

# MAJOR INDEX STATISTICS: CYCLIC SIEVING, BRANCHING RULES, AND ASYMPTOTICS

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ABSTRACT. These are lecture notes for the author's thesis defense talk given in the University of Washington combinatorics seminar on May 23rd, 2018.

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## 1. INTRODUCTION

My thesis has five content chapters:

- (1) Refined Cyclic Sieving on Words for the Major Index Statistic
- (2) Cyclic Sieving, Branching Rules, and Higher Lie Modules
- (3) On the Existence of Tableaux with Given Modular Major Index
- (4) Distribution of major index for standard tableaux and asymptotic normality
- (5) On a theorem of Baxter and Zeilberger via a result of Roselle

It's also 210 pages long and covers a range of topics including word, necklace, and tableaux combinatorics; cyclic sieving; symmetric group and general linear group representation theory; higher Lie modules; weak convergence of random variables; cumulants; and estimates of characteristic functions. Bowing to the pressures of time, today I'll summarize pieces of two of the chapters.

## 2. BACKGROUND

**Definition 2.1.** For a permutation  $w = w_1 w_2 \cdots w_n$  in the symmetric group  $S_n$ , the *major index* of  $w$  is the sum of all  $i$  such that  $w_i > w_{i+1}$ .

**Example 2.2.**

- $\text{maj}(\text{id}) = \text{maj}(1\ 2\ \cdots\ n) = 0$ .
- $\text{maj}(w_0) = \text{maj}(n\ \cdots\ 2\ 1) = 1 + 2 + \cdots + (n - 1) = \binom{n}{2}$ .

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**Remark 2.3.** Baxter and Zeilberger call this the “second most important permutation statistic” (after inversion number). In my opinion, the major index has richer connections to representation theory and asymptotic combinatorics than the inversion number.

**Definition 2.4.** Recall that the conjugacy classes of  $S_n$  are determined by cycle types. Cycle types correspond to (integer) *partitions* of  $n$ , which are weakly decreasing sequences  $\lambda \vdash n$  of non-negative integers summing to  $n$ .

By elementary character theory, complex irreducible representations of a finite group  $G$  are in general equinumerous with the conjugacy classes of  $G$ .

**Definition 2.5.** The cyclic group  $C_n = \langle \sigma_n \rangle$  of order  $n$  has irreducibles

$$\begin{aligned}\chi^r : C_n &\rightarrow \mathbb{C}^\times \\ \chi^r(\sigma_n) &:= \omega_n^r\end{aligned}$$

where  $\omega_n$  is a primitive  $n$ th root of unity.

**Remark 2.6.** When  $G = S_n$ , something amazing happens: the irreducibles  $S^\lambda$  are called *Specht modules* and are *canonically* indexed by partitions  $\lambda \vdash n$ . (Indeed, this construction can be categorified using the notion of *Schur functors*, which are endofunctors of  $\text{Vec}$  interpolating between the exterior and symmetric power functors.)

**Definition 2.7.** The *Young diagram* of  $\lambda$  is the northwest justified grouping of square cells where the  $i$ th row from the top has length  $\lambda_i$ .

**Definition 2.8.** A *standard tableau* of shape  $\lambda \vdash n$  is a filling of the Young diagram of  $\lambda$  with  $1, 2, \dots, n$  (each used once) which strictly increases along rows and columns. Write  $\text{SYT}(\lambda)$  for the set of standard Young tableaux of shape  $\lambda$ .

**Definition 2.9.** The *descent set* of  $T \in \text{SYT}(\lambda)$  is the set of all  $i$  such that  $i + 1$  appears in  $T$  in a row strictly below the row of  $i$ .

**Definition 2.10.** The *major index* of  $T$  is the sum of the descents.

In fact,  $\text{SYT}(\lambda)$  indexes a basis for  $S^\lambda$ . The number  $f^\lambda := \#\text{SYT}(\lambda) = \dim S^\lambda$  is usually very large. (It’s typically thought of as  $n!^\epsilon$  for some  $\epsilon$ .)

