On the distribution of the major index on standard Young tableaux

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Based on joint work with
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Papers: [BKS20a], [BKS20b] / arXiv: 1905.00975, 1809.07386
Slides: http://www.math.ucsd.edu/~jswanson/talks/2020_FPSAC.pdf

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Outline

- Background/motivation
- Existence classification for maj on $SYT(\lambda)$
- Distribution classification
- Unimodality conjectures
- Further directions

Descents and major index of standard Young tableaux (SYT)

$$T = \begin{array}{|c|c|c|c|c|c|} \hline 1 & \underline{2} & \underline{4} & \underline{7} & \underline{9} & 12 \\ \hline 3 & 6 & \underline{10} & \multicolumn{3}{c}{} \\ \cline{1-3} 5 & 8 & 11 & \multicolumn{3}{c}{} \\ \cline{1-3} \end{array} \in SYT(6,3,3) \text{ has }$$

$$Des(T) = \{2, 4, 7, 9, 10\}$$

$$des(T) = |Des(T)| = 5$$

$$maj(T) = 2+4+7+9+10 = 32$$

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Notation Write g.f. as

$$SYT(\lambda/\mu)^{maj}(q) := \sum_{T \in SYT(\lambda/\mu)} q^{maj(T)} =: \sum_k \underbrace{b_{\lambda/\mu, k}}_{\#\{T \in SYT(\lambda/\mu) \,:\, maj(T) = k\}} q^k.$$

Why study maj on $SYT(\lambda/\mu)$?

A1 Naturally generalizes beautiful special cases:

- **1.** Words/q-multinomials:

  Thm (MacMahon) $$\sum_{\substack{w \text{ with} \\ \text{content } \alpha}} q^{\mathrm{maj}(w)} = \binom{n}{\alpha}_q =: \frac{[n]_q!}{[\alpha_1]_q! \cdots [\alpha_m]_q!}$$

  $\left(\text{where } [n]_q! := \prod_{i=1}^{n} [i]_q, \quad [i]_q := \frac{1-q^i}{1-q} = 1+q+\cdots+q^{i-1}\right)$

- **2.** q-binomials: $SYT(n-k+1, 1^k)^{\mathrm{maj}}(q) = q^{\binom{k+1}{2}} \binom{n}{k}_q$

- **3.** q-Catalans: $SYT(n,n)^{\mathrm{maj}}(q) = q^n \frac{1}{[n+1]_q} \binom{2n}{n}_q$

Why study maj on $SYT(\lambda/\mu)$?

A2 Lots of connections to algebra, geometry, rep. theory, symmetric functions!

Thm (Lusztig-Stanley, [Sta79, Prop. 4.11])

$$\text{mult. of } \underbrace{S^\lambda}_{\substack{\text{Specht module,} \\ \text{(an } S_n\text{-irrep)}}} \text{ in deg. } k \text{ component of } \underbrace{\frac{\mathbb{C}[x_1, \ldots, x_n]}{\langle e_1, e_2, \ldots, e_n \rangle}}_{\text{Type A coinvariant algebra}} = b_{\lambda, k}.$$

Lusztig called $SYT(\lambda)^{maj}(q)$ fake degrees (type A) since $SYT(\lambda)^{maj}(1) = \dim S^\lambda =: f^\lambda$.

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Note See e.g. Green [Gre55, Lem. 7.4], Stembridge [Ste89, Thm. 3.3], Manivel [Man01, §3.6], Kraśkiewicz–Weyman [KW01], Stanley [EC2, Prop. 7.19.11] for more.

Modular maj results

Let $a_{\lambda,r} := \#\{T \in SYT(\lambda) : \mathrm{maj}(T) \equiv r \pmod{n}\}$ where $\lambda \vdash n$. That is,

$$a_{\lambda,r} = \sum_{k \equiv r \pmod{n}} b_{\lambda,k}.$$

$\boxed{Q}$ (Sundaram [Sun18, Rem. 4.8] $r=0$; Klyachko [Kly74] $r=1$; S. general r)

When is $a_{\lambda,r} = 0$?

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When is $a_{\lambda,r} = 0$?

