q}[\underline{x}_n, \underline{\theta}_n] : \partial_{e_i} f = 0, \overbrace{\partial_{de_i} f}^{\sum_{j=1}^{n} \partial_{\partial_{x_j} e_i} \partial_{\theta_j} f} = 0 \ \forall i = 1, \dots, n \}$$

Lem $SH_n \xrightarrow{\sim} SR_n$ is a bigraded S_n-module isomorphism

Harmonic polynomial differential forms

Claim SH_n is combinatorially rich!

Generalized exterior derivatives

- The generalized exterior derivatives are the operators

$$d_i = \sum_{j=1}^{n} \partial_{x_j}^{i}\, dx_j \qquad (d_1 = d)$$

Thm (Wallach-S.) The alternant harmonic polynomial differential forms are

$$SH_n^{sgn} = \{F \in SH_n : \sigma \cdot F = sgn(\sigma) F \ \ \forall \sigma \in S_n\}$$

$$= \{\underset{d_I \Delta_n}{\underline{d_{i_1} \cdots d_{i_k} \Delta_n}} : 1 \leq i_1 < \cdots < i_k \leq n\}$$

Differential operator conjecture

Conj (Wallach-S.)

$$SH_n = \mathbb{Q}[\partial_{x_1}, \ldots, \partial_{x_n}] \, SH_n^{sgn}$$

Thm
- True for θ-degree $0, 1, n-1, n$
- Supports of $\mathrm{Hilb}(-;q,t)$ agree, for all θ-degrees.
  - Proof uses Artin and Gröbner bases

Flip action modules

- The flip action of $\mathbb{Q}[\underline{x}_n]$ on $\mathbb{Q}[\underline{x}_n,\underline{\theta}_n]$ is $f\cdot g=\partial_f g$.

- Let $SH_I=\mathbb{Q}[\underline{x}_n]\cdot d_I\Delta_n=\mathbb{Q}[\partial_{x_1},\ldots,\partial_{x_n}]d_I\Delta_n$

- Conjecturally, SH_n is the $\mathbb{Q}[\underline{x}_n]$-module generated by $\{d_I\Delta_n : I\in 2^{[n-1]}\}$

Q What are the $\mathbb{Q}[\underline{x}_n]$-linear relations between the $d_I\Delta_n$?

Flip action modules

Ex $\partial_{e_i} d_I \Delta_n = 0$ since $d_I \Delta_n$ is harmonic

Ex The additional $\mathbb{Q}[\underline{x}_n]$-linear relations for $n=3$, θ-degree 1 are generated by

$$\left.\begin{array}{l} \partial_{x_1 x_2} d_2 \Delta_3 = 0 \\ \partial_{x_1 x_2} d_1 \Delta_3 = \partial_{x_1 + x_2} d_2 \Delta_3 \end{array}\right] \text{ and } S_3\text{-images}$$

Hence the <u>filtration</u> $0 < SH_{\{2\}} < SH_{\{1\}} + SH_{\{2\}} = SH^1_3$ has <u>composition factors</u> isomorphic to $\mathbb{Q}[x_1,x_2,x_3] / \left\langle \substack{e_1, e_2, e_3 \\ x_1x_2,\, x_1x_3,\, x_2x_3} \right\rangle$

Tanisaki ideals

- Tanisaki gave a presentation of $H^*(\underbrace{\overline{X}_\mu}_{\text{a Springer fiber}}) \cong R_\mu = \mathbb{Q}[\underline{x}_n]/\mathcal{I}_\mu$

