Thrall's problem: cyclic sieving, necklaces, and branching rules

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Slides: http://www.math.ucsd.edu/~jswanson/talks/2019_FPSAC.pdf

Outline

- We first apply the *cyclic sieving phenomenon* of Reiner–Stanton–White to prove Schur expansions due to Kraśkiewicz–Weyman related to *Thrall's problem*.
- The resulting argument is *remarkably simple and nearly bijective*. It is a rare example of the CSP being used *to prove other results*, rather than vice-versa.
- We then apply our approach to prove other results of Stembridge and Schocker.
- Guided by our experience, we *suggest a new approach* to Thrall's problem.

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Thrall's problem

What is Thrall's problem?

Definition

Let...

- V be a finite-dimensional vector space over $\mathbb{C}$;
- $T(V) := \oplus_{n \geq 0} V^{\otimes n}$ be the *tensor algebra of V*;
- $\mathcal{L}(V)$ be the *free Lie algebra on V*, namely the Lie subalgebra of $T(V)$ generated by V;
- $\mathcal{L}_n(V) := \mathcal{L}(V) \cap V^{\otimes n}$ be the *nth Lie module*;
- $\mathfrak{U}(\mathcal{L}(V))$ be the *universal enveloping algebra* of $\mathcal{L}(V)$; and
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By an appropriate version of the Poincaré–Birkhoff–Witt Theorem,

$$T(V) \cong \mathcal{U}(\mathcal{L}(V)) \cong \bigoplus_{\lambda=1^{m_1}2^{m_2}\cdots} \mathrm{Sym}^{m_1}(\mathcal{L}_1(V)) \otimes \mathrm{Sym}^{m_2}(\mathcal{L}_2(V)) \otimes \cdots$$

as graded $\mathrm{GL}(V)$-modules.

Definition (Thrall [Thr42])

The *higher Lie module* associated to $\lambda = 1^{m_1}2^{m_2}\cdots$ is

$$\mathcal{L}_\lambda(V) := \mathrm{Sym}^{m_1}(\mathcal{L}_1(V)) \otimes \mathrm{Sym}^{m_2}(\mathcal{L}_2(V)) \otimes \cdots .$$

Thus we have a *canonical* $\mathrm{GL}(V)$-module decomposition

$$T(V) \cong \oplus_{\lambda\in\mathrm{Par}}\mathcal{L}_\lambda(V).$$

Question (*Thrall's Problem*)

What are the irreducible decompositions of the $\mathcal{L}_\lambda(V)$?

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- The Littlewood–Richardson rule reduces Thrall's problem to the *rectangular case* $\lambda = (a^b)$ with b rows of length a.
- In the rectangular case,

$$\mathcal{L}_{(a^b)}(V) = \mathrm{Sym}^b \mathcal{L}_a(V)$$

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Kraśkiewicz–Weyman [KW01] solved Thrall's problem *in the one-row case*. We next describe their answer.

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A *partition* λ of n is a sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots$ such that $\sum_i \lambda_i = n$. Partitions can be visualized by their *Ferrers diagram*

$\lambda = (5, 3, 1) \leftrightarrow$

Theorem

(Young, early 1900's) *The complex inequivalent irreducible representations* S^λ *of* S_n *are* **canonically** *indexed by partitions of* n.

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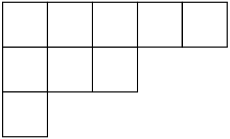
$\lambda = (5, 3, 1) \leftrightarrow$ 

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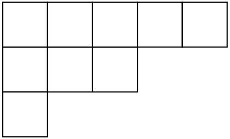
$\lambda = (5, 3, 1) \leftrightarrow$ 

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Standard tableaux

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A *standard Young tableau* (*SYT*) of shape $\lambda \vdash n$ is a filling of the cells of the Ferrers diagram of λ with $1, 2, \ldots, n$ which **increases along rows** and **decreases down columns**.

$T =$

1	3	6	7	9
2	5	8		
4				

$\in \mathrm{SYT}(\lambda)$

Descent set: $\{1, 3, 7\}$. Major index: $1 + 3 + 7 = 11$.