$\boxed{Thm}$ (S. [Swa18, Thm. 1.4]) $a_{\lambda,r} > 0$ when λ is <u>not</u>

$$(2,2),\ (2,2,2),\ (3,3),\qquad (n),\ (1^n),\ (n-1,1),\ (2,1^{n-2}).$$

$\boxed{Thm}$ (S. [Swa18, Thm. 1.8]) If $f^\lambda \geq n^5$, then for all $0 \leq r < n$,

$$\left| \frac{a_{\lambda,r}}{f^\lambda} - \frac{1}{n} \right| < \frac{1}{n^2}.$$

Our three motivating questions!

Q1 (Existence) For which λ, k is $b_{\lambda,k} = 0$?

Q2 (Distribution) What does $(b_{\lambda,0}, b_{\lambda,1}, b_{\lambda,2}, \ldots)$ "look like"?

Q3 (Unimodality) For which λ is there some m for which

$$b_{\lambda,0} \leq b_{\lambda,1} \leq \cdots \leq b_{\lambda,m} \geq b_{\lambda,m+1} \geq \cdots ?$$

Existence: internal zeros classification

Thm (Billey–Konvalinka–S. [BKS20b, Thm. 1.1])
If $\lambda \vdash n$ and $b(\lambda) \leq k \leq \binom{n}{2} - b(\lambda')$, then

$$b_{\lambda,k} > 0$$

except when λ is a rectangle with at least two rows and columns and $k \in \{b(\lambda)+1, \binom{n}{2} - b(\lambda') - 1\}$. Furthermore, $b_{\lambda,k} = 0$ for $k < b(\lambda)$ or $k > \binom{n}{2} - b(\lambda')$,

where $b(\lambda) := \sum_{i \geq 1} (i-1)\lambda_i$ and λ' is the conjugate partition of λ.

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Ex $\mathrm{SYT}(4,4)^{\mathrm{maj}}(q) = q^4+q^6+q^7+2q^8+q^9+2q^{10}+q^{11}+2q^{12}+q^{13}+q^{14}+q^{16}$

$$\Rightarrow (0,0,0,0,1,0,1,1,2,1,2,1,2,1,1,0,1,0,0,0,\dots)$$

$b(4,4)=4$

internal zeros

$\binom{8}{2}-b(2,2,2,2)=16$

Existence: internal zeros proof

Strategy Construct a map $\phi: SYT(\lambda) - \mathcal{E}(\lambda) \to SYT(\lambda)$ s.t.

$$\mathrm{maj}(\phi(T)) = \mathrm{maj}(T) + 1$$

where $\mathcal{E}(\lambda)$ is a very small, explicit set of exceptional tableaux. Now iterate ϕ!

Involves two types of operations:

- "rotation rules"
- "block rules"

Existence: rotation rules

Def A rotation rule increments a single descent, leaving the others alone. See [BKS20b, §4.1].

Ex

$$T = \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & 9 \\ \hline 6 \\ \cline{1-1} 7 \\ \cline{1-1} 8 \\ \cline{1-1} \end{array} \xmapsto{(543)} T' = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 & 9 \\ \hline 6 \\ \cline{1-1} 7 \\ \cline{1-1} 8 \\ \cline{1-1} \end{array}$$

has

$$\mathrm{Des}(T) = \{1, 2, 4, 5, 6, 7\}$$

$$\mathrm{Des}(T') = \{1, 3, 4, 5, 6, 7\}$$

$$\Rightarrow \mathrm{maj}(T') = \mathrm{maj}(T) + 1$$

Rem Combinatorial characterization of rotation rules: [BKS20b, Lem. 4.5–4.6]

Existence: rotation rules

Rem Relatively plentiful, e.g. $1 \in \mathrm{Des}(T)$ ensures existence.

When $\lambda = (5,4,4,2)$, all but 24 out of 81,081 have a rotation rule.

Ex

$$
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\xmapsto{(32)}
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$$

Des: $\quad 2 \mapsto 3 \qquad 7 \mapsto 8 \qquad 1 \mapsto 2 \qquad 6 \mapsto 7 \qquad 5 \mapsto 6$

Existence: block rules

Problem Rotation rules fix $\mathrm{des}(T) := \#\mathrm{Des}(T)$ — incomplete!

Def The block rules B1–B5 apply certain permutations, increasing maj and des by 1. See [BKS20b, Def. 4.13] for details.

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Note

- Rotation rules increase (maj − des, des) by (1,0),
- Block rules " " by (0,1).

We explicitly construct ϕ from rotation and block rules by cases. Technical!