<u>where</u> $\mathcal{I}_\mu = S_n \cdot \langle e_1(\underline{n}), \dots, e_n(\underline{n}), e_{\underline{d}_1}(\underline{n-1}), \dots, e_{\underline{d}_{\mu_1'-1}}(\underline{n-\mu_1+1}) \rangle$

is a Tanisaki ideal

<u>where</u> $\underline{d}_0 = 1$ and

$\underline{d}_i = \underline{d}_{i-1} + (\mu_i' - 1)$

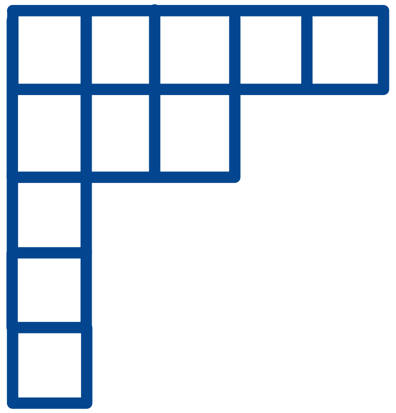
$\boxed{\text{Ex}}$ $\dim SH_3^1 = 2 \dim R_{(2,1)}$

$\boxed{\text{Ex}}$

$i = 0, 1, 2, 3, 4$

$\underline{d}_i = 1, 5, 6, 7, 7$

$\mu =$



$n = 11$

$\Rightarrow$ $\mathcal{I}_{(5,3,1,1,1)} =$

$S_n \cdot \langle e_1(\underline{n}), \dots, e_n(\underline{n}),$

$e_5(\underline{n-1}), e_6(\underline{n-2}),$

$e_7(\underline{n-3}), e_7(\underline{n-4}) \rangle$

A potential Filtration

- Have 2^{n-1} generators $d_I \Delta_n$ vs. $\#\{\overbrace{\alpha \models n}^{\text{weak compositions}}\} = 2^{n-1}$

- Combining conjectures and results of Zabrocki, Wallach-S., and Haglund-Rhoades-Shimozono suggests...

A potential Filtration

Q Is there a total order $I_1 < I_2 < \cdots$ on $2^{[n-1]}$ and a bijection

$$\Phi_n : \{I \subset [n-1]\} \to \{\overbrace{\alpha \vDash n}^{\text{weak compositions}}\}$$

where the composition factor

$$\sum_{j \le m} SH_{I_j} \Big/ \sum_{j<m} SH_{I_j}$$

is annihilated precisely by $\mathcal{I}_{\Phi_n(I_m)}$?

Call a relation

$$\partial_{e_r(\underline{s})} d_{I_m} \Delta_n = \sum_{j<m} \partial_{f_j} d_{I_j} \Delta_n$$

for $f_j \in \mathbb{Q}[x_1, \dots, x_n]$ a

Tanisaki witness relation

A potential Filtration

- True for $n \leq 8$ by computer

Ex At $n=7$, $k=2$, have

$$0 = 5\partial_{e_5(\underline{5})} d_{16}\Delta_7 - 4\partial_{e_4(\underline{5})} d_{26}\Delta_7 + 3\partial_{e_3(\underline{5})} d_{36}\Delta_7 - 2\partial_{e_2(\underline{5})} d_{46}\Delta_7 + \partial_{e_1(\underline{5})} d_{56}\Delta_7$$

$$+ 3\partial_{e_5(\underline{5})} d_{25}\Delta_7 - 2\partial_{e_4(\underline{5})} d_{35}\Delta_7 + \partial_{e_3(\underline{5})} d_{45}\Delta_7$$

$$+ \partial_{e_5(\underline{5})} d_{34}\Delta_7$$

A potential Filtration

- True for $n \leq 8$ by computer

- This approach would give an explicit (though complicated) basis using the Garsia-Procesi monomial basis for R_μ

- When all conjectures hold, get

$$(*) \qquad \sum_{I \subseteq [n]} z^{|I|} q^{\binom{n}{2} - \mathrm{sum}(I)} = \sum_{\alpha \models n} z^{n-\ell(\alpha)} q^{2\overbrace{b(\mu(\alpha))}^{\sum_{i\geq 1}(i-1)\underline{\mu}_i} - \binom{\ell(\alpha)}{2} + \overbrace{\mathrm{coinv}(\alpha)}^{\#\{i<j \,:\, \alpha_i < \alpha_j\}}}$$