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The *descent set* of $T \in \mathrm{SYT}(\lambda)$ is the set

$$\mathrm{Des}(T) := \{1 \leq i < n : i + 1 \text{ is in a lower row of } T \text{ than } i\}.$$

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$$a_{\lambda,r} := \#\{T \in \mathrm{SYT}(\lambda) : \mathrm{maj}(T) \equiv_n r\}.$$

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Kraśkiewicz–Weyman's argument hinges on the following key formula:

$$\mathrm{SYT}(\lambda)^{\mathrm{maj}}(\omega_n^r) = \chi^\lambda(\sigma_n^r) \tag{1}$$

for all $r \in \mathbb{Z}$, where:

$$\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) := \sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(T)},$$

- ω_n is any primitive nth root of unity,
- $\chi^\lambda(\sigma)$ is the character of S^λ at σ, and
- $\sigma_n = (1\ 2\ \cdots\ n) \in S_n$.

Their approach involves results of Lusztig and Stanley on coinvariant algebras and an intricate though beautiful argument involving ℓ-decomposable partitions. The key formula bears a striking resemblance to the *cyclic sieving phenomenon* of Reiner–Stanton–White, which we describe next.

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Words

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- A *word* is a sequence

$$w = w_1 w_2 \cdots w_n \qquad \text{s.t.} \qquad w_i \in \mathbb{Z}_{\geq 1}.$$

- W_n is the set of words of length n.
- The *content* of w is the weak composition $\alpha = (\alpha_1, \alpha_2, \ldots)$ where $\alpha_j = \#\{i : w_i = j\}$.
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Definition (MacMahon, early 1900's)

The *descent set* of $w \in \mathrm{W}_n$ is

$$\mathrm{Des}(w) := \{1 \leq i \leq n-1 : w_i > w_{i+1}\}.$$

The *major index* is

$$\mathrm{maj}(w) := \sum_{i \in \mathrm{Des}(w)} i.$$

For example,

$$\mathrm{Des}(412144) = \mathrm{Des}(4.12.144) = \{1, 3\}$$
$$\mathrm{maj}(412144) = 1 + 3 = 4.$$

Major index on words

Theorem (MacMahon [Mac])

The major index generating function on $\mathsf{W}_\alpha \subset \mathsf{W}_n$ *is*

$$\mathsf{W}_\alpha^{\mathsf{maj}}(q) := \sum_{w \in \mathsf{W}_\alpha} q^{\mathsf{maj}(w)} = \frac{[n]_q!}{\prod_{i \geq 1} [\alpha_i]_q!} = \binom{n}{\alpha}_q$$

where $[n]_q := (1-q^n)/(1-q) = 1 + q + \cdots + q^{n-1}$ *and* $[n]_q! := [n]_q [n-1]_q \cdots [1]_q$.

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$$\mathrm{W}_{\alpha}^{\mathrm{maj}}(q) = \binom{n}{\alpha}_q$$

We have $\binom{n}{\alpha}_{q=1} = \binom{n}{\alpha} = \#\,\mathrm{W}_{\alpha}$.

Exercise

Let ω_d be any primitive dth root of unity. If $d \mid n$,

$$\binom{n}{\alpha}_{q=\omega_d} = \begin{cases} \binom{n/d}{\alpha_1/d, \alpha_2/d, \ldots} & \text{if } d \mid \alpha_1, \alpha_2, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

Question

What does $\binom{n}{\alpha}_{q=\omega_d}$ count?

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Let $\sigma_n := (1\,2\,\cdots\,n) \in S_n$ be the standard n-cycle. Let $C_n := \langle \sigma_n \rangle$, which acts on each $\mathrm{W}_\alpha \subset \mathrm{W}_n$ by rotation.

