CYCLOTOMIC GENERATING FUNCTION ASYMPTOTICS

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Contents

1. Motivation

Theorem 1.1 (de Moivre–Laplace; the O.G. Central Limit Theorem)**.** *Let $\mathcal{X}_n$ be the number of heads after n fair coin tosses. Then for $t \in \mathbb{R}$,*

$$\lim_{n \to \infty} \mathbb{P}\left[\frac{X_n - n/2}{\sqrt{n/4}} \leq t\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-x^2}\, dx = \mathbb{P}[\mathcal{N}(0,1) \leq t],$$

i.e. $\mathcal{X}_n$ is asymptotically normal.

Note that size: $2^{[n]} \to \mathbb{Z}_{\geq 0}$ has the same distribution as $\mathcal{X}_n$. Given stat: $W \to \mathbb{Z}_{\geq 0}$, let

$$W^{\mathrm{stat}}(q) \coloneqq \sum_{w \in W} q^{\mathrm{stat}(w)} = \sum_{k \geq 0} c_k q^k = |W| \cdot \mathbb{E}[q^{\mathcal{X}}] \in \mathbb{Z}_{\geq 0}[q],$$

where $\mathbb{P}[\mathcal{X} = k] = c_k / |W|$.

Question 1.2. *What do the coefficients c_k look like for your favorite statistic?*

Theorem 1.3 (Bender '73, Harper '67)**.** *Suppose $\mathcal{X}_1, \mathcal{X}_2, \ldots$ has $\mathbb{E}[q^{\mathcal{X}_n}] \in \mathbb{R}_{\geq 0}[q]$ with all real roots. Then $\mathcal{X}_n$ is asymptotically normal if and only if $\sigma_n \to \infty$.*

Example 1.4.

- For size on $2^{[n]}$, $(1+q)^n$ has all real roots and $\sigma_n^2 = n/4 \to \infty$.
- Number of blocks on set partitions of $[n]$ (i.e. Stirling numbers of the second kind).
- Number of descents in permutations of S_n (Eulerian numbers, Eulerian statistics).
- Non-example: inversions in S_n (Mahonian statistics).

Theorem 1.5 (Hwang–Zacharovas '15)**.** *Suppose $\mathcal{X}_1, \mathcal{X}_2, \ldots$ have $\mathbb{E}[q^{\mathcal{X}_n}] \in \mathbb{R}_{\geq 0}[q]$ with all roots on the unit circle $\{z \in \mathbb{C} : |z| = 1\}$. Then $\mathcal{X}_n$ is asymptotically normal if and only if $\kappa_4/\sigma^4 \to 0$.*

Example 1.6. Inversions in S_n have

$$S_n^{\mathrm{inv}}(q) = [n]_q! \coloneqq [n]_q [n-1]_q \cdots [1]_q$$

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where $[k]_q := 1 + q + \cdots + q^{k-1} = (1-q^k)/(1-q)$. It turns out that

$$\kappa_4/\sigma^4 = -\frac{36}{25}\frac{31 + 31n + 21n^2 + 6n^3}{n(n-1)(2n+5)^2} \approx 1/n \to 0,$$

so Mahonian statistics are asymptotically normal.

Remark 1.7. In the real-rooted case, *Newton's inequalities* say

$$\left(\frac{c_k}{\binom{n}{k}}\right)^2 \geq \frac{c_{k-1}}{\binom{n}{k-1}}\frac{c_{k+1}}{\binom{n}{k+1}},$$

i.e. the coefficients are *ultra log-concave*. They are in particular log-concave, unimodal, and have no internal zeroes. The unit root case by contrast is more complex. Such polynomials need not have any of the above properties. Nonetheless, they do enjoy other interesting properties!

2. Cyclotomic generating functions and cumulants

Definition 2.1 (Billey–Konvalinka–S.)**.** A cyclotomic generating function is a polynomial of the form

$$f(q) = q^\beta \prod_{k=1}^m \frac{[a_k]_q}{[b_k]_q} \in \mathbb{Z}_{\geq 0}[q]$$

for multisets $\{a_1, \ldots, a_m\}$ and $\{b_1, \ldots, b_m\}$ of positive integers and some offset $\beta \in \mathbb{Z}_{\geq 0}$.