A new bijection

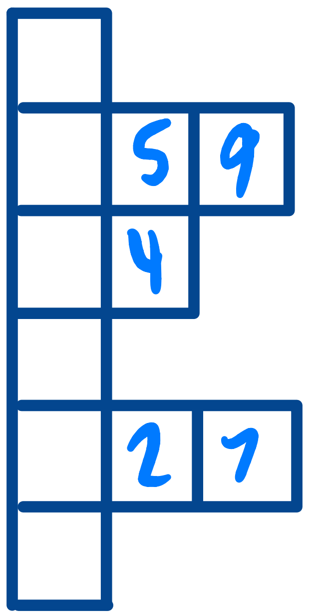
Thm (S.) There is an explicit $\Phi_n : \{I \subset [n-1]\} \to \{\alpha \vDash n\}$ where

$$n - \ell(\alpha) = |I|$$

$$2bl_{\mu}(\alpha) - \binom{\ell(\alpha)}{2} + \text{coinv}(\alpha) = \binom{n}{2} - \text{sum}(I).$$

Cor (*) is true.

Ex



$(1,3,2,1,3,1) \vDash 11$

$\updownarrow$

$\{2,4,5,7,9\} \subset [11-1]$

Generic Pieri Rule

Thm (5.) Let $I=\{i_1<\cdots<i_k\}\subset[n-1]$. Then

$$\sum (-1)^d \partial_{e_{n-k-d}(\underline{n-1})} d_{j_1\cdots j_k} \Delta_n = 0$$

where the sum is over all subsets $J=\{j_1<\cdots<j_k\}\subset[n-1]$ where

$$1\leq i_1\leq j_1<i_2\leq j_2<\cdots<i_k\leq j_k<n$$

and

$$d=(j_1-i_1)+\cdots+(j_k-i_k).$$

Generic Pieri Rule

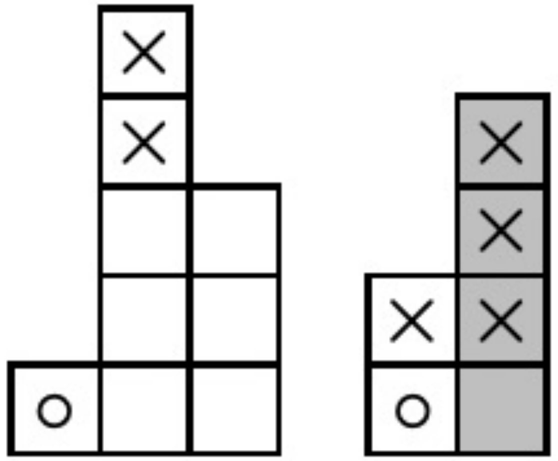
Cor $\{n-1\} < \{n-2\} < \cdots < \{1\}$ gives a filtration of SH_n' by $SH_{\{i\}}$'s where the composition factors are annihilated precisely by the Tanisaki ideals $\mathcal{I}_{(2,1^{n-2})}$. Furthermore,

$$\mathrm{GrFrob}(SH_n'; q) = [n-1]_q \, \omega Q'_{(2,1^{n-2})}(\underline{x}; q)$$

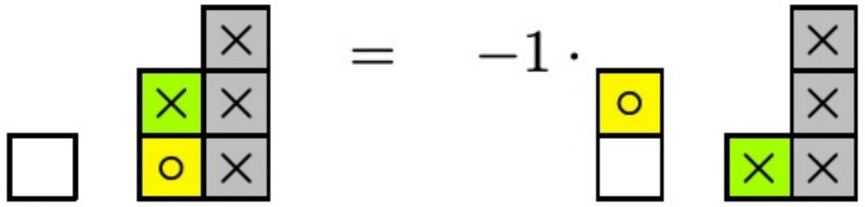
- These relations are "almost all" in a certain asymptotic sense (k fixed, $n \to \infty$)

