Asymptotics of Mahonian statistics

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Based partly on joint work with
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Inversion number definition

Definition

The *symmetric group* is

$$S_n := \{\text{bijections } \pi\colon \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}\}.$$

The elements are *permutations*.

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$$\mathrm{inv}(35142) = \#\{(1,3),(1,5),(2,3),(2,4),(2,5),(4,5)\} = 6.$$

Zeilberger: this is the "**most important permutation statistic**".

Inversion number bijection

Lemma (Classical)

The map

$$\Phi\colon S_n \to \{\alpha \in \mathbb{Z}_{\geq 0}^n : (\alpha_1, \ldots, \alpha_n) \leq (n-1, n-2, \ldots, 1, 0)\}$$
$$\alpha_i := \#\{j : i < j \leq n, \pi(i) > \pi(j)\}$$

is a **bijection**.

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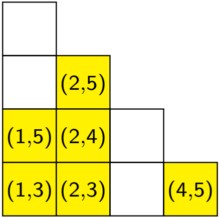
$$\mathrm{inv}(35142) = \#\{(1,3), (1,5), (2,3), (2,4), (2,5), (4,5)\} = 6$$



$$\Rightarrow \Phi(35142) = (2,3,0,1,0)$$

Inversion number distribution

Clearly $\mathrm{inv}(\pi) = \alpha_1 + \cdots + \alpha_n$. Hence:

Corollary

- inv *on* S_n *is* **symmetrically distributed** *with mean* $((n-1)+(n-2)+\cdots+0)/2 = \frac{1}{2}\binom{n}{2}$.

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- inv *on* S_n *is* **symmetrically distributed** *with mean* $((n-1)+(n-2)+\cdots+0)/2 = \frac{1}{2}\binom{n}{2}$.
- *The ordinary generating function of* inv *on* S_n *is*

$$\sum_{\pi \in S_n} q^{\mathrm{inv}(\pi)} = \prod_{i=1}^{n} \sum_{\alpha_i=0}^{n-i} q^{\alpha_i}$$
$$= (1+q+\cdots+q^{n-1})(1+q+\cdots+q^{n-2})\cdots(1)$$
$$=: [n]_q!$$

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- $\mathcal{X}_{\mathrm{inv}} \sim \mathcal{U}_{n-1} + \cdots + \mathcal{U}_1$ *is the* **sum of independent discrete uniform random variables**.

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Theorem (Feller '45; implicit earlier)

As $n \to \infty$, $\mathcal{X}_{\mathsf{inv}}$ *is asymptotically normal.*

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As $n \to \infty$, $\mathcal{X}_{\text{inv}}$ is asymptotically normal. That is, for all $u \in \mathbb{R}$,

$$\lim_{n\to\infty} \mathbb{P}[\mathcal{X}^*_{\text{inv}} \le u] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} e^{-x^2/2}\, dx$$

where

$$\mathcal{X}^*_{\text{inv}} := \frac{\mathcal{X}_{\text{inv}} - \mu_n}{\sigma_n}$$

with

$$\mu_n = \frac{n(n-1)}{4}, \qquad \sigma_n^2 = \frac{2n^3 + 3n^2 - 5n}{72}.$$

Inversion number application

Theorem (Feller '45; implicit earlier)

As $n \to \infty$, $\mathcal{X}_{\mathrm{inv}}$ is asymptotically normal.

Example (*Kendall's τ test*)

Say some process generated distinct real numbers $x_1, x_2, \ldots, x_n$, one per day. You want to know if the process is **independent of time**. Turn the data into a permutation π while preserving the relative order of data points and compute $\mathrm{inv}(\pi)$. Since $\mathcal{X}_{\mathrm{inv}} \approx \mathcal{N}(\mu_n, \sigma_n)$, independent data would have

$$|(\mathrm{inv}(\pi) - \mu_n)/\sigma_n| \leq 3$$

$\approx 99.7\%$ of the time. So, **if this z-score is too big**, say larger than 3, the process is very likely time-dependent.

Major index definition

Definition (MacMahon, early 1900's)

The *descent set* of $\pi \in S_n$ is

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$$\mathrm{maj}(25143) = \mathrm{maj}(25.14.3) = 2 + 4 = 6.$$

Zeilberger: this is the "**second most important permutation statistic**".