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If $\sigma \in C_n$ has order $d \mid n$, then

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The cyclic sieving phenomenon

Definition (Reiner–Stanton–White [RSW04])

Let X be a finite set on which a cyclic group C of order n acts and suppose $X(q) \in \mathbb{Z}[q]$. The triple $(X, C, X(q))$ *exhibits the cyclic sieving phenomenon* (*CSP*) if for all elements $\sigma_d \in C$ of order d,

$$X(\omega_d) = \#X^{\sigma_d}.$$

Remark

- $d = 1$ gives $X(1) = \#X$, so $X(q)$ *is a q-analogue* of $\#X$.
- $\#X^{\sigma_d} = \mathrm{Tr}_{\mathbb{C}\{X\}}(\sigma_d)$, so the CSP says that evaluations of $X(q)$ *encode the isomorphism type* of the C-action on X.
- $X(q)$ is uniquely determined *modulo* $q^n - 1$. If $\deg X(q) < n$, the kth coefficient of $X(q)$ is the number of elements of X whose *stabilizer has order dividing* k.

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Theorem ([RSW04, Prop. 4.4])

The triple $(\mathrm{W}_\alpha, C_n, \mathrm{W}_\alpha^{\mathrm{maj}}(q))$ *exhibits the CSP.*

That is, maj is a *"universal" cyclic sieving statistic* on words W_n for the S_n-action in the following sense:

Corollary ([BER11, Prop. 3.1])

Let W *be a finite set of length n words closed under the* S_n*-action. Then, the triple*

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Corollary

By "changing basis" from Schur functions and irreducible characters to homogeneous symmetric functions and induced trivial characters, Kraśkiewicz–Weyman's key formula (1) *holds:*

$$\mathrm{SYT}(\lambda)^{\mathrm{maj}}(\omega_n^r) = \chi^\lambda(\sigma_n^r).$$

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Schur–Weyl duality

To connect cyclic sieving to Thrall's problem, we require some standard $\mathrm{GL}(V)$-representation theory.

Definition

The *Schur character* of a $\mathrm{GL}(V)$-module E is

$$(\mathrm{ch}\, E)(x_1, \ldots, x_m) := \mathrm{Tr}_E(\mathrm{diag}(x_1, \ldots, x_m)),$$

where $m = \dim(V)$.

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Let M be an S_n-module. The *Schur–Weyl dual* of M is the $\mathrm{GL}(V)$-module

$$E(M) := V^{\otimes n} \otimes_{\mathbb{C}S_n} M.$$

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For any S_n-module M,

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Thrall's problem and necklaces

Definition

- A *necklace* is a C_n-orbit $[w]$ of a word $w \in \mathrm{W}_n$, e.g.

$$[221221] = \{221221, 122122, 212212\}.$$

- [221] has trivial stabilizer so is *primitive*.
- [221221] is not primitive and has *frequency* 2 since it's made of two copies of a primitive word.

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Proposition (Klyachko [Kly74])

There is a weight space basis for $E(\exp(2\pi i/n)\uparrow_{C_n}^{S_n})$ *indexed by primitive necklaces of length n words.*

Theorem (Marshall Hall [Hal59, Lem. 11.2.1])

$\mathcal{L}_n$ *also has a weight space basis indexed by primitive necklaces.*

Corollary (Klyachko [Kly74])

The Schur–Weyl dual of $\exp(2\pi i/n)\uparrow_{C_n}^{S_n}$ *is* $\mathcal{L}_n$.

To apply cyclic sieving, we need generating functions over words, not primitive necklaces.

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$\mathrm{NFD}_{n,r} := \{\underline{n}\text{ecklaces of length } n \text{ words with } \underline{f}\text{requency } \underline{d}\text{ividing } r\}.$

Hence $\mathrm{NFD}_{n,1}$ consists of primitive necklaces.

Proposition ([AS18a])

There is a weight space basis for $E(\exp(2\pi ir/n)\uparrow_{C_n}^{S_n})$ *indexed by* $\mathrm{NFD}_{n,r}$.

Corollary

We have

$$\sum_{r=1}^{n} q^r \operatorname{ch} \exp(2\pi ir/n)\uparrow_{C_n}^{S_n} = \sum_{r=1}^{n} q^r \,\mathrm{NFD}_{n,r}^{\mathrm{cont}}(\mathbf{x}).$$

However, as r varies, the $\mathrm{NFD}_{n,r}$ are *not disjoint*.

Flex

To fix this, we use the following.