Lemma 2.2 (Kronecker 1857)**.** *If $f \in \mathbb{Z}[q]$ is monic and all complex roots z have $|z| \leq 1$, then all roots are roots of unity or 0.*

Definition 2.3. Let

$$\mu_d := \mathbb{E}[\mathcal{X}^d] \qquad \text{and} \qquad \alpha_d := \mathbb{E}[(\mathcal{X}-\mu)^d].$$

Note that

$$\sum_{d=0}^\infty \mu_d \frac{t^d}{d!} = \mathbb{E}[e^{t\mathcal{X}}] = \frac{1}{|W|} W^{\text{stat}}(e^t).$$

The *cumulants* of $\mathcal{X}$ are defined by

$$\sum_{d=1}^\infty \kappa_d \frac{t^d}{d!} := \log \mathbb{E}[e^{t\mathcal{X}}] = \log \sum_{d=0}^\infty \mu_d \frac{t^d}{d!}.$$

Observation 2.4. *The characteristic function of $\mathcal{X}$ is*

$$\mathbb{E}[e^{it\mathcal{X}}] = \frac{1}{|W|} W^{\text{stat}}(e^{it}),$$

which is essentially the Fourier transform of the density/mass of $\mathcal{X}$. Hence identities for generating functions yield relations between characteristic functions!

Remark 2.5. Cumulants are like moments, but better in every way:

(1) $\kappa_1 = \mu$, $\kappa_2 = \sigma^2$, $\kappa_3 = \alpha_3$.
(2) $\kappa_d^{\mathcal{X}} = \kappa_d^{\mathcal{X}-c}$ for all $d \geq 2$ and $c \in \mathbb{R}$.
(3) $\kappa_d^{c\mathcal{X}} = c^d \kappa_d^{\mathcal{X}}$.
(4) $\kappa_d^{\mathcal{X}+\mathcal{Y}} = \kappa_d^{\mathcal{X}} + \kappa_d^{\mathcal{Y}}$ if $\mathcal{X}$ and $\mathcal{Y}$ are independent.
(5) μ's, α's, κ's are all polynomials in each other.

Example 2.6. Let $\mathcal{X} = \mathcal{N}(\mu, \sigma)$. Then $\kappa_1 = \mu, \kappa_2 = \sigma^2, \kappa_d = 0$ for $d \geq 3$. Contrast with

$$\alpha_d = \begin{cases} 0 & d \text{ odd} \\ \sigma^d (d-1)!! & d \text{ even.} \end{cases}$$

Example 2.7. Let $\mathcal{X} = \mathcal{U}_n$ be the uniform discrete distribution supported on $\{0, 1, \ldots, n-1\}$. Note

$$\mathbb{E}[q^{\mathcal{U}_n}] = \frac{1}{n}[n]_q.$$

In fact,

$$\kappa_d = \frac{B_d}{d}(n^d - 1)$$

where

$$B_0 = 1, B_1 = 1/2, B_2 = 1/6, B_3 = 0, B_4 = -1/30, \ldots$$

are the *Bernoulli numbers* defined by $\sum_{d=0}^{\infty} B_d/d!t^d = t/(1-e^{-t})$.