Major index bijection

Lemma (Gupta, '78)

For a given $\pi \in S_{n-1}$*, let* $C_\pi \subset S_n$ *be the* n *permutations obtained by inserting* n *into* π *in all possible ways. Then*

$$\{\mathrm{maj}(\pi') - \mathrm{maj}(\pi) : \pi' \in C_\pi\} = \{0, 1, \ldots, n-1\}.$$

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Corollary

There is a bijection

$$\Psi\colon \{\alpha \in \mathbb{Z}^n_{\geq 0} : (\alpha_1, \ldots, \alpha_n) \leq (n-1, n-2, \ldots, 1, 0)\} \to S_n$$

for which $\mathrm{maj}(\Psi(\alpha)) = \alpha_1 + \cdots + \alpha_n$.

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Corollary

The bijection $\Psi \circ \Phi\colon S_n \to S_n$ sends inv *to* maj. *Hence $\mathcal{X}_{\mathrm{inv}} \sim \mathcal{X}_{\mathrm{maj}}$ and $\mathcal{X}_{\mathrm{maj}}$ is asymptotically normal as $n \to \infty$.*

Major index bijection

Example

Use $\alpha = (2, 3, 0, 1, 0)$. Then:

	maj	Δ maj
1	0	0

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25.14.3	6	2

Hence $\Psi((2, 3, 0, 1, 0)) = 25143$.

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	maj	Δ maj
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2.1	1	1
2.13	1	0
2.14.3	4	3
25.14.3	6	2

Hence $\Psi((2, 3, 0, 1, 0)) = 25143$. Since $\Phi(35142) = (2, 3, 0, 1, 0)$, we have

$$(\Psi \circ \Phi)(35142) = 25143$$
$$\text{inv}(35142) = 6 = \text{maj}(25143).$$

Inv and maj

Question (Svante Janson)

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Theorem (Baxter–Zeilberger)

inv *and* maj *on* S_n *are jointly independently asymptotically normally distributed as* $n \to \infty$*. That is, for all* $u, v \in \mathbb{R}$*,*

$$\lim_{n\to\infty} \mathbb{P}[\mathcal{X}^*_{\mathrm{inv}} \leq u, \mathcal{X}^*_{\mathrm{maj}} \leq v] = \frac{1}{2\pi} \int_{-\infty}^{u} \int_{-\infty}^{v} e^{-x^2/2} e^{-y^2/2}\, dy\, dx$$

where $\mathcal{X}^* := (\mathcal{X} - \mu_n)/\sigma_n$ *with* μ_n, σ_n *from Feller's theorem.*

Inv and maj

The Baxter–Zeilberger proof can be summarized as follows:

1. The *method of moments* says it suffices to show that for each fixed $(s, t) \in \mathbb{Z}_{\geq 0}^2$, the (s, t)-mixed moment $\mathbb{E}[(\mathcal{X}_{\text{inv}}^*)^s(\mathcal{X}_{\text{maj}}^*)^t]$ tend to the (s, t)-mixed moment of $\mathcal{N}(0, 1) \times \mathcal{N}(0, 1)$ as $n \to \infty$.

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2. Let $F_{n,i}(p,q) := \sum_{\substack{\pi \in S_n \\ \pi_n = i}} p^{\mathrm{inv}(\pi)} q^{\mathrm{maj}(\pi)}$. Derive a recurrence for $F_{n,i}(p,q)$ by considering the effect of removing the last letter.

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The details are involved and are perhaps best handled by a computer, which can easily compute all the relevant quantities using the recursions. The approach gives me no intuition for *why* the result should be true.

The \$300 question

"Referee Dan Romik believe[s] that we should mention, at this point, the 'explicit' formula of Roselle (mentioned by Knuth) in terms of a certain infinite double product for the q-exponential generating function of $\sum_{\pi\in S_n} p^{\mathrm{inv}(\pi)}q^{\mathrm{maj}(\pi)}$*. Romik believes that this may lead to an alternative proof, that would even imply a stronger result (a local limit law). We strongly doubt this, and [Doron Zeilberger] is hereby offering* \$300 *for the first person to supply such a proof, whose length should not exceed the length of this article [13 pages]." (***)*

Roselle's formula

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Theorem (Roselle)

We have

$$\sum_{n \geq 0} \frac{H_n(p, q) z^n}{(p)_n (q)_n} = \prod_{a, b \geq 0} \frac{1}{1 - p^a q^b z}$$

where $(p)_n := (1 - p)(1 - p^2) \cdots (1 - p^n)$.