Definition ([AS18b])

The statistic flex: $W_n \to \mathbb{Z}_{\geq 0}$ is $\text{flex}(w) := \text{freq}(w) \cdot \text{lex}(w)$ where $\text{lex}(w)$ is the position at which w appears in the lexicographic order of its rotations, starting at 1.

Example

$\text{flex}(221221) = 2 \cdot 3 = 6$ since 221221 is the concatenation of 2 copies of the primitive word 221 and 221221 is third in lexicographic order amongst its 3 cyclic rotations.

Lemma

We have

$$\sum_{r=1}^{n} q^r \, \text{NFD}_{n,r}^{\text{cont}}(\mathbf{x}) = W_n^{\text{cont;flex}}(\mathbf{x}; q).$$

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Flex is a *"universal" cyclic sieving statistic* on words W_n for C_n-actions in the following sense:

Lemma ([AS18b])

Let W be a finite set of length n words closed under the C_n-action, where C_n acts by cyclic rotations. Then, the triple $(W, C_n, W^{\mathrm{flex}}(q))$ exhibits the CSP.

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We have

$$\mathrm{W}_n^{\mathrm{cont};\mathrm{flex}}(q) = \mathrm{W}_n^{\mathrm{cont};\mathrm{maj}_n}(q)$$

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Proving Kraśkiewicz–Weyman's theorem

We finally have the following *remarkably direct, largely bijective proof* of Kraśkiewicz–Weyman's result using cyclic sieving.

1. Using *Schur–Weyl duality* and *Hall's basis*, $\operatorname{ch} \mathcal{L}_n$ can be replaced by $\operatorname{ch} \exp(2\pi i/n)\uparrow_{C_n}^{S_n}$.
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3. Using *universal cyclic sieving* on words for S_n- or C_n-actions,

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4. Using the *RSK algorithm* $w \mapsto (P, Q)$ where $\mathrm{Des}(w) = \mathrm{Des}(Q)$,

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There are multiple published proofs of Kraśkiewicz–Weyman's theorem. However, none of them give a bijective explanation for the following symmetry:

Corollary

Let $\lambda \vdash n$*. Then* $\#\{T \in \mathrm{SYT}(\lambda) : \mathrm{maj}(T) \equiv_n r\}$ *depends only on* λ *and* $\gcd(n, r)$*.*

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Such a bijection Φ would give a bijective proof of the identity

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Is there a refinement of Kraśkiewicz–Weyman's Schur expansion *involving cyclic descent types*?

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Stembridge generalized Kraśkiewicz–Weyman's result to describe all branching rules for any $\langle\sigma\rangle \hookrightarrow S_n$ where σ is of cycle type ν and order ℓ:

Theorem (Stembridge [Ste89])

$$\sum_{r=1}^{\ell} q^r \operatorname{ch}(\exp(2\pi i r/\ell)\uparrow_{\langle\sigma\rangle}^{S_n}) = \sum_{\substack{\lambda \vdash n \\ T \in \mathrm{SYT}(\lambda)}} q^{\mathrm{maj}_\nu(T)} s_\lambda(\mathbf{x})$$

where maj_ν *is a generalization of* maj_n.

We give a cyclic sieving-based proof of Stembridge's result. The first step is a natural generalization of Klyachko's basis:

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$$\operatorname{ch}\exp(2\pi i r/\ell)\uparrow_{\langle\sigma\rangle}^{S_n} = \mathrm{OFD}_{n,r}^{\mathrm{cont}}(\mathbf{x})$$

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Recall that $\mathcal{L}_{(a^b)} = \operatorname{Sym}^b \mathcal{L}_a$. Consequently,

$$\operatorname{ch} \mathcal{L}_{(a^b)} = h_b[\mathcal{L}_a].$$

Thus Thrall's problem is an instance of a *plethysm problem*. Such problems are notoriously difficult.

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One may show that $\mathcal{L}_{(a^b)}$ is the Schur–Weyl dual of a certain induced one-dimensional representation $\chi^{r,1}\uparrow_{C_a \wr S_b}^{S_{ab}}$ of the *wreath product* $C_a \wr S_b$. Here $C_a \wr S_b$ can be thought of as the subgroup of permutations on ab letters which permute the b size-a intervals in $[ab]$ amongst themselves and cyclically rotate each size-a interval independently.

