A GENTLE INTRODUCTION TO COINVARIANT ALGEBRAS

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ABSTRACT. These notes were for a lecture given in the informal post-doc seminar at the University of California, San Diego on November 12th, 2019.

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1. SYMMETRIC POLYNOMIALS

Definition 1.1. The *symmetric group* S_n is the group of bijections on $\{1, \ldots, n\}$. The symmetric group acts on the polynomial ring $\mathbb{Q}[\mathbf{x}_n] \coloneqq \mathbb{Q}[x_1, \ldots, x_n]$ by

$$\sigma(x_i) \coloneqq x_{\sigma(i)}.$$

The S_n-invariants of $\mathbb{Q}[\mathbf{x}_n]$ are the *symmetric polynomials*.

Example 1.2. If we wanted to come up with many examples of symmetric polynomials, we would quickly stumble upon the idea of "symmetrizing":

$$f \mapsto \sum_{\sigma \in S_n} f.$$

For example, $x_1 \in \mathbb{Q}[\mathbf{x}_3] \mapsto 2(x_1 + x_2 + x_3)$.

Definition 1.3. Symmetrizing the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ yields (up to a scale factor) the *monomial symmetric polynomial*

$$m_\alpha \coloneqq \sum_{\sigma \in S_n} x_1^{\alpha_{\sigma(1)}} \cdots x_n^{\alpha_{\sigma(n)}}.$$

Note that m_α has degree $d \coloneqq \alpha_1 + \cdots + \alpha_n$. We may as well assume $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n \geq 0$. Such a sequence is called an *integer partition* of d of length at most n, written $\alpha \vdash d$ with $\ell(\alpha) \leq n$.

Remark 1.4. It is quite straightforward to see that $\{m_\alpha : \alpha \textit{ is a partition of length } n\}$ is a $\mathbb{Q}$-basis for $\mathbb{Q}[\mathbf{x}_n]^{S_n}$.

Question 1.5. *"How many" symmetric polynomials are there? "What fraction" of polynomials are symmetric?*

Definition 1.6. Let $W = \oplus_{j=0}^\infty W_j$ be a graded vector space with $\dim W_j < \infty$. The *Hilbert series* of W is

$$\mathrm{Hilb}(W; q) \coloneqq \sum_{j=0}^\infty q^j \dim W_j.$$

Date: November 12, 2019.

Example 1.7. We have

$$\mathrm{Hilb}(\mathbb{Q}[x];q) = \sum_{j=0}^{\infty} q^j = \frac{1}{1-q}.$$

One may check that Hilbert series multiply over tensor products, so for instance

$$\begin{aligned}\mathrm{Hilb}(\mathbb{Q}[\mathbf{x}_n];q) &= \mathrm{Hilb}(\mathbb{Q}[x_1] \otimes \cdots \otimes \mathbb{Q}[x_n];q) \\ &= \mathrm{Hilb}(\mathbb{Q}[x_1];q) \cdots \mathrm{Hilb}(\mathbb{Q}[x_n];q) \\ &= (1-q)^{-n}.\end{aligned}$$

(From the binomial series, it follows that $\dim \mathbb{Q}[\mathbf{x}_n]_d = (-1)^d \binom{-n}{d} = (-1)^d \frac{(-n)(-n-1)\cdots(-n-d+1)}{d!} = \binom{n+d-1}{d}$.)

Remark 1.8. What's $\mathrm{Hilb}(\mathbb{Q}[\mathbf{x}_n]^{S_n};q)$? How many integer partitions have a fixed degree? We would need to count

$$\{\alpha_1 \geq \cdots \geq \alpha_n \geq 0 : \alpha_1 + \cdots + \alpha_n = d\},$$

i.e. we would need to count the lattice points in a polytope. This should discourage us from expecting a completely explicit answer.

Remark 1.9. After working with partitions for any length of time, you'll stumble upon "exponential notation": $\alpha = 1^{e_1} 2^{e_2} \cdots$ where e_i is the number of times i appears in α. For instance, $4 \geq 2 \geq 2 \geq 1$ becomes $1^1 2^2 3^0 4^1 5^0 6^0 \cdots$. Note that $d = e_1 + 2e_2 + \cdots + ne_n$ is more complicated in exponential notation, but we've virtually removed the restrictions on the e_i's. Inspiration strikes!

$$\begin{aligned}\mathrm{Hilb}(\mathbb{Q}[\mathbf{x}_n]^{S_n};q) &= \sum_{d \geq 0} q^d \sum_{\substack{\alpha \vdash d \\ \ell(\alpha) \leq n}} 1 = \sum_{e_1,\ldots,e_n \geq 0} q^{e_1+2e_2+\cdots+ne_n} \\ &= \mathrm{Hilb}(\mathbb{Q}[x_1, x_2^2, x_3^3, \ldots, x_n^n];q) = \mathrm{Hilb}(\mathbb{Q}[x_1] \otimes \cdots \otimes \mathbb{Q}[x_n^n];q) \\ &= \mathrm{Hilb}(\mathbb{Q}[x_1];q) \cdots \mathrm{Hilb}(\mathbb{Q}[x_n^n];q) = (1-q)^{-1}(1-q^2)^{-1} \cdots (1-q^n)^{-1}.\end{aligned}$$

(From this, we can say the dth coefficient is a convolution of binomial coefficients, but not much more.)