Existence: consequences

Def The strong SYT poset $P(\lambda)$ on the set

- $SYT(\lambda)$, if there are no internal zeros, and
- $SYT(\lambda) - \{\mathrm{minmaj}(\lambda), \mathrm{maxmaj}(\lambda)\}$ otherwise

is the transitive closure of covering relations given by all applicable rotation rules, block rules, and inverse-transpose block rules.

Ex Hasse diagrams: https://sites.math.washington.edu/~billey/papers/syt.posets

Cor $P(\lambda)$ is ranked by maj (and (maj-des, des)).

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Rem Cyclage ranks $SYT(n)$ by maj, but it does not at all restrict to $SYT(\lambda)$.

Existence: consequences

Cor • $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q)$ has almost no internal zeros.

• $\mathrm{SYT}(\lambda)^{\mathrm{des}}(q)$ has no internal zeros (answering [AER18, Problem 7.5] for $\lambda \vdash n$).

• $\mathrm{SYT}(\lambda)^{\mathrm{maj}-\mathrm{des}}(q)$ has no internal zeros.

Existence: consequences

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- $SYT(\lambda)^{\mathrm{maj}}(q)$ has almost no internal zeros.
- $SYT(\lambda)^{\mathrm{des}}(q)$ has no internal zeros (answering [AER18, Problem 7.5] for $\lambda \vdash n$).
- $SYT(\lambda)^{\mathrm{maj}-\mathrm{des}}(q)$ has no internal zeros.
- Classification of internal zeros of fake degrees of <u>all</u> Shephard–Todd complex reflection groups $G(m,d,n)$ (see [BKS20b, Thm. 6.3 and 8.3])!
  - Includes types B and D. (See [BKS20b, Cor. 6.4 and 8.4]).
- New proof of $a_{\lambda,r}=0$ classification.

Distribution: motivating examples

Pick some λ, compute $SYT^{maj}(q) = \sum_k b_{\lambda,k} q^k$, plot list $(b_{\lambda,0}, b_{\lambda,1}, b_{\lambda,2}, \ldots)$:

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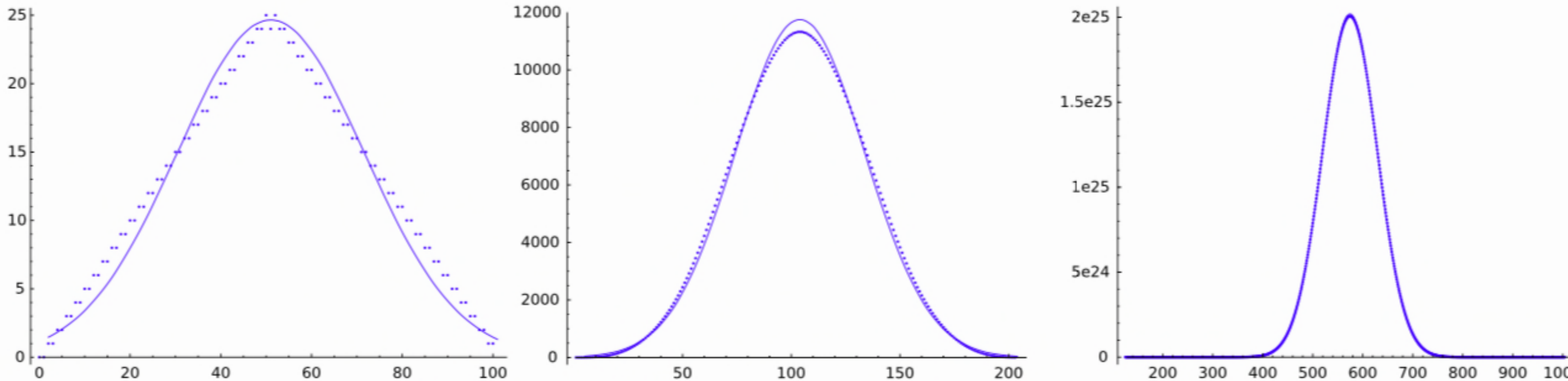


(a) $\lambda = (50, 2)$ (b) $\lambda = (50, 3, 1)$ (c) $\lambda = (8, 8, 7, 6, 5, 5, 5, 2, 2)$

Fig. 1. Plots of $\#\{T \in \mathrm{SYT}(\lambda) : \mathrm{maj}(T) = k\}$ as a function of k for three partitions λ, overlaid with scaled Gaussian approximations using the same mean and variance.