A correction factor

If inv and maj on S_n were independent, we would have

$$\frac{H_n(p,q)}{n!} = \frac{[n]_p![n]_q!}{n!^2}.$$

In this case, joint asymptotic normality would follow trivially from individual asymptotic normality.

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$$\frac{H_n(p,q)}{n!} = \frac{[n]_p![n]_q!}{n!^2} F_n(p,q)$$

where

$$F_n(p,q) = \frac{n! \cdot \text{g.f. of size-}n\text{ multisets from } \mathbb{Z}_{\geq 0}^2}{\text{g.f. of size-}n\text{ lists from } \mathbb{Z}_{\geq 0}^2}.$$

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Intuitively, F_n is **"1 to first order"**. This explains "why" Baxter–Zeilberger's result holds and suggests an alternate proof.

Explicit correction factor

Theorem (S.)

There are constants $c_\mu \in \mathbb{Z}$ indexed by integer partitions μ such that

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$$F_n(p,q) = \sum_{d=0}^{n} [(1-p)(1-q)]^d \sum_{\substack{\mu \vdash n \\ \ell(\mu)=n-d}} \frac{c_\mu}{\prod_i [\mu_i]_p [\mu_i]_q}.$$

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Explicitly,

$$c_\mu = \sum_{\lambda \vdash n} \lambda! \sum_{\substack{\Lambda : \Pi(\lambda) \leq \Lambda \\ \text{type}(\Lambda)=\mu}} \mu(\Pi(\lambda), \Lambda).$$

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The $d = 0$ contribution is 1. Hence, $H_n(1,q) = [n]_q!$.

Estimating the correction factor

Theorem (S.)

Uniformly on compact subsets of $\mathbb{R}^2$, we have

$$F_n(e^{is/\sigma_n}, e^{it/\sigma_n}) \to 1 \qquad as \qquad n \to \infty$$

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The argument uses the explicit form of c_μ, the explicit form of the Möbius function on the lattice of set partitions, and some estimates to bound the $d > 0$ contributions to F_n.

Estimating the correction factor

Technical details: easy manipulations give, for $|s|, |t| \leq M$ and n large,

$$|F_n(e^{is/\sigma_n}, e^{it/\sigma_n}) - 1| \leq \sum_{d=1}^{n} \frac{|st|^d}{\sigma_n^{2d}} \sum_{\lambda \vdash n} \lambda! \sum_{\substack{\Lambda : \Pi(\lambda) \leq \Lambda \\ \#\Lambda = n-d}} |\mu(\Pi(\lambda), \Lambda)|.$$

Lemma

Suppose $\lambda \vdash n$ with $\ell(\lambda) = n - k$, and fix d. Then

$$\sum_{\substack{\Lambda : \Pi(\lambda) \leq \Lambda \\ \#\Lambda = n-d}} \mu(\Pi(\lambda), \Lambda) = (-1)^{d-k} \sum_{\substack{\Lambda \in P[n-k] \\ \#\Lambda = n-d}} \prod_{A \in \Lambda} (\#A - 1)!$$

and the terms on the left all have the same sign $(-1)^{d-k}$. The sums are empty unless $n \geq d \geq k \geq 0$.

Estimating the correction factor

Lemma

Let $\lambda \vdash n$ with $\ell(\lambda) = n - k$ and $n \geq d \geq k \geq 0$. Then

$$\sum_{\substack{\Lambda : \Pi(\lambda) \leq \Lambda \\ \#\Lambda = n-d}} |\mu(\Pi(\lambda), \Lambda)| \leq (n-k)^{2(d-k)}.$$

Lemma

For $n \geq d \geq k \geq 0$, we have

$$\sum_{\substack{\lambda \vdash n \\ \ell(\lambda) = n-k}} \lambda! \sum_{\substack{\Lambda : \Pi(\lambda) \leq \Lambda \\ \#\Lambda = n-d}} |\mu(\Pi(\lambda), \Lambda)| \leq (n-k)^{2d-k}(k+1)!.$$