Corollary 1.10. *We have*

$$\begin{aligned}\frac{\mathrm{Hilb}(\mathbb{Q}[\mathbf{x}_n];q)}{\mathrm{Hilb}(\mathbb{Q}[\mathbf{x}_n]^{S_n};q)} &= \frac{1-q^n}{1-q} \frac{1-q^{n-1}}{1-q} \cdots \frac{1-q}{1-q} \\ &= (1+q+\cdots+q^{n-1})(1+q+\cdots+q^{n-2}) \cdots 1 =: [n]_q!.\end{aligned}$$

Taking $q \to 1$, we may heuristically conclude that $1/n!$ of polynomials in $\mathbb{Q}[\mathbf{x}_n]$ are actually symmetric.

2. Classical invariant theory

We just saw $\mathbb{Q}[\mathbf{x}_n]^{S_n}$ has the same Hilbert series as the polynomial ring $\mathbb{Q}[x_1, x_2^2, \ldots, x_n^n]$. The invariants are a $\mathbb{Q}$-algebra. Dare we hope they're actually freely generated? Yes!

Theorem 2.1 (Fundamental theorem of symmetric polynomials)**.** *Every element of $\mathbb{Q}[\mathbf{x}_n]^{S_n}$ can be written uniquely as a polynomial in the power-sum symmetric polynomials $p_i(\mathbf{x}_n) := \sum_{j=1}^n x_j^i$ for $i = 1, \ldots, n$.*

(Most proofs rely on a variation of the idea of picking off a "leading monomial" m_α recursively.)

Question 2.2 ("First problem of invariant theory", late 1800's)**.** *Let $G \leq \mathrm{GL}(V)$ be a finite group acting naturally on $S(V) := \mathrm{Sym}(V)$ for a finite-dimensional vector space V over a field k. If $S(V)^G$ finitely generated?*

(If V has basis $x_1, \ldots, x_n$, we may identify $S(V)$ with the polynomial ring $k[x_1, \ldots, x_n]$.)

Hilbert famously solved this problem in 1890 *and* introduced Hilbert's Basis Theorem to do it! We'll sketch the argument now.

Definition 2.3. The *Reynolds operator* on $S(V)$ is

$$R(f) \coloneqq \frac{1}{|G|}\sum_{\sigma\in G}\sigma\cdot f.$$

Note that $R\colon S(V)\twoheadrightarrow S(V)^G$ is in fact an $S(V)^G$-algebra morphism: if $f\in S(V)^G$ and $g\in S(V)$, then

$$R(fg)=\frac{1}{|G|}\sum_{\sigma\in G}\sigma\cdot fg=\frac{1}{|G|}\sum_{\sigma\in G}(\sigma\cdot f)(\sigma\cdot g)=\frac{1}{|G|}\sum_{\sigma\in G}f(\sigma\cdot g)=f\frac{1}{|G|}\sum_{\sigma\in G}\sigma\cdot g=fR(g).$$

Definition 2.4. The *coinvariant ideal* of $S(V)$ is the ideal $S(V)^G_+$ generated by all homogeneous, non-constant G-invariants.

Lemma 2.5. *Let $f_1,\ldots,f_r\in S(V)^G$ be non-constant and homogeneous. Then $S(V)^G=k[f_1,\ldots,f_r]$ if and only if $S(V)^G_+=\langle f_1,\ldots,f_r\rangle$.*

Proof. ($\Rightarrow$) If $S(V)^G=k[f_1,\ldots,f_r]$, then we may replace every generator of $S(V)^G_+$ with a polynomial combination of $f_1,\ldots,f_r$ with non-constant coefficient, so $S(V)^G_+=\langle f_1,\ldots,f_r\rangle$.

($\Leftarrow$) Suppose $S(V)^G_+=\langle f_1,\ldots,f_r\rangle$. Clearly $S(V)^G\supset k[f_1,\ldots,f_r]$, so we must show the reverse containment. We do so by induction on the degree d, i.e. we show $S(V)^G_{\le d}=k[f_1,\ldots,f_r]_{\le d}$. The base case $d=0$ is trivial, so take $d>0$ and suppose $S(V)^G_{<d}=k[f_1,\ldots,f_r]_{<d}$. Pick $f\in S(V)^G$ homogeneous of degree d. Since $S(V)^G\subset S(V)^G_+=\langle f_1,\ldots,f_r\rangle$,

$$f=f_1s_1+\cdots+f_rs_r$$

for some homogeneous elements $s_1,\ldots,s_r\in S(V)$ of degree $<d$. Apply the Reynolds operator to get

$$f=R(f)=f_1R(s_1)+\cdots+f_rR(s_r).$$

But $R(s_i)\in S(V)^G_{<d}=k[f_1,\ldots,f_r]_{<d}$, so indeed $f\in k[f_1,\ldots,f_r]$! ☐

Example 2.6. By the Fundamental Theorem of Symmetric Polynomials and the lemma, $\mathbb{Q}[x_1,\ldots,x_n]^{S_n}_+=\langle p_1,\ldots,p_n\rangle$.