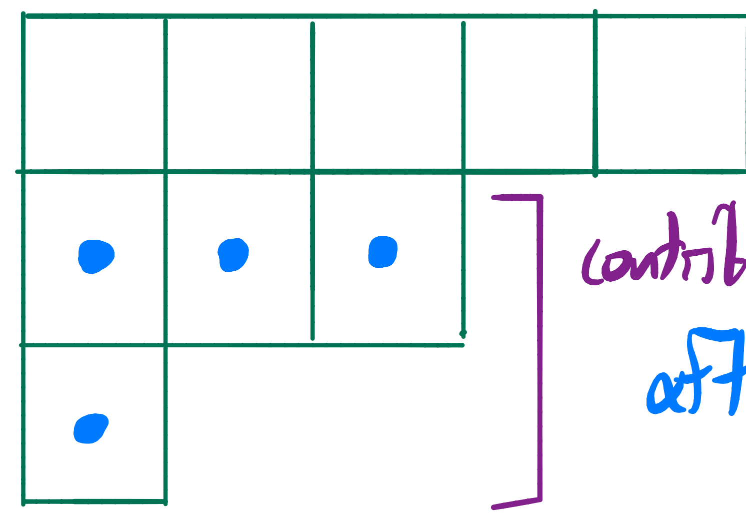
$\boxed{\text{Rem}}$ Clearly a Central Limit Theorem is occurring! But, some subtlety: "generically" normal, sometimes not in "degenerate" cases?

Distribution: $\mathrm{aft}(\lambda)$

Def Given a partition $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$, let

$$\mathrm{aft}(\lambda) := n - \max\{\lambda_1, k\}.$$

Ex $\lambda = (5,3,1) \Rightarrow \mathrm{aft}(\lambda) = 4$



contribute to aft

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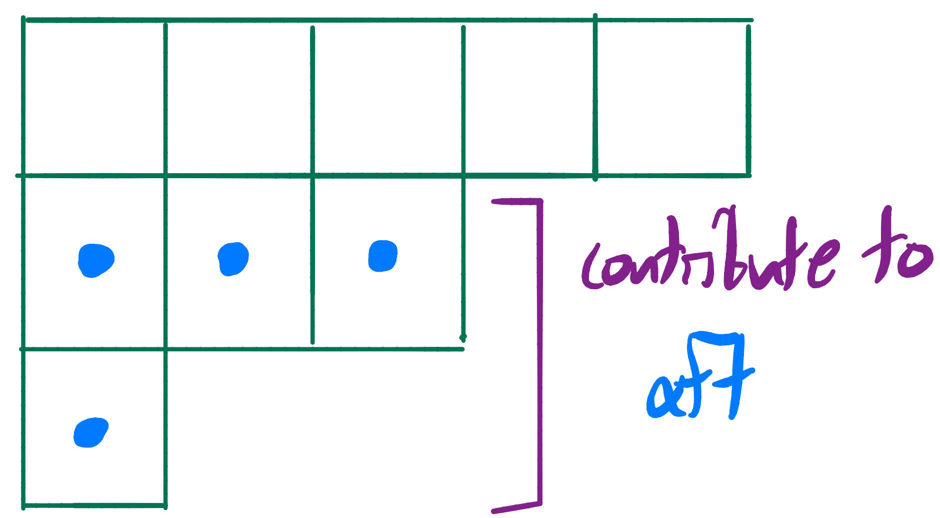

Ex $\mathrm{aft}(\lambda) = 1 \iff \lambda = (n-1,1)$ or $(2,1^{n-2})$ — two degenerate families for $a_{\lambda,r}$!

Rem On FindStat as St001214.
Reminiscent of representation stability of Church–Farb [CF13, Thm. 7.1].

Distribution: random variables

Notation Write $X^* := \frac{X-\mu}{\sigma}$ for the standardization of X with mean 0, variance 1.

Def A sequence of random variables $X_1, X_2, \ldots$ is asymptotically normal if

$$\forall t \in \mathbb{R}, \quad \lim_{n\to\infty} \mathbb{P}[X_i^* \le t] = \mathbb{P}[\underbrace{N}_{\text{standard normal}} \le t] \qquad \left(= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-x^2/2}\, dx \right).$$

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Notation Let $X_\lambda[\mathrm{maj}]$ be the random variable for the maj statistic on $SYT(\lambda)$, where each tableau is equally likely.

Rem maj is sum of descents, but they're <u>not</u> independent!