**Example 2.11.** There are two standard tableaux of shape  $\lambda = (2, 1)$ , namely  $T = 12/3$  and  $T = 13/2$ . The descent set of  $1267/35/48$  is  $\{2, 3, 7\}$  and major index  $2 + 3 + 7 = 12$ .

### 3. MODULAR MAJOR INDEX ESTIMATES

The corresponding paper has been published [Swa18]. Sundaram’s paper [Sun18] has also now been published.

**Question 3.1** (Sundaram). *Let  $S_n$  act by conjugation  $\mathbb{C}$ -linearly on permutations of cycle type  $\mu$ . For which  $\mu$  does every  $S_n$ -irreducible appear? (When is  $\mu$  a global class?)*

**Conjecture 3.2** (Sundaram). *Take  $n \geq 8$ .  $\mu$  is a global class if and only if  $\mu$  has at least 2 parts and all parts are odd and distinct.*

**Remark 3.3.** She proved the conjecture contingent on a classification of which irreducibles appear when  $\mu = (n)$  (all but  $(n - 1, 1)$  and  $(2, 2^{n-2})$  when  $n$  is odd, and all but  $(n - 1, 1)$  and  $(1^n)$  when  $n$  is even). The first part of the talk describes asymptotics strong enough to answer this question and, hence, prove Sundaram’s conjecture. The second part describes related work on major index statistic asymptotics.

**Remark 3.4.** When  $\mu = (n)$ , the module in question is  $1 \uparrow_{C_n}^{S_n}$  where  $C_n = \langle (1 \ 2 \ \dots \ n) \rangle$ . We consider more generally  $\chi^r \uparrow_{C_n}^{S_n}$ , as above.

**Definition 3.5.** Set

$$a_{\lambda,r} := \langle S^\lambda, \chi^r \uparrow_{C_n}^{S_n} \rangle = \langle S^\lambda \downarrow_{C_n}^{S_n}, \chi^r \rangle.$$

Sundaram was interested in the  $r = 0$  case.

**Theorem 3.6** (Kraskiewicz–Weyman). *Let  $\lambda \vdash n$ . Then*

$$a_{\lambda,r} = \#\{T \in \text{SYT}(\lambda) : \text{maj}(T) \equiv_n r\}.$$

**Remark 3.7.** Klyachko classified when  $a_{\lambda,1} = 0$  by finding faithful representations of  $C_n$  in  $S^\lambda$ , though this argument doesn't generalize to other  $r$  in any obvious way. Marianne Johnson gave a combinatorial argument re-proving Klyachko's result from the K-W theorem, though it relied on the representation-theoretic result that  $a_{\lambda,r}$  depends only on  $\lambda$  and  $\text{gcd}(r, n)$  and was relatively ad-hoc.

We give the following stronger result, answering Sundaram's conjecture in the affirmative and hence completing her classification of the global conjugacy classes of  $S_n$ .

**Theorem 3.8** ([Swa18, Thm. 1.4]). *Let  $\lambda \vdash n$  and  $r \in \mathbb{Z}$ . Then  $a_{\lambda,r} \neq 0$  except possibly when  $\lambda = (2, 2), (2, 2, 2), (3, 3)$  or when  $\lambda$  is in one of the four infinite families  $(1^n), (n), (2, 1^{n-1}), (n-1, 1)$ .*

Moreover, the argument is more general, more conceptual, and offers vastly more precise estimates of each  $a_{\lambda,r}$  than in earlier work. The key idea is obtaining the following type of bound.

**Theorem 3.9** (S.). *Let  $\lambda \vdash n$ . Independent of  $r$ , we have*

$$\left| \frac{a_{\lambda,r}}{f^\lambda} - \frac{1}{n} \right| \leq \frac{2n^{3/2}}{\sqrt{f^\lambda}}.$$

**Remark 3.10.** Intuitively, since  $f^\lambda$  is typically enormous compared to  $n^{3/2}$ , this says that the “maj mod  $n$ ” statistic on  $\text{SYT}(\lambda)$  is approximately uniformly distributed, with  $a_{\lambda,r} \approx f^\lambda/n$  independent of  $r$ . Some ingredients in the proof are as follows.