Schocker [Sch03] gave a formula for the Schur expansion of $\operatorname{ch} \mathcal{L}_{(a^b)}$, though it involves many divisions and subtractions in general. We generalized Schocker's result to all induced *one-dimensional* representations of $C_a \wr S_b$ using cyclic sieving.

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For all $a, b \geq 1$ *and* $r = 1, \ldots, a$, *we have*

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and $\mu_f(d, e)$ *is a generalization of the classical Möbius function.*

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A new approach

Our generalization of Schocker's formula involves considering only the one-dimensional representations of $C_a \wr S_b$, which may explain its failure to be cancellation-free.

The earlier statistics flex, maj_n, and maj_ν gave *monomial expansions* of the branching rules in question as generating functions on words. We have identified the monomial expansion for $C_a \wr S_b \hookrightarrow S_{ab}$ as a statistic generating function as follows.

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Fix integers $a, b \geq 1$. We have

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The statistics flex_a^b and maj_a^b are somewhat involved. For flex_a^b:

1. Write $w \in \mathrm{W}_{ab}$ in the form $w = w^1 \cdots w^b$ where $w_j \in \mathrm{W}_a$.
2. Let $w^{(r)}$ denote the *subword* of w whose letters are those $w^j \in \mathrm{W}_a$ such that $\mathrm{flex}(w^j) = r$.
3. *Totally order* W_a lexicographically, so that RSK is well-defined for words with letters from W_a.
4. Set

$$\mathrm{flex}_a^b(w) := (\mathrm{sh}(w^{(1)}), \ldots, \mathrm{sh}(w^{(a)}))$$

where sh denotes the shape under RSK.

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A new approach

Previously, we were able to simply use RSK to go from the *monomial to the Schur basis*, since maj_ν depends only on $Q(w)$. However, flex_a^b and maj_a^b do not have the corresponding property.

Open Problem

Fix $a, b \geq 1$. Find a statistic

$$\mathrm{mash}_a^b \colon \mathrm{W}_{ab} \to \{a\text{-tuples of partitions with total size } b\}$$

with the following properties.

(i) For all $\alpha \vDash ab$, maj_a^b (or equivalently flex_a^b) and mash_a^b are *equidistributed* on W_α.

(ii) If $v, w \in \mathrm{W}_{ab}$ satisfy $Q(v) = Q(w)$, then $\mathrm{mash}_a^b(v) = \mathrm{mash}_a^b(w)$.

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A new approach

Finding such a statistic mash_a^b would *determine all branching rules* for $C_a \wr S_b \hookrightarrow S_{ab}$, in particularly *solving Thrall's problem*, as follows.

Corollary

Suppose mash_a^b *satisfies Properties (i) and (ii). Then*

$$\mathrm{ch}(S^{\underline{\lambda}}\uparrow_{C_a \wr S_b}^{S_{ab}}) = \sum_{\nu \vdash ab} \frac{\#\{Q \in \mathrm{SYT}(\nu) : \mathrm{mash}_a^b(Q) = \underline{\lambda}\}}{\dim(S^{\underline{\lambda}})} s_\nu(\mathbf{x}),$$

where $\mathrm{mash}_a^b(Q) := \mathrm{mash}_a^b(w)$ *for any* $w \in \mathrm{W}_{ab}$ *with* $Q(w) = Q$*.*

A new approach

When $a = 1$ and $b = n$, $\mathrm{maj}_1^n(w)$ essentially reduces to $\mathrm{sh}(w)$, the *shape of w under RSK*. When $a = n$ and $b = 1$, $\mathrm{maj}_n^1(w)$ essentially reduces to $\mathrm{maj}_n(w)$. Both of these satisfy (i) and (ii). In this sense mash_a^b, *interpolates between the major index* maj_n *and the shape under RSK*, hence the name.

Question

Could a useful notion of *"group sieving"* for the wreath products $C_a \wr S_b$ be missing?

THANKS!

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