Distribution: classification

Thm (Billey–Konvalinka–S. [BKS20a, Thm. 1.3])

Suppose $\lambda^{(1)}, \lambda^{(2)}, \ldots$ is a sequence of partitions. Then $\chi_{\lambda^{(1)}}[\mathrm{maj}], \chi_{\lambda^{(2)}}[\mathrm{maj}], \ldots$ is asymptotically normal if and only if

$$\mathrm{aft}(\lambda^{(N)}) \to \infty \quad \text{as} \quad N \to \infty.$$

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Q What if $\mathrm{aft}(\lambda^{(N)}) \not\to \infty$? Non-normal limits?

Distribution: classification

Thm (Billey–Konvalinka–S. [BKS20a, Thm. 1.7])

Suppose $\lambda^{(1)}, \lambda^{(2)}, \ldots$ is a sequence of partitions. Then $\mathcal{X}_{\lambda^{(1)}}[\mathrm{maj}]^*, \mathcal{X}_{\lambda^{(2)}}[\mathrm{maj}]^*, \ldots$ converges in distribution if and only if

- (i) $\mathrm{aft}(\lambda^{(N)}) \to \infty$; or
- (ii) $|\lambda^{(N)}| \to \infty$ and $\mathrm{aft}(\lambda^{(N)}) \to M < \infty$; or
- (iii) the distribution of $\mathcal{X}_{\lambda^{(N)}}[\mathrm{maj}]^*$ is eventually constant.

The limit law is $\mathcal{N}$ in case (i), $\underbrace{\boxed{\mathcal{IH}_M^*}}$ in case (ii), and discrete in case (iii).

Irwin–Hall distribution: sum of M i.i.d. continuous uniform $[0,1]$ r.v.'s

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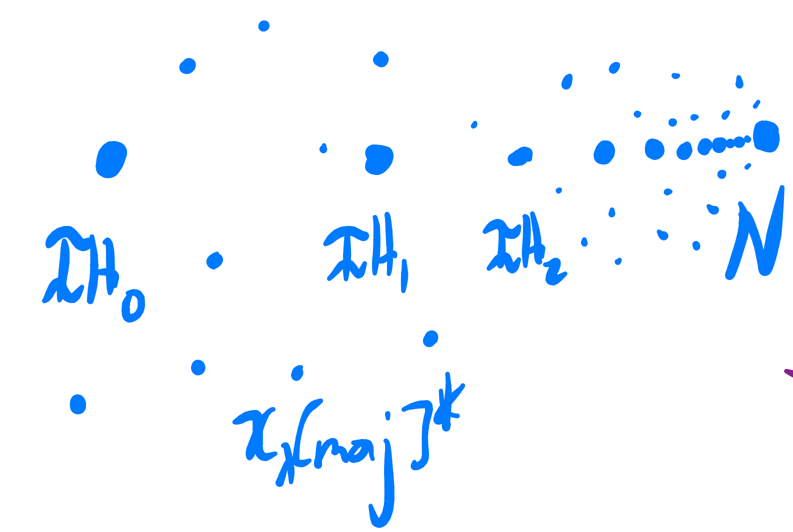
Compactification w.r.t. Lévy metric:



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Distribution: moments and cumulants

Thm (Method of moments) If

$$\lim_{n\to\infty} \mu_d^{X_n} = \mu_d^{X} \qquad \forall d \in \mathbb{Z}_{\geq 1},$$

then $X_1, X_2, \ldots$ converges in distribution to X.

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then $\mathcal{X}_1, \mathcal{X}_2, \ldots$ converges in distribution to $\mathcal{X}$.

Def The cumulants of $\mathcal{X}_\lambda[\mathrm{maj}]$ are like moments but have better formal properties; see [BKS20a, §2.2].

[Formal power series: $\displaystyle\sum_{d=1}^{\infty} \kappa_d^\lambda \frac{t^d}{d!} := \log \frac{1}{f^\lambda} \mathrm{SYT}^{\mathrm{maj}}(e^t)$.]

Rem $\kappa_1 = \mu$ and $\kappa_2 = \sigma^2$. Have $\kappa_d^{\mathcal{X}} = 0 \ \forall d \geq 3$ if and only if $\mathcal{X}$ is normal.

Can replace moments with cumulants!

Distribution: proof overview

Thm ([BKS20a]) $K_d^\lambda = \dfrac{B_d}{d}\left[\sum_{j=1}^{|\lambda|} j^d - \sum_{c\in\lambda} h_c^d\right]$ where h_c is the hook length, B_d is a Bernoulli number.

