Cyclotomic generating function asymptotics

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Based on joint work with
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Slides: `http://www.math.ucsd.edu/~jswanson/talks/2019_CombinaTexas.pdf`

Outline

- Classical asymptotics
- Roots of unity and unit roots
- Cyclotomic generating functions
- maj on $\mathrm{SYT}(\lambda)$ limit law classification
- Cumulants
- Further limit laws

Classical asymptotics

Theorem (de Moivre–Laplace: the O.G. C.L.T.)

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i.e. $\mathcal{X}_n$ is <u>asymptotically normal</u>.

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i.e. $\mathcal{X}_n$ is asymptotically normal.

Note that size: $2^{[n]} \to \mathbb{Z}_{\geq 0}$ has the same distribution as $\mathcal{X}_n$ and generating function

$$\sum_{S\subset[n]} q^{\text{size}(S)} = (1+q)^n$$

with *all real roots.*

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Given a statistic $\mathrm{stat}\colon W \to \mathbb{Z}_{\geq 0}$, form the generating function

$$W^{\mathrm{stat}}(q) := \sum_{w \in W} q^{\mathrm{stat}(w)} = \sum_{k \geq 0} c_k q^k.$$

Question

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- What's the distribution of stat where W is sampled uniformly?

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Suppose $\mathcal{X}_1, \mathcal{X}_2, \ldots$ *are random variables where* $\mathbb{E}[q^{\mathcal{X}_n}] \in \mathbb{R}_{\geq 0}[q]$ *has all real roots.*

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Example

If $\mathcal{X}_n$ is size on $2^{[n]}$, then $\mathbb{E}[q^{\mathcal{X}_n}] = \frac{1}{2^n}(1+q)^n$ and $\sigma_n^2 = n/4 \to \infty$.

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- Number of blocks on set partitions of $[n]$ (i.e. Stirling numbers of the second kind).
- Number of descents in permutations of S_n (Eulerian numbers, Eulerian statistics).

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$$S_n^{\mathrm{inv}}(q) = \sum_{w \in S_n} q^{\mathrm{inv}(w)} = [n]_q! = [n]_q[n-1]_q \cdots [1]_q$$

where $[c]_q := 1 + q + \cdots + q^{c-1}$.

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Since $[c]_q = (1 - q^c)/(1 - q)$, the roots are all *roots of unity* and are almost never real.

Unit roots

Theorem (Hwang–Zacharovas '15)

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For inv on S_n, it turns out that

$$\kappa_4/\sigma^4 = -\frac{36}{25}\frac{31 + 31n + 21n^2 + 6n^3}{n(n-1)(2n+5)^2} \approx -1/n \to 0,$$

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- Can have internal zeros, though they usually don't (see arXiv:1809.07386)
- Need not be unimodal, though they often are (e.g. q-binomials)
- Even less likely to be log-concave or ultra log-concave (though related γ-expansions are both!)

Cyclotomic generating functions

Definition (Billey–Konvalinka–S.)

A cyclotomic generating function is a polynomial of the form

$$f(q) = \alpha q^{\beta} \prod_{k=1}^{m} \frac{[a_k]_q}{[b_k]_q} \in \mathbb{Z}_{\geq 0}[q]$$

for multisets $\{a_1, \ldots, a_m\}$ and $\{b_1, \ldots, b_m\}$ of positive integers and $\alpha, \beta \in \mathbb{Z}_{\geq 0}$.

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They are also a particularly nice family of random variables. For instance, the characteristic functions are piecewise log-concave, and $\log \mathbb{E}[e^{it\mathcal{X}^*}]$ always converges in a complex neighborhood of 0 of radius at least $2\pi\sqrt{1/\zeta(2) - 1/4} \approx 3.76\ldots$.

Cyclotomic generating functions

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For $f(q) \in \mathbb{Z}_{\geq 0}[q]$, TFAE:

(i) *(Rational form.) f is a cyclotomic generating function, i.e.*

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Moreover, the factorization in (i) is unique if the multisets are disjoint and $f \neq 0$.

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Yes!

maj on SYT(λ) limit law classification

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Let $\mathrm{aft}(\lambda) := |\lambda| - \max\{\lambda_1, \lambda_1'\}$ and let

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Let $\lambda^{(1)}, \lambda^{(2)}, \ldots$ be a sequence of partitions. Let $\mathcal{X}_n$ denote the major index statistic on standard tableaux of shape $\lambda^{(n)}$ sampled uniformly, and let $\mathcal{X}_n^* := (\mathcal{X}_n - \mu_n)/\sigma_n$.

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(i) $\mathrm{aft}(\lambda^{(n)}) \to \infty$; *or*

(ii) $|\lambda^{(n)}| \to \infty$ *and* $\mathrm{aft}(\lambda^{(n)}) \to M < \infty$; *or*

(iii) *the distribution of* $\mathcal{X}_{\lambda^{(n)}}^*[\mathrm{maj}]$ *is eventually constant.*

The limit law is $\mathcal{N}(0,1)$ in case (i), $\mathcal{IH}_M^*$ in case (ii), and discrete in case (iii).

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(3) (Additivity) $\kappa_d^{\mathcal{X}+\mathcal{Y}} = \kappa_d^{\mathcal{X}} + \kappa_d^{\mathcal{Y}}$ if $\mathcal{X}$ and $\mathcal{Y}$ are independent.

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(5) (Polynomiality) μ's, α's, κ's are all polynomials in each other.

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Let $\mathcal{X} = \mathcal{N}(\mu, \sigma)$. Then

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μ_d is even messier!

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Theorem (Billey–Konvalinka–S. '19, Hwang–Zacharovas '15, Chen–Wang–Wang '08, ...)

Let $B_0 = 1, B_1 = 1/2, B_2 = 1/6, B_3 = 0, B_4 = -1/30, \ldots$ *be the Bernoulli numbers. Given a cyclotomic generating function with* $\beta = 0$,

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- Asymptotic normality is "obvious" when $\sum_{j=1}^{n} j^d$ dominates, though for small aft(λ), there is enormous cancellation resulting in degenerate cases with Irwin–Hall distributions.

Further limit laws

- For size on $\mathrm{PP}(a \times b \times c)$, we get asymptotic normality if and only if $\mathrm{median}\{a, b, c\} \to \infty$.

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- For rank on $\mathrm{SSYT}_{\leq m}(\lambda)$, we get asymptotic normality in many cases, $\mathcal{IH}_M$ in others, and $\mathcal{D}^*$ where

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Further questions

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Example

For maj on $\mathrm{SYT}(\lambda)$, the fact that the corresponding rational function is in $\mathbb{Z}[q]$ can be interpreted in terms of ℓ-quotients of λ. Why are the coefficients positive?

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- Enumerative properties of the set of cyclotomic generating functions (e.g. limiting unimodal fraction?).
- Connections to regular sequences and Hilbert series of complete intersections in weighted projective space (e.g. which such Hilbert series are possible?).

Thanks!

$\mathcal{THANKS}$!