Theorem 2.8 (Billey–Konvalinka–S. '19, Hwang–Zacharovas '15, Chen–Wang–Wang '08)**.** *Given a cyclotomic generating function with $\beta = 0$, then*

$$\kappa_d = \frac{B_d}{d}\sum_{k=1}^{m}(a_k^d - b_k^d)$$

$$\alpha_d = \sum_{\substack{\lambda \vdash n \\ \textit{even parts}}} \frac{d!}{z_\lambda}\prod_{i=1}^{\ell(\lambda)}\frac{B_{\lambda_i}}{\lambda_i!}\left(\sum_{k=1}^{m} a_k^{\lambda_i} - b_k^{\lambda_i}\right)$$

$$\mu_d = \sum_{\substack{\lambda \vdash n \\ \textit{even parts} \\ \textit{or singletons}}} \frac{d!}{z_\lambda}\prod_{i=1}^{\ell(\lambda)}\frac{B_{\lambda_i}}{\lambda_i!}\left(\sum_{k=1}^{m} a_k^{\lambda_i} - b_k^{\lambda_i}\right).$$

Example 2.9. For $(1+q)^n$, have $\sigma_n^2 = \kappa_2 = \frac{1}{12}n(2^2-1) = n/4$. These are the unique cyclotomic generating functions with all real roots, up to shifting.

Theorem 2.10 (Method of moments)**.** *If for all $d \geq 1$,*

$$\lim_{n\to\infty} \mu_d^{\mathcal{X}_n} = \mu_d^{\mathcal{X}},$$

then

$$\lim_{n\to\infty} \mathbb{P}[\mathcal{X}_n \leq t] = \mathbb{P}[\mathcal{X} \leq t]$$

for all (continuity points) $t \in \mathbb{R}$, i.e. $\mathcal{X}_n \Rightarrow \mathcal{X}$.

Corollary 2.11. *$\mathcal{X}_1, \mathcal{X}_2, \ldots$ is asymptotically normal if for all $d \geq 3$,*

$$\lim_{n\to\infty} \kappa_d^{\mathcal{X}_n}/\sigma_n^d = 0.$$

3. An easy example: $\mathrm{baj} - \mathrm{inv}$ on S_n

Definition 3.1. Set $\mathrm{baj}\colon S_n \to \mathbb{Z}_{\geq 0}$ by

$$\mathrm{baj}(\sigma) := \sum_{i \in \mathrm{Des}(\sigma)} i(n-i).$$

Theorem 3.2 (Stembridge–Waugh '98)**.**

$$S_n^{\mathrm{baj}\,-\,\mathrm{inv}}(q) = n\prod_{i=1}^{n-1}\frac{[i(n-i)]_q}{[i]_q}.$$

Corollary 3.3.

$$\kappa_d^n = \frac{B_d}{d}\left(\sum_{i=1}^{n-1}[i(n-i)]^d - i^d\right)$$

Lemma 3.4.
$$\kappa_d^n \sim \frac{B_d}{d} \int_0^1 x^d (1-x)^d \, dx \cdot n^{2d+1}.$$
Hence
$$\kappa_d^n / \sigma_n^d = \Theta(n^{1-d/2}) \to 0 \qquad \text{for } d \geq 3,$$
so $\mathrm{baj} - \mathrm{inv}$ *on* S_n *is asymptotically normal.*

4. A medium example: maj on $\mathrm{SYT}(\lambda)$

Definition 4.1. Let λ be a *partition* of n, $\mathrm{SYT}(\lambda)$ the set of *standard Young tableaux* of shape λ, $\mathrm{Des}\colon \mathrm{SYT}(\lambda) \to 2^{[n-1]}$ the *descent set* (e.g. $\mathrm{Des}(125/368/47) = \{2, 3, 5, 6\}$), the *major index* $\mathrm{maj}\colon \mathrm{SYT} \to \mathbb{Z}_{\geq 0}$ by $\mathrm{maj}(T) \coloneqq \sum_{i \in \mathrm{Des}(T)} i$, $c \in \lambda$ for the *cells* c of λ, and h_c for the *hook length* of c.

Theorem 4.2 (Billey–Konvalinka–S., arXiv:1809.07386)**.** $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q)$ *has internal zeroes if and only if* λ *is a rectangle (not a single row or column).*

Theorem 4.3 (Stanley/Lusztig '70's)**.**
$$\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) = q^{r(\lambda)} \frac{[n]_q!}{\prod_{c \in \lambda} [h_c]_q}$$
where $r(\lambda) \coloneqq \sum_{i \geq 1} (i-1)\lambda_i$.