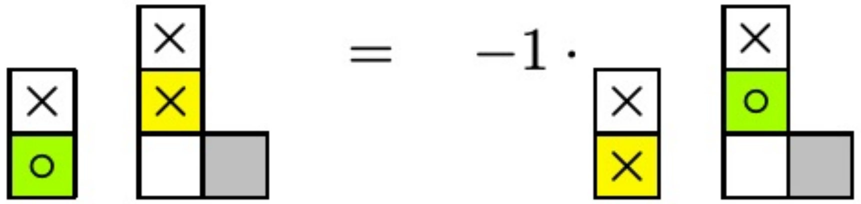
Monomial model

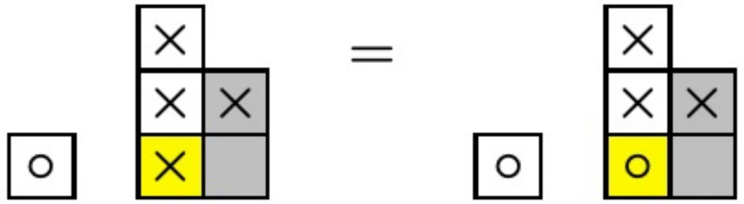
- Proof uses marked staircase diagrams for terms in $\partial_{e_d(\underline{u})} d_I \Delta_n$:

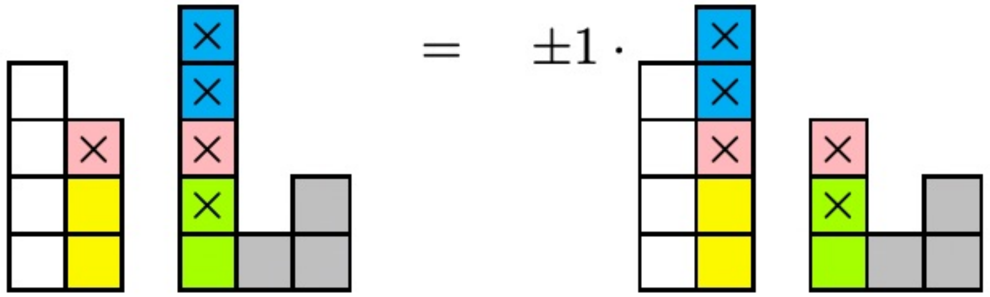


- They have various relations:





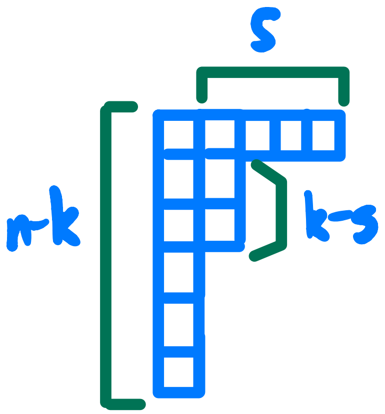




Extreme hook relations

Thm (S.) Suppose $I=\{i_1<\cdots<i_k\}\subset[n-1]$ is such that for some $1\le s\le k$,

$$\left.\begin{array}{r} i_1,\dots,i_{k-s+1}\le n-k \\ i_{k-s+2}=n-s+1 \\ i_{k-s+3}=n-s+2 \\ \vdots \\ i_k=n-1. \end{array}\right] \Leftrightarrow \Phi_n(I)=\alpha\models n \text{ with } \begin{array}{l} \ell(\alpha)=n-k \\ \bar{\alpha}=(s,1^{k-s}) \end{array}$$



Pick $0\le u\le s$. Then

$$\sum(-1)^d\Delta_s(j_{k-s+1},\dots,j_k)\binom{d+u}{u}\partial_{e_{n-s+d}(\underline{n-s+u})}d_J\Delta_n=0$$