Ex

8	7	6	3	2	1
4	3	2			
3	2	1			

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Ex

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Rem Follows from Stanley's q-hook length formula [EC2, Cor. 7.21.5]

$$\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) = q^{b(\lambda)} \frac{[n]_q!}{\prod_{c \in \lambda} [h_c]_q}$$

and a general formula for cumulants of $\displaystyle\prod_{j=1}^{m} \frac{[a_j]_q}{[b_j]_q}$ first stated explicitly by Hwang–Zacharovas [HZ15, §4.1]; see [BKS20a, Rem. 2.10] for more history.

See [BKS20a, Thm. 2.9] for general statement, including moments.

Distribution: consequences

$\boxed{\text{Cor}}$

- New proof of asymptotic normality for $[n]_q!$ due to Feller [Fel45].
- New proof of asymptotic normality of $\binom{n}{k}_q$ due to Mann–Whitney [MW47].
- New proof of asymptotic normality of $\frac{1}{[n+1]_q}\binom{2n}{n}_q$ due to Chen–Wang–Wang [CWW08].
- New proof of asymptotic normality for $\binom{n}{\alpha}_q$ due to Diaconis [Dia88, p.128-129], Canfield–Janson–Zeilberger [CJZ11].
- Common framework for all of these!

Unimodality: background

- Unimodality for coefficients of $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q)$ goes back implicitly all the way to Sylvester's 1878 proof for $\binom{n}{k}_q$ [Syl78]!

- True for $\binom{n}{\alpha}_q$ by Hard Lefschetz Theorem from algebraic geometry on flag manifolds.

- Fails for general $n!$ (e.g. when $\lambda=(a^b)$ for $a,b\geq 2$)

Unimodality: conjectures

Conj 1 ([BKS20a, Conj. 8.1]) The coefficients of $SYT(\lambda)^{maj}(q)$ are unimodal if λ has at least 4 $\underbrace{\text{corners}}_{\text{cells with } h_c = 1}$. There is an explicit list of exceptions for ≤ 3 corners.

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Conj 2 ([BKS20a, Conj. 8.5]) Uniformly for all $\lambda \vdash n$ and k,

$$\left| \mathbb{P}[X_\lambda[\mathrm{maj}] = k] - \underbrace{\mathcal{N}(k; \mu^\lambda, \sigma^\lambda)}_{\text{normal density}} \right| = O\left(\frac{1}{\sigma^\lambda \mathrm{aft}(\lambda)} \right).$$

Rem This would give a local limit theorem for $b_{\lambda,r}$; contrast with $a_{\lambda,r}$.

Notoriously technical! Since $f^\lambda \gg n$, would "mostly" imply unimodality.

Unimodality: conjectures

Def A sequence $(b_1, b_2, \ldots, b_k)$ is parity-unimodal if $(b_1, b_3, b_5, \ldots)$ and $(b_2, b_4, b_6, \ldots)$ are each unimodal.

Thm (Stucky [Stu18, Thm. 1.3]) The coefficients of $SYT(n,n)^{maj}(q)$ are parity-unimodal.

