Alternant polynomial differential forms and coinvariants of finite pseudo-reflection groups

SIAM TX-LA Section at SMU, Dallas, Texas

November 3rd, 2019

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Based on joint work with

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Slides: http://www.math.ucsd.edu/~jswanson/talks/2019_SMU.pdf

Outline

- The *Delta conjecture*, Zabrocki's *super diagonal coinvariant conjecture*, and *alternants*
- *Pseudo-reflection groups* and *invariant theory*
- Description of the *alternant invariants* and *coinvariants* via *harmonics*
- *Perfect pairings* and *differential actions*

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The Delta conjecture and Zabrocki's conjecture

- $\mathbb{Q}[\boldsymbol{x}_n, \boldsymbol{y}_n, \boldsymbol{\theta}_n]$ = polynomial ring with commuting indeterminates $x_1, \ldots, x_n$, $y_1, \ldots, y_n$, and anti-commuting indeterminates $\theta_1, \ldots, \theta_n$, so $\theta_i \theta_j = -\theta_j \theta_i$.
- $\mathfrak{S}_n$ = the symmetric group of order n, acting via simultaneous permutation: $\sigma(x_i) := x_{\sigma(i)}, \sigma(y_i) := y_{\sigma(i)}, \sigma(\theta_i) := \theta_{\sigma(i)}$.
- $\mathbb{Q}[\boldsymbol{x}_n, \boldsymbol{y}_n, \boldsymbol{\theta}_n]_+^{\mathfrak{S}_n}$ = the *coinvariant ideal*, generated by homogeneous, non-constant $\mathfrak{S}_n$-invariants
- $\mathbb{Q}[\boldsymbol{x}_n, \boldsymbol{y}_n, \boldsymbol{\theta}_n]/\mathbb{Q}[\boldsymbol{x}_n, \boldsymbol{y}_n, \boldsymbol{\theta}_n]_+^{\mathfrak{S}_n}$ = the *super diagonal coinvariant algebra*, a tri-graded $\mathfrak{S}_n$-module.

The θ-degree 0 component is the classic Garsia–Haiman diagonal coinvariant algebra with intimate connections to Macdonald theory. Everything will be hard in full generality!

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Question

What is the tri-graded $\mathfrak{S}_n$-isomorphism type of the super diagonal coinvariant algebra?

Conjecture (Zabrocki '19)

We have

$$\mathrm{GrFrob}(\mathbb{Q}[\boldsymbol{x}_n, \boldsymbol{y}_n, \boldsymbol{\theta}_n]/\mathbb{Q}[\boldsymbol{x}_n, \boldsymbol{y}_n, \boldsymbol{\theta}_n]_+^{\mathfrak{S}_n}; q, t, z) = \sum_{i=1}^{n} z^{n-k} \Delta'_{e_{k-1}} e_n$$

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- The $z = 0$ case recovers the famous "*master formula*" conjecture of Garsia–Haiman, which was later proved by Haiman after solving the *n! conjecture* and *Macdonald positivity conjecture*.
- The right-hand side is half (one-third) of the *Delta conjecture* of Haglund–Remmel–Wilson, which gives two explicit monomial expansions for this expression as a generating function for labeled Dyck paths.
- Full version is likely to be very hard, so *restrict to $t = 0$* for the rest of the talk. See work of Pawlowski, Rhoades, Wilson, . . . for this case.

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Alternants

- Marino Romero noted that, for the sgn representation at $t = 0$, Zabrocki's conjecture reduces to

$$\mathrm{Hilb}((\mathbb{Q}[\boldsymbol{x}_n, \boldsymbol{\theta}_n]/\mathbb{Q}[\boldsymbol{x}_n, \boldsymbol{\theta}_n]_+^{\mathfrak{S}_n})^{\mathrm{sgn}}; q, z) \stackrel{?}{=} \prod_{i=1}^{n-1}(z + q^i).$$

Why?

- First hint: $\Delta := \prod_{1 \le i < j \le n}(x_i - x_j)$ is non-zero in $(\mathbb{Q}[\boldsymbol{x}_n, \boldsymbol{\theta}_n]/\mathbb{Q}[\boldsymbol{x}_n, \boldsymbol{\theta}_n]_+^{\mathfrak{S}_n})^{\mathrm{sgn}}$.
- $\prod_{i=1}^{n-1}(z + q^i) = q^{\binom{n}{2}} + q^{\binom{n}{2}-1}z + \cdots$ where $\cdots$ has lower total degree (q-degree plus z-degree)
- Δ contributes $q^{\binom{n}{2}}$. *What about* $q^{\binom{n}{2}-1}z$?
- Candidate for $q^{\binom{n}{2}-1}z$: $\mathrm{d}\Delta$ where d is the *exterior derivative*

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Main results

The rest of the talk is devoted to explaining the following.