**Theorem 3.11** (Foulkes). *We have*

$$\text{ch } \chi^r \uparrow_{C_n}^{S_n} = \frac{1}{n} \sum_{\lambda \vdash n} c_\ell(r) p_{(\ell^n/\ell)}$$

where

$$\begin{aligned} c_\ell(r) &:= \text{sum of } r\text{th powers of primitive } \ell\text{th roots of unity} \\ & (= \mu(\ell/(\ell, r))\phi(\ell)/\phi(\ell/(\ell, r)) ) \end{aligned}$$

is a so-called Ramanujan sum.

**Corollary 3.12.** *For  $\lambda \vdash n$ , let  $f^\lambda := \#\text{SYT}(\lambda) = \chi^\lambda(1^n)$ . Then*

$$\frac{a_{\lambda,r}}{f^\lambda} = \frac{1}{n} + \frac{1}{n} \sum_{\substack{\ell|n \\ \ell \neq 1}} \frac{\chi^\lambda(\ell^n/\ell)}{f^\lambda} c_\ell(r).$$

**Theorem 3.13** (Fomin–Lulov). *Let  $\lambda \vdash n = \ell s$ . Then*

$$|\chi^\lambda(\ell^s)| \leq \frac{s! \ell^s}{(n!)^{1/\ell}} (f^\lambda)^{1/\ell}.$$

**Remark 3.14.** The theorem follows from combining the corollary, the Fomin–Lulov bound, and Stirling's approximation (carefully). To actually show  $a_{\lambda,r} \neq 0$  using this sort of estimate requires lower bounds of the form  $f^\lambda \geq n^d$  for fixed  $d$ . This is accomplished by introducing an “opposite hook product” inequality (discovered independently by Morales–Pak–Panova) and using a certain recursive procedure to reduce to the case of hook shapes. The relevant estimates are strong enough when  $n \geq 34$ , with the remainder being brute-forced on computer. Since only partitions with “very few” standard tableaux are left to check, the brute-force check can be done using even a naive implementation in under a minute.

## 4. NON-MODULAR MAJOR INDEX ESTIMATES

This work is joint with Sara Billey and Matjaž Konvalinka. The corresponding paper is in preparation (should be on arXiv very soon). Let

$$b_{\lambda,i} := \#\{T \in \text{SYT}(\lambda) : \text{maj}(T) = i\}.$$

These constants appear in a number of contexts: the graded Frobenius series of the type A coinvariant algebra; stable principal specializations of Schur functions; and coefficients of certain degree polynomials for  $\text{GL}_n(\mathbb{F}_q)$ -representations. We won't describe these connections further here.

**Question 4.1.**

- (1) What does the distribution of  $\text{maj}$  on  $\text{SYT}(\lambda)$  look like?
- (2) When is  $b_{\lambda,i} = 0$ ?

We'll begin by describing an answer to the first question.

**Definition 4.2.** Given a random variable  $X$  with mean  $\mu$  and standard deviation  $\sigma$ , define the corresponding normalized random variable by

$$X^* := \frac{X - \mu}{\sigma}.$$

$X^*$  has mean 0 and variance 1.

**Definition 4.3.** Let  $X_1, X_2, \dots$  be a sequence of real-valued random variables. Suppose  $X_N^*$  has cumulative distribution function  $F_N(t) := \mathbb{P}[X_N^* \leq t]$ . We say the sequence  $X_1, X_2, \dots$  is *asymptotically normal* if for all  $t \in \mathbb{R}$ ,

$$\lim_{N \rightarrow \infty} F_N(t) = F(t)$$

where  $F(t)$  is the CDF of the standard normal distribution.