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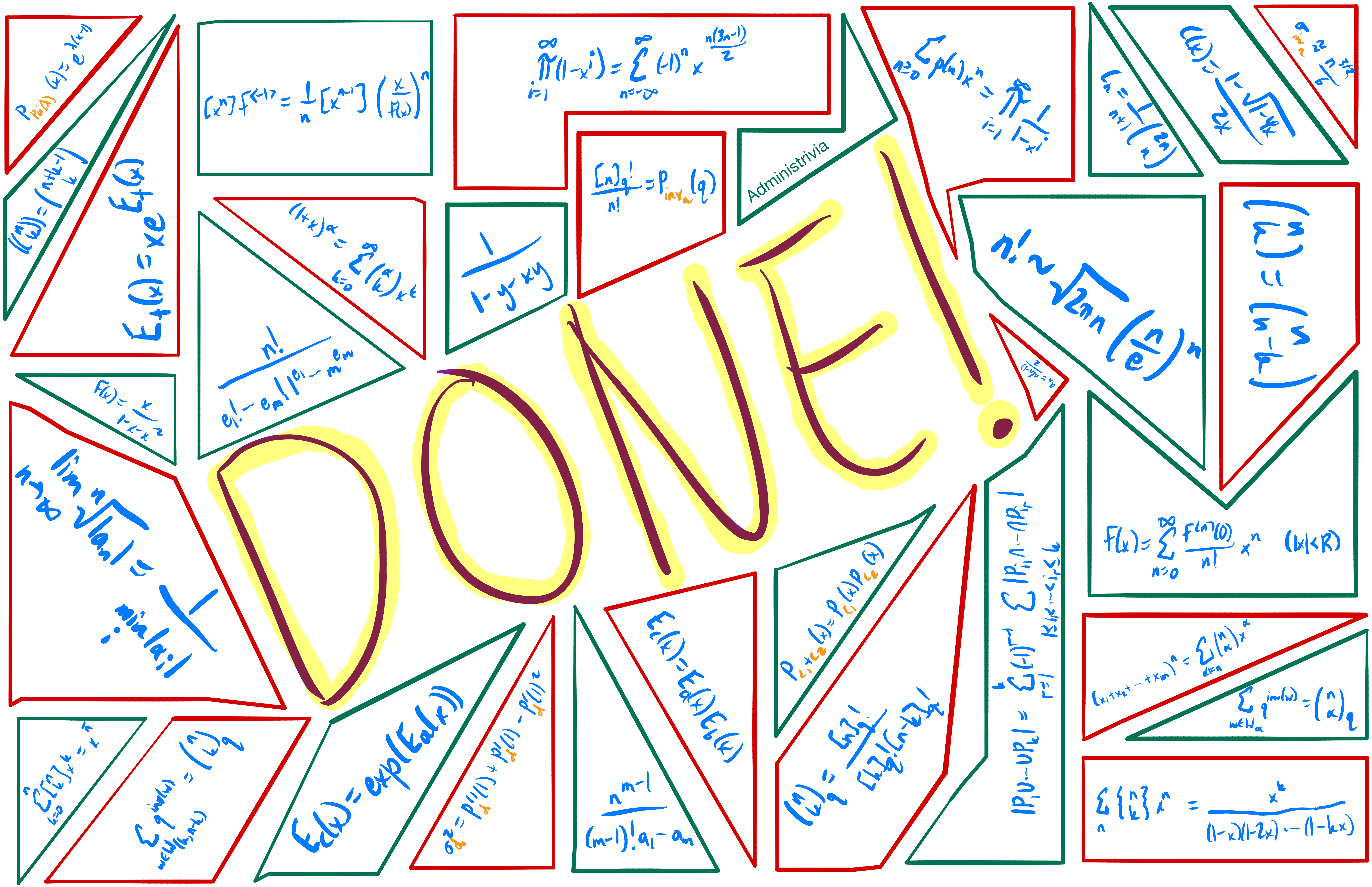
Thm (Stucky [Stu18, Thm. 1.3]) The coefficients of $\mathrm{SYT}(n,n)^{\mathrm{maj}}(q)$ are parity-unimodal.

Conj 3 ([BKS20b, Conj. 9.1]) The coefficients of $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q)$ are parity-unimodal for all λ.

Rem True for $n \leq 50$. Would imply internal zeros classification.
Stucky's argument constructed an $\mathfrak{sl}_2$-action on rational Cherednik algebras.
The $\mathfrak{sl}_2$-action approach goes back to Sylvester [Syl87]!
See [Hai94, §3.1] for a prototype in a related context.


DONE!
Administrivia

Main papers

- [BKS20a] Sara C. Billey, Matjaž Konvalinka, and Joshua P. Swanson. "Asymptotic normality of the major index on standard tableaux". Adv. in Appl. Math. 113 (2020)
- [BKS20b] Sara C. Billey, Matjaž Konvalinka, and Joshua P. Swanson. "Tableau posets and the fake degrees of coinvariant algebras". Advances in Mathematics 371 (2020)
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Further directions

Q1 Distribution of maj on $SYT(\lambda/\mu)$ in general skew case?

Rem See q-Naruse formula of Morales-Pak-Panova [MPP18, (3.4)].

Our argument works for the "formal cumulants" of the "leading term." See [BKS20a, Conj. 8.6].

Q2 Distribution of coefficients of $\mathfrak{S}_w(1, q, q^2, \ldots)$? ("Right-skewed"?)

Q3 When is $(b_{\lambda,0}, b_{\lambda,1}, \ldots)$ log-concave? ("Usually"?)

Q4 Geometric explanation a la Hard Lefschetz? (Does not naively restrict!)