Let $G \leq \mathrm{GL}(V)$ be a pseudo-reflection group over a subfield of $\mathbb{C}$ and let M be an r-dimensional G-module.

Theorem

Suppose $J_{M^*} \doteq \Delta_{M^*}$ and $M^G = 0$. Then the 2^r elements

$$\{d^*_{i_1} \cdots d^*_{i_k} \Delta_{M^*} \mid \{i_1, \ldots, i_k\} \subset [r]\}$$

form a basis of $\mathcal{H}(\mathrm{S}(V^*) \otimes \wedge M^*)^{\det_{M^*}}$, and their images form a basis of $(\mathrm{S}(V^*) \otimes \wedge M^* / \mathcal{J}^*_M)^{\det_{M^*}}$.

Corollary

In this situation,

$$\mathrm{Hilb}((\mathrm{S}(V^*) \otimes \wedge M^* / \mathcal{J}^*_M)^{\det_{M^*}}; q, z) = \prod_{i=1}^{r} (z + q^{e_i^{M^*}}).$$

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Main results

- The $t = 0$, sgn component of Zabrocki's conjecture follows when G consists of $n \times n$ permutation matrices and M is the *standard representation*.
- As an intermediate step, we also get:

Theorem

Suppose $J_{M^*} \doteq \Delta_{M^*}$. Then either of the sets

$$\{f_1^{a_1} \cdots f_n^{a_n} d_{i_1}^* \cdots d_{i_k}^* \Delta_{M^*} : \{i_1, \ldots, i_k\} \subset [n], a_j \in \mathbb{Z}_{\geq 0}\}$$

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Invariant theory of pseudo-reflection groups

Classical theory:

- F = a field of characteristic 0
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- Chevalley and Shephard–Todd showed $S(V^*)^G = F[f_1, \ldots, f_n]$ for *algebraically independent* $f_1, \ldots, f_n$ called *basic invariants*
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Jacobians and discriminants

- The *Jacobian* J_M of M is defined by $\omega_1^M \wedge \cdots \wedge \omega_r^M \doteq J_M \eta$ where $0 \neq \eta \in \wedge^r M^*$ and $\doteq$ means equality up to a non-zero constant
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- We have $\Delta_M = J_{\det_M}$
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Corollary

If the pseudo-reflections of G act on M as pseudo-reflections or trivially, then $\Delta_M \doteq J_M$. In particular, $\Delta_V \doteq J_V$ and $\Delta_{V^*} \doteq J_{V^*}$.

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Suppose $J_{M^*} \doteq \Delta_{M^*}$ *and* $M^G = 0$*. Then the* 2^r *elements*

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form a basis of $\mathcal{H}(\mathrm{S}(V^*) \otimes \wedge M^*)^{\det_{M^*}}$*, and their images form a basis of* $(\mathrm{S}(V^*) \otimes \wedge M^* / \mathcal{J}_M^*)^{\det_{M^*}}$*.*

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Perfect pairings and an action

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There is a canonical G-invariant perfect pairing

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which extends the perfect pairing $V \times V^* \to F$ given by $(v, \lambda) \mapsto \lambda(v)$.

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There is a canonical G-equivariant action of $\mathsf{S}(V) \otimes \wedge M$ on $\mathsf{S}(V^*) \otimes \wedge M^*$, where $\widetilde{\omega} \in \mathsf{S}(V) \otimes \wedge M$ acts by the adjoint of multiplication by $\widetilde{\omega}$:

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These actions are very old: when G consists of permutation matrices and we identify the standard basis of $\mathbb{Q}^n$ and its dual, $\mathsf{S}(V)$ acts on $\mathsf{S}(V^*)$ as *partial differentiation operators*. Similarly M acts on $\wedge M^*$ by *interior products*.