Corollary 4.4.
$$\kappa_d^\lambda = \frac{B_d}{d} \left(\sum_{k=1}^m k^d - \sum_{c \in \lambda} h_c^d \right).$$

Theorem 4.5 (Billey–Konvalinka–S.)**.** *Let* $\lambda^{(1)}, \lambda^{(2)}, \ldots$ *be a sequence of partitions. Then* $(\mathcal{X}_{\lambda^{(N)}}[\mathrm{maj}]^*)$ *converges in distribution if and only if*

(i) $\mathrm{aft}(\lambda^{(N)}) \to \infty$*; or*
(ii) $|\lambda^{(N)}| \to \infty$ *and* $\mathrm{aft}(\lambda^{(N)}) \to M < \infty$*; or*
(iii) the distribution of $\mathcal{X}^*_{\lambda^{(N)}}[\mathrm{maj}]$ *is eventually constant.*

The limit law is $\mathcal{N}(0,1)$ *in case (i),* $\mathcal{IH}^*_M$ *in case (ii), and discrete in case (iii).*

Here
$$\mathcal{IH}_M \coloneqq \mathcal{U}[0,1] + \cdots + \mathcal{U}[0,1]$$
is the *Irwin–Hall distribution*, namely the sum of M independent continuous uniform random variables.

Question 4.6. *What are the possible limiting distributions of* maj *on* $\mathrm{SYT}(\lambda/\mu)$*?*

5. A hard example: rank on $\mathrm{SSYT}_{\leq m}(\lambda)$

Theorem 5.1. *The size statistic on* $\mathrm{PP}(a \times b \times c)$*, i.e. plane partitions in a box, is asymptotically normal if and only if*
$$\mathrm{median}\{a, b, c\} \to \infty.$$
If ab *converges and* $c \to \infty$*, the limit law is* $\mathcal{IH}^*_{ab}$.

Theorem 5.2 (classical; Littlewood?)**.** *We have*
$$s_\lambda(1, q, \ldots, q^{m-1}) = q^{r(\lambda)} \prod_{1 \leq i < j \leq m} \frac{[\lambda_i - \lambda_j + j - i]_q}{[j-i]_q}.$$

Call the corresponding statistic on $\mathrm{SSYT}_{\leq m}(\lambda)$ "rank". We have many particular limits, though no complete classification. Here $\lambda = \ell_1^{e_1} \cdots \ell_k^{e_k}$ with $\ell_1 > \cdots > \ell_k \geq 0$ and $e_i \geq 1$. Also, $e^{[i]}$ is the ith largest element amongst $e_1, e_2, \ldots, e_k$.

Summary 5.3.

(i) If $|\lambda| \to n$ *and* $m \to \infty$*, then* $\mathcal{X}_{\lambda;m}[\mathrm{rank}]^* \Rightarrow \mathcal{IH}^*_n$.
(ii) If $|\lambda| \to \infty$ *and* $|\lambda| = o(m)$*, then* $\mathcal{X}_{\lambda;m}[\mathrm{rank}]^* \Rightarrow \mathcal{N}$.

(iii) *If $|\lambda| \to \infty$ and m is fixed, then $\mathcal{X}_{\lambda;m}[\mathrm{rank}]^* \Rightarrow \mathcal{D}^*$ where*

$$\mathcal{D} = \sum_{1 \le i < j \le m} \mathcal{U}[x_j, x_i]$$

and $x_k := \lambda_k/\lambda_1$, provided the x_k converge. Moreover, a converse holds.