Estimating the correction factor

Lemma

For n sufficiently large, for all $0 \le d \le n$ we have

$$\sum_{\lambda \vdash n} \lambda! \sum_{\substack{\Lambda : \Pi(\lambda) \le \Lambda \\ \#\Lambda = n-d}} |\mu(\Pi(\lambda), \Lambda)| \le 3n^{2d}.$$

Putting it all together:

$$|F_n(e^{is/\sigma_n}, e^{it/\sigma_n}) - 1| \le 3 \sum_{d=1}^{n} \frac{(Mn)^{2d}}{\sigma_n^{2d}}$$

$$(Mn)^{2d}/\sigma_n^{2d} \sim (36^2 M^2/n)^d$$

$$\lim_{n\to\infty} \sum_{d=1}^{n} \frac{(Mn)^{2d}}{\sigma_n^{2d}} = 0.$$

Finishing up

Definition

The *characteristic function* of a real-valued random variable $\mathcal{X}$ is

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$$\phi_{\mathcal{X}}(t) := \mathbb{E}[e^{iXt}].$$

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If $\mathcal{X}$ has a density function, $\phi_{\mathcal{X}}$ is its Fourier transform.

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The *characteristic function* of a real-valued random variable $\mathcal{X}$ is

$$\phi_{\mathcal{X}} : \mathbb{R} \to \mathbb{C}$$
$$\phi_{\mathcal{X}}(t) := \mathbb{E}[e^{iXt}].$$

If $\mathcal{X}$ has a density function, $\phi_{\mathcal{X}}$ is its Fourier transform.

Theorem (*Lévy Continuity*)

$\mathcal{X}_1, \mathcal{X}_2, \ldots$ converges in distribution to $\mathcal{X}$ if and only if for all $t \in \mathbb{R}$,

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A similar result holds for $\mathbb{R}^k$-valued random variables.

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The equation

$$\frac{H_n(p,q)}{n!} = \frac{[n]_p![n]_q!}{n!^2} F_n(p,q)$$

can be reinterpreted as

$$\phi_{(\mathcal{X}^*_{\mathrm{inv}},\mathcal{X}^*_{\mathrm{maj}})}(s,t) = \phi_{\mathcal{X}^*_{\mathrm{inv}}}(s)\phi_{\mathcal{X}^*_{\mathrm{maj}}}(t) F_n(e^{is/\sigma_n}, e^{it/\sigma_n}).$$

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For fixed s, t, using the theorem above and Feller's result gives

$$\lim_{n\to\infty} \phi_{(\mathcal{X}^*_{\text{inv}},\mathcal{X}^*_{\text{maj}})}(s,t) = e^{-s^2/2} e^{-t^2/2} = \phi_{(\mathcal{N}(0,1),\mathcal{N}(0,1))}(s,t).$$

This completes the proof of the Baxter–Zeilberger theorem using Roselle's formula.

Done!

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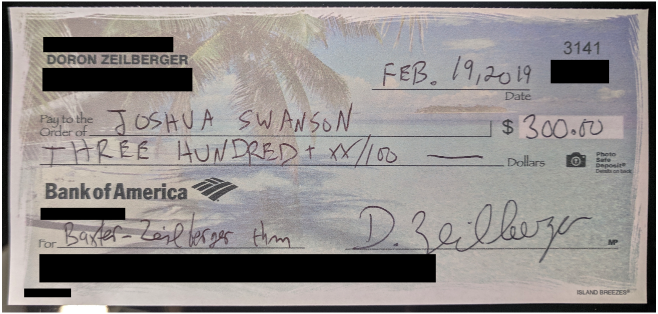

Local limit theorem?

Romik's question was largely motivated by a desire to find a *local limit theorem*. Here, this would be a statement of the form

$$\mathbb{P}[\mathrm{inv} = u, \mathrm{maj} = v] = \frac{1}{2\pi\sigma_n} e^{-(u-\mu_n)^2/\sigma_n-(v-\mu_n)^2/\sigma_n} + O(f(n))$$

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The method of moments has no hope of proving such a result. A standard approach to local limit theorems is to use the Cauchy integral formula on the generating function, though such arguments are typically lengthy and technical. A local limit theorem in this context will be the subject of a future article.

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There are many generalizations and variations of Mahonian statistics. **What can be said of their distributions**?

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What are the **possible normalized limit laws** for maj on **tableaux**?