Perfect pairings and an action

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Harmonics

The space of *G-harmonic* elements of $\mathsf{S}(V^*) \otimes \wedge M^*$ is the *orthogonal complement* of $\mathcal{J}_M$ with respect to the natural perfect pairing $\langle -, - \rangle$:

$$\mathcal{H}(\mathsf{S}(V^*) \otimes \wedge M^*) := \{\omega \in \mathsf{S}(V^*) \otimes \wedge M^* : \langle \widetilde{\omega}, \omega \rangle = 0 \text{ for all } \widetilde{\omega} \in \mathcal{J}_M\}$$

- If $F \subset \mathbb{C}$, then $\mathsf{S}(V^*) \otimes \wedge M^* = \mathcal{H}(\mathsf{S}(V^*) \otimes \wedge M^*) \oplus \mathcal{J}_M^*$
- Hence *natural projection* gives

$$(\mathsf{S}(V^*) \otimes \wedge M^*)/\mathcal{J}_M^* \cong \mathcal{H}(\mathsf{S}(V^*) \otimes \wedge M^*)$$

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Another action

Let $v_1, \ldots, v_n \in V$ be a basis with dual basis $\lambda_1, \ldots, \lambda_n$. The *exterior derivative* is

$$\mathrm{d} = \sum_i \partial_{v_i} \otimes m_{\lambda_i} \in \mathrm{End}_F(\mathrm{S}(V^*) \otimes \wedge V^*)$$

where m_λ is the wedge-by-λ operator. This is *not obtained* from the $\mathrm{S}(V) \otimes \wedge M$ action on $\mathrm{S}(V^*) \otimes \wedge M^*$. What operators are missing?

Corollary

There is a canonical G-equivariant action of $\mathrm{S}(V) \otimes \wedge M^*$ on $\mathrm{S}(V^*) \otimes \wedge M^*$, where $\wedge M^*$ acts on $\wedge M^*$ by wedging and $\mathrm{S}(V)$ acts on $\mathrm{S}(V^*)$ as before.

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The final piece

- Let d_i^* be the action of $\widetilde{\omega}_i{}^{M^*} \in (\mathrm{S}(V) \otimes M^*)^G$ on $\mathrm{S}(V^*) \otimes \wedge M^*$.
- This *lowers* $\mathrm{S}(V^*)$*-degree by* $e_i^{M^*}$ and *raises* $\wedge M^*$*-degree by* 1.
- For Zabrocki's conjecture, we can use $d_i^* = \sum_{j=1}^{n} \frac{\partial^i}{\partial x_j^i} \theta_j$.
- *Key insight:* apply the d_i^*'s in all possible ways to Δ_{M^*}!
- Must verify result *stays in* $\mathcal{H}(\mathrm{S}(V^*) \otimes \wedge M^*)^{\det_{M^*}}$ (it does), is *linearly independent* (it is), and *spans* (it does).

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Main results, one last time

Theorem

Suppose $F \subset \mathbb{C}$, $J_{M^*} \doteq \Delta_{M^*}$ and $M^G = 0$. Then the 2^r elements

$$\{d_{i_1}^* \cdots d_{i_k}^* \Delta_{M^*} \mid \{i_1, \dots, i_k\} \subset [r]\}$$

form a basis of $\mathcal{H}(\mathrm{S}(V^*) \otimes \wedge M^*)^{\det_{M^*}}$, and their images form a basis of $(\mathrm{S}(V^*) \otimes \wedge M^* / \mathcal{J}_M^*)^{\det_{M^*}}$.

Corollary

In this situation,

$$\mathrm{Hilb}((\mathrm{S}(V^*) \otimes \wedge M^* / \mathcal{J}_M^*)^{\det_{M^*}}; q, z) = \prod_{i=1}^{r} (z + q^{e_i^{M^*}}).$$

Main results, one last time

Theorem

Suppose $F \subset \mathbb{C}$ *and* $J_{M^*} \doteq \Delta_{M^*}$*. Then either of the sets*

$$\{f_1^{a_1} \cdots f_n^{a_n} d_{i_1}^* \cdots d_{i_k}^* \Delta_{M^*} : \{i_1, \ldots, i_k\} \subset [n], a_j \in \mathbb{Z}_{\geq 0}\}$$

or

$$\{d_{i_1}^* \cdots d_{i_k}^* f_1^{a_1} \cdots f_n^{a_n} \Delta_{M^*} : \{i_1, \ldots, i_k\} \subset [n], a_j \in \mathbb{Z}_{\geq 0}\}$$

form bases of $(\mathrm{S}(V^*) \otimes \wedge M^*)^{\det_{M^*}}$.

$\mathcal{FIN}$

THANKS!