summed over $J=\{j_1<\cdots<j_k\}\subset[n-1]$ for which

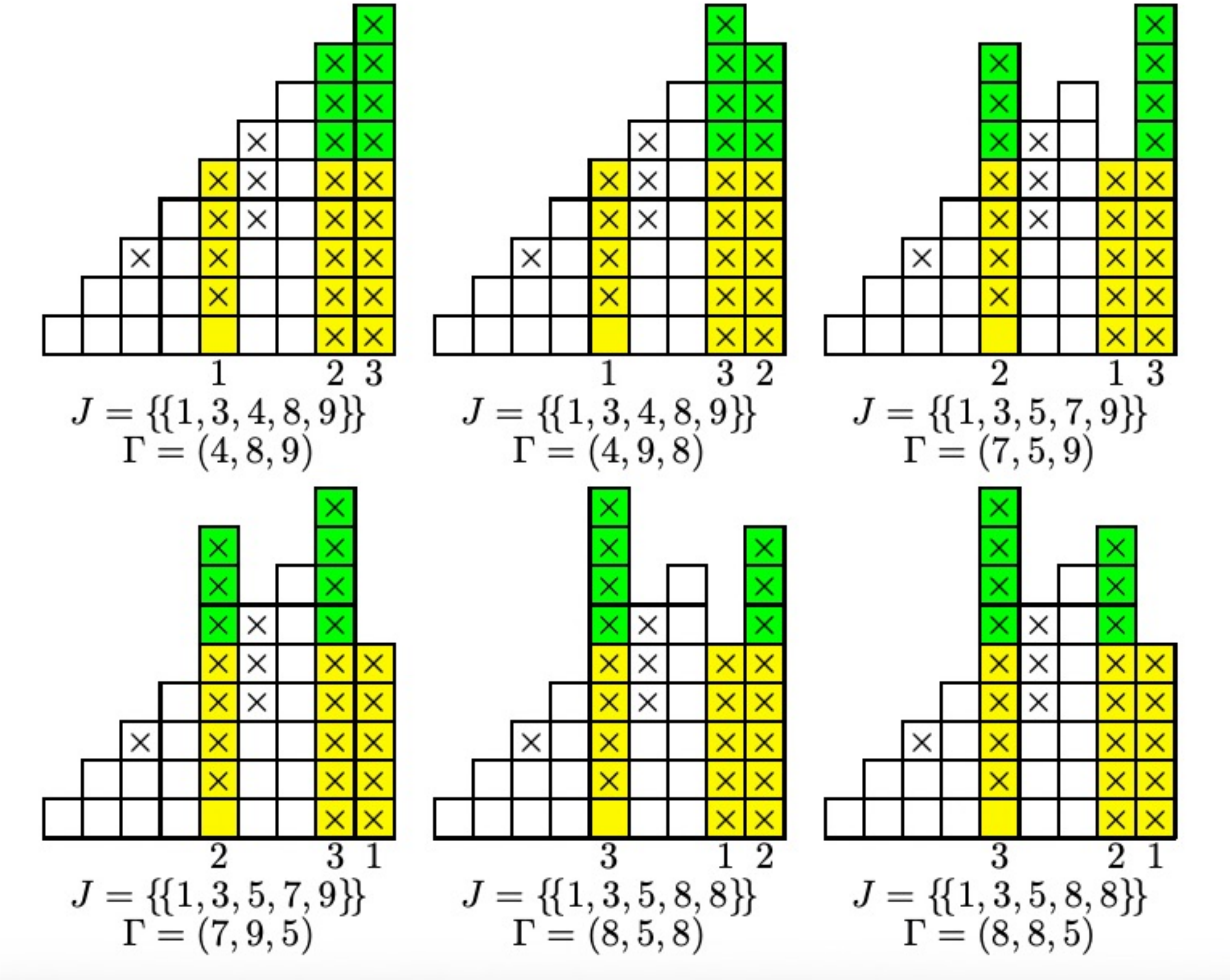
$j_1=i_1,\dots,j_{k-s}=i_{k-s}$ and

$d=\mathrm{sum}(J)-\mathrm{sum}(I)\ge \underline{0}$.

Extreme hook relations

- Proof introduces S_s-action on relevant marked staircases

Complicated!



$J = \{\{1,3,4,8,9\}\}$ $\Gamma = (4,8,9)$

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$J = \{\{1,3,5,7,9\}\}$ $\Gamma = (7,5,9)$

$J = \{\{1,3,5,7,9\}\}$ $\Gamma = (7,9,5)$

$J = \{\{1,3,5,8,8\}\}$ $\Gamma = (8,5,8)$

$J = \{\{1,3,5,8,8\}\}$ $\Gamma = (8,8,5)$

Tanisaki coefficients

Q| What are <u>all</u> the Tanisaki witness relations?

What is a combinatorial description for their coefficients?

Is there a geometric/algebraic/topological interpretation?

THANKS!