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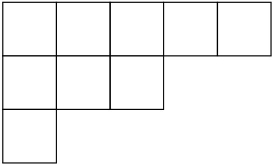

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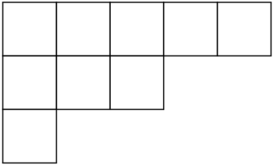

Theorem

(Young, early 1900's) *The complex inequivalent irreducible representations* S^λ *of* S_n *are* **canonically** *indexed by partitions of* n*.*

Standard tableaux

Definition

A *standard Young tableau* (*SYT*) of shape $\lambda \vdash n$ is a filling of the cells of the Ferrers diagram of λ with $1, 2, \ldots, n$ which **increases along rows** and **decreases down columns**.

$$T = \begin{array}{|c|c|c|c|c|} \hline 1 & 3 & 6 & 7 & 9 \\ \hline 2 & 5 & 8 \\ \cline{1-3} 4 \\ \cline{1-1} \end{array} \in \mathrm{SYT}(\lambda)$$

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The *descent set* of $T \in \mathrm{SYT}(\lambda)$ is the set

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Descent set: $\{1, 3, 7\}$. Major index: $1 + 3 + 7 = 11$.

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The *major index* of $T \in \mathrm{SYT}(\lambda)$ is $\mathrm{maj}(T) := \sum_{i \in \mathrm{Des}(T)} i$.

Standard tableaux

For $\lambda = (5, 3, 1)$,

$$\sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(T)} = q^5(q^{18} + 2q^{17} + 4q^{16} + 5q^{15} + 8q^{14} + 10q^{13} + 13q^{12} + 14q^{11} + 16q^{10} + 16q^9 + 16q^8 + 14q^7 + 13q^6 + 10q^5 + 8q^4 + 5q^3 + 4q^2 + 2q + 1).$$

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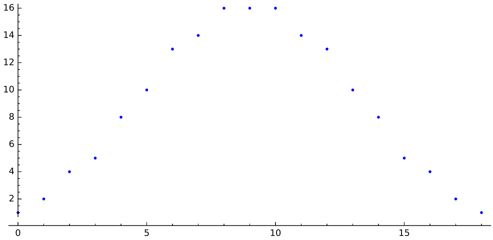
The coefficients of $q^{-5} f^{(5,3,1)}(q)$:

$$(1, 2, 4, 5, 8, 10, 13, 14, 16, 16, 16, 14, 13, 10, 8, 5, 4, 2, 1)$$



maj on SYT(λ) limit law classification

Definition

Let aft(λ) $:= |\lambda| - \max\{\lambda_1, \lambda'_1\}$ and let

$$\mathcal{IH}_M := \mathcal{U}[0,1] + \cdots + \mathcal{U}[0,1]$$

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(i) $\mathrm{aft}(\lambda^{(n)}) \to \infty$*; or*

(ii) $|\lambda^{(n)}| \to \infty$ *and* $\mathrm{aft}(\lambda^{(n)}) \to M < \infty$*; or*

(iii) *the distribution of* $\mathcal{X}_{\lambda^{(n)}}^*[\mathrm{maj}]$ *is eventually constant.*

The limit law is $\mathcal{N}(0,1)$ in case (i), $\mathcal{IH}_M^*$ in case (ii), and discrete in case (iii).

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- Consequently, $\kappa_d^{\mathcal{X}^*} = \Theta(\mathrm{aft}(\lambda)^{1-d/2}) \to 0$ if $\mathrm{aft}(\lambda) \to \infty$ and $d > 2$, which agrees with $\kappa_d^{\mathcal{N}(0,1)}$ in the limit. Now apply the *method of moments*.

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- Asymptotic normality is "obvious" when $\sum_{j=1}^{n} j^d$ dominates, though for small $\mathrm{aft}(\lambda)$, there is enormous cancellation resulting in degenerate cases with Irwin–Hall distributions.

Further limit laws

- For size on plane partitions in an a by b by c box, we get asymptotic normality if and only if $\mathrm{median}\{a, b, c\} \to \infty$.

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- For "rank" on $\mathrm{SSYT}_{\leq m}(\lambda)$, we get asymptotic normality in many cases, $\mathcal{IH}_M$ in others, and $\mathcal{D}^*$ where

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Thanks!

$$\mathcal{T}\,\mathcal{H}\,\mathcal{A}\,\mathcal{N}\,\mathcal{K}\,\mathcal{S}\,!$$