Theorem 2.7 (Hilbert, 1890)**.** *$S(V)^G$ is finitely generated.*

Proof. By Hilbert's Basis Theorem, the ideal $S(V)^G_+$ is finitely generated, and we may use a homogeneous, non-constant, G-invariant set of generators $f_1,\ldots,f_r$. By the lemma, $S(V)^G=k[f_1,\ldots,f_r]$. ☐

3. Coinvariant algebras

Question 3.1. *Are the coinvariant ideals $S(V)^G_+$ or the corresponding quotients "interesting"?*

Definition 3.2. The *classical coinvariant algebra* of G is $S(V)/S(V)^G_+$. This is a graded F-algebra and a graded G-module.

Example 3.3. The "original" coinvariant algebra is

$$\frac{\mathbb{Q}[x_1,\ldots,x_n]}{\langle p_1,\ldots,p_n\rangle}.$$

This is finite-dimensional. In fact, $x_i^n\in\langle p_1,\ldots,p_n\rangle$! Here's a slick argument:

$$(t-x_1)\cdots(t-x_n)=t^n+(\text{lower order terms in } t \text{ whose coefficients in } \mathbb{Q}[x_1,\ldots,x_n] \text{ are symmetric}).$$

Now let $t=x_i$, giving $x_i^n\in\langle p_1,\ldots,p_n\rangle$. Thus $\dim\mathbb{Q}[x_1,\ldots,x_n]/\langle p_1,\ldots,p_n\rangle\le n^n$.

Remark 3.4. Emil Artin gave a clever argument in his *Galois Theory* text which, when unwound, more generally shows that $h_r(x_r,\ldots,x_n)\in\mathbb{Q}[\mathbf{x}_n]^{S_n}_+$, using the *complete homogeneous symmetric polynomials*. For our purposes, we only need to know that

$$h_r(x_r,\ldots,x_n)=x_r^r+(\text{terms of lower } x_r\text{-degree}).$$

It follows that $\{x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} : 0 \leq a_i < i\}$ descends to a spanning set for the coinvariant algebra, giving a maximum dimension of $n!$ with a corresponding maximum (coefficient-wise) Hilbert series of $[n]_q!$.

Remark 3.5. Indeed, for quite generic reasons concerning regular sequences, it follows from the $[n]_q!$ upper bound and our calculations concerning the "fraction" of polynomials which are symmetric polynomials that

$$\{x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} : 0 \leq a_i < i\}$$

descends to a basis, called the *Artin basis*.

Question 3.6. *When is $S(V)^G$ free?*

Theorem 3.7 (Chevalley, Shephard–Todd, Serre). *Suppose* $\mathrm{char}(F) \nmid |G|$. *Then $S(V)^G$ is a polynomial ring if and only if G is generated by pseudo-reflections, namely elements $\sigma \in \mathrm{GL}(V)$ such that* $\dim \ker(\sigma - I) = \dim(V) - 1$.

Theorem 3.8 (Shephard–Todd). *There is an explicit classification of such G (in characteristic 0, at least), consisting of one infinite family $G(m, p, n)$ and 34 exceptional groups.*

Theorem 3.9 (Chevalley). *In this case, $S(V)/S(V)^G_+$ carries the regular representation of G and $S(V) \cong S(V)^G \otimes S(V)/S(V)^G_+$ as graded G-modules.*

Remark 3.10. $\mathbb{Q}[x_1, \ldots, x_n]/\langle p_1, \ldots, p_n\rangle$ is a *graded analogue* of the regular representation of S_n! Much of my research recently has been motivated by understanding aspects of the graded irreducible decomposition of this and related quotients.

Theorem 3.11 (Borel). *The cohomology of the complete flag manifold $H^*(G/B, \mathbb{C})$ is isomorphic to $\mathbb{C}[x_1, \ldots, x_n]/\langle p_1, \ldots, p_n\rangle$.*

Remark 3.12. The $n!$ dimensions of the quotient are reflected by the $n!$ *Schubert varieties* comprising the Schubert cell decomposition of the complete flag manifold. The coinvariant algebra consequently has intimate connections to both toplogy (Borel–Moore homology) and algebraic geometry (Chow rings).

Theorem 3.13 (Lascoux–Schützenberger). *There is an explicitly defined set of polynomials called Schubert polynomials representing the classes of Schubert varieties in $\mathbb{C}[x_1, \ldots, x_n]/\langle p_1, \ldots, p_n\rangle$. These polynomials have the remarkable property that they are "stable" as $n \to \infty$ in a natural sense.*