(iv) *If $(\lambda_1 - \lambda_m)/m^3 \to \infty$, then $\mathcal{X}_{\lambda;m}[\mathrm{rank}]^* \Rightarrow \mathcal{N}$.*

(v) *If the sequence λ is obtained by successively replacing the cells of a fixed partition with $c \times c$ grids of cells and $c \to \infty$, then $\mathcal{X}_{\lambda;m}[\mathrm{rank}]^* \Rightarrow \mathcal{N}$.*

(vi) *If $\delta_N := (N, N-1, \ldots, 2, 1)$, then $\mathcal{X}_{\delta_N;N}[\mathrm{rank}]^* \Rightarrow \mathcal{N}$.*

(vii) *In the following situations, $\mathcal{X}_{\lambda;m}[\mathrm{rank}]^* \Rightarrow \mathcal{N}$:*

(a) $\liminf(\ell_1 - \ell_k)/m > 0$ *and* $k \to \infty$

(b) $\liminf k/m > 0$ *and* $m \to \infty$

(c) $\limsup(\ell_1 - \ell_k)/m < \infty$ *and* $e^{[2]}/k \to \infty$

(d) $\limsup \ell_1 - \ell_k < \infty$ *and* $e^{[2]} \to \infty$.

Question 5.4. *Are there any other continuous limiting distributions aside from $\mathcal{N}, \mathcal{IH}_M^*, \mathcal{D}^*$?*

These cyclotomic generating functions are effectively rank generating functions for type A crystals. More generally, we have a large number of cyclotomic generating functions as follows.

Theorem 5.5 (Stembridge '94)**.** *Let V_λ be the irreducible $\mathfrak{g}$-module of highest weight λ. Let r be the rank inherited from the weight poset on the weights Λ_λ of V_λ, namely $r(\mu) = \langle \mu, \rho^\vee \rangle$. Then*

$$\sum_{\mu \in \Lambda_\lambda} \dim(V_\lambda)_\mu \cdot q^{r(\mu)} = q^{-\langle \lambda, \rho^\vee \rangle} \prod_{\alpha \in \Phi^+} \frac{1 - q^{\langle \lambda + \rho, \alpha^\vee \rangle}}{1 - q^{\langle \rho, \alpha^\vee \rangle}}.$$

6. Final remarks

Using maj or inv on linear extensions of forests, we've found families of cyclotomic generating functions with limiting distributions $\mathcal{S}^*$ where

$$\mathcal{S} = \sum_{k=1}^{\infty} \mathcal{U}[0, t_k] \qquad \text{where} \qquad \sum_{k=1}^{\infty} t_k^2 < \infty.$$

These include $\mathcal{IH}_n$, $\mathcal{D}$, and in a "limiting" sense $\mathcal{N}$.

Question 6.1. *Are there any other continuous limiting distributions of cyclotomic generating functions?*

Question 6.2. *Which $\{a_1, \ldots, a_m\}$, $\{b_1, \ldots, b_m\}$ have $\prod_{k=1}^m [a_k]_q/[b_k]_q \in \mathbb{Z}_{\ge 0}[q]$? What is special about $\{h_c : c \in \lambda\}$? Can we efficiently compute a set of monoid generators? What is the growth rate as a function of degree? What about analogous questions for the "unimodal" and "log-concave" submonoids?*

We have a number of necessary conditions and a variety of sufficient conditions. Necessary conditions:

- $\sum_{k=1}^m a_k^d \ge \sum_{k=1}^m b_k^d$ for $d \in \{1, 2, 4, 6, 8, \ldots\}$.
- $\max\{a_k\} \ge \max\{b_k\}$ and $\min\{a_k\} \ge \min\{b_k\}$.
- $\#\{k : \ell \mid a_k\} \ge \#\{k : \ell \mid b_k\}$ for all $\ell \ge 2$.
- $\frac{\sum_{k=1}^m (a_k^4 - b_k^4)}{\left(\sum_{k=1}^m a_k^2 - b_k^2\right)^2} \le \frac{5}{3}$.

Sufficient conditions:

- q-Weyl dimension formula
- $\mathcal{X} + \mathcal{U}_{b_1} + \cdots + \mathcal{U}_{b_m} = \beta + \mathcal{U}_{a_1} + \cdots + \mathcal{U}_{a_m}$.
- Hilbert series of homogeneous coordinate rings of complete intersections in weighted projective space.