**Definition 4.4.** Define a statistic

$$\begin{aligned} \text{aft}(\lambda) &:= |\lambda| - \max\{\lambda_1, \tilde{\lambda}_1\} \\ &= (\text{if } \lambda \text{ is at least as wide as it is tall}) \text{ number of cells not in the largest row.} \end{aligned}$$

**Theorem 4.5** (Billey–Konvalinka–S.). *Suppose  $\lambda^{(1)}, \lambda^{(2)}, \dots$  is a sequence of partitions. Let  $X_N$  be the major index statistic on  $\text{SYT}(\lambda^{(N)})$ . Then, the sequence  $X_1, X_2, \dots$  is asymptotically normal if and only if*

$$\lim_{N \rightarrow \infty} \text{aft}(\lambda^{(N)}) = \infty.$$

As an example, the theorem recovers the following earlier result by letting  $\lambda^{(N)} := (N, N)$ , since then  $\text{aft}(\lambda^{(N)}) = 2N - N = N \rightarrow \infty$ .

**Corollary 4.6** (Chen–Wang–Wang). *The coefficients of the  $q$ -Catalan numbers  $\frac{1}{[N+1]_q} \binom{2N}{N}_q$  are asymptotically normal.*

**Remark 4.7.** The proof uses Stanley's  $q$ -hook length formula, a beautifully explicit cumulant formula, and direct combinatorial growth rate estimates of normalized cumulants. For instance, we show for fixed  $d$  and uniformly for all  $\lambda$  that

$$\left| \kappa_d^{\lambda^*} \right| = \Theta(\text{aft}(\lambda)^{1-d/2}).$$

**Remark 4.8.** Actually, our result is more general and I've given the above simplified version in the interest of time. Our full result allows one to use “block diagonal” skew partitions. Special cases then include  $\text{maj}$  on words of content  $\alpha$ . This allows us to simultaneously generalize a series of earlier asymptotic normality results by Canfield–Janson–Zeilberger, Diaconis, Mann–Whitney, and Feller.

**Remark 4.9.** One may ask what happens when  $\text{aft}(\lambda^{(N)})$  does not tend to  $\infty$ . We have the following result which, together with the above result, completely classifies all possible limiting normalized distributions of  $\text{maj}$  on  $\text{SYT}(\lambda)$  for any sequence of  $\lambda$ 's (or, more generally, block diagonal skew partitions).

**Theorem 4.10** (Billey–Konvalinka–S.). *In the notation above, suppose  $\text{aft}(\lambda^{(N)}) = k$  for all  $N$  and  $|\lambda^{(N)}| \rightarrow \infty$ . Then  $X_1, X_2, \dots$  is asymptotically distributed according to  $X^*$  where  $X = \sum_{i=1}^k U[0, 1]$ .*

These are reasonably satisfying answers to question (1). As for (2), we have the following.

**Theorem 4.11** (Billey–Konvalinka–S.). *The generating function  $\sum_{T \in \text{SYT}(\lambda)} q^{\text{maj}(T)}$  has “no internal zeros,” except for two particular exceptions when  $\lambda$  is a rectangle with more than 1 row and column.*

**Remark 4.12.** I have further ongoing work on a “local limit theorem,” attempting to give an estimate for each  $b_{\lambda_i}$  akin to the estimate  $a_{\lambda,r} \approx f_{\lambda,r}/n$ . The arguments are much harder and progress has stalled. I’ve proven a generalization of the Fomin–Lulov bound outside of the  $r$ -decomposable case and used it to give a sufficiently powerful bound of the characteristic function in a neighborhood of  $-1$ , but attempts to extend this have not been successful.

## REFERENCES

- [Sun18] Sheila Sundaram. On conjugacy classes of  $S_n$  containing all irreducibles. *Israel Journal of Mathematics*, Mar 2018.
- [Swal8] Joshua P. Swanson. On the existence of tableaux with given modular major index. *Algebraic Combinatorics*, 1(1):3–21, 2018.