Refined Cyclic Sieving on Words and Tableaux

San Diego JMM, January 13th, 2018

Josh Swanson
University of Washington

based on joint work with
Connor Ahlbach and Brendon Rhoades

arXiv:1706.08631

Outline

- The cyclic sieving phenomenon (CSP) and refinements

Outline

- The cyclic sieving phenomenon (CSP) and refinements
- Refined CSP on words

Outline

- The cyclic sieving phenomenon (CSP) and refinements
- Refined CSP on words
- Refined CSP on tableaux

The Cyclic Sieving Phenomenon (CSP)

Definition (Reiner–Stanton–White, 2004)

Take $(X, C, f(q))$ where X is a finite set, C is a finite cyclic group acting on X, and $f(q) \in \mathbb{Z}_{\geq 0}[q]$.

The Cyclic Sieving Phenomenon (CSP)

Definition (Reiner–Stanton–White, 2004)

Take $(X, C, f(q))$ where X is a finite set, C is a finite cyclic group acting on X, and $f(q) \in \mathbb{Z}_{\geq 0}[q]$.

We say $(X, C, f(q))$ exhibits *the cyclic sieving phenomenon (CSP)* if for all $c \in C$ and roots of unity $\omega \in \mathbb{C}$ of the same order as c,

$$\#\{x \in X : c \cdot x = x\} = f(\omega).$$

The Cyclic Sieving Phenomenon (CSP)

Definition (Reiner–Stanton–White, 2004)

Take $(X, C, f(q))$ where X is a finite set, C is a finite cyclic group acting on X, and $f(q) \in \mathbb{Z}_{\geq 0}[q]$.

We say $(X, C, f(q))$ exhibits *the cyclic sieving phenomenon (CSP)* if for all $c \in C$ and roots of unity $\omega \in \mathbb{C}$ of the same order as c,

$$\#\{x \in X : c \cdot x = x\} = f(\omega).$$

(Equivalently, $f(\omega)$ is $\mathrm{Tr}_{\mathbb{C}\{X\}}(c)$. Note $f(1) = \#X$.)

The Cyclic Sieving Phenomenon (CSP)

Example

Let $X = \binom{[n]}{k}$ and let $C = \mathbb{Z}/n$ act on X by addition mod n: if $n = 6, k = 3$, then

$$\overline{2} \cdot \{2, 3, 5\} = \{4, 5, 1\}.$$

The Cyclic Sieving Phenomenon (CSP)

Example

Let $X = \binom{[n]}{k}$ and let $C = \mathbb{Z}/n$ act on X by addition mod n: if $n = 6, k = 3$, then

$$\overline{2} \cdot \{2, 3, 5\} = \{4, 5, 1\}.$$

Theorem (RSW)

The triple $(\binom{[n]}{k}, \mathbb{Z}/n, \binom{n}{k}_q)$ *exhibits the CSP.*

The Cyclic Sieving Phenomenon (CSP)

Example

Let $X = \binom{[n]}{k}$ and let $C = \mathbb{Z}/n$ act on X by addition mod n: if $n = 6, k = 3$, then

$$\overline{2} \cdot \{2,3,5\} = \{4,5,1\}.$$

Theorem (RSW)

The triple $(\binom{[n]}{k}, \mathbb{Z}/n, \binom{n}{k}_q)$ *exhibits the CSP.*

Recall:

- $\binom{n}{k}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}$
- $[n]_q! := [n]_q[n-1]_q \cdots [1]_q$
- $[c]_q := 1 + q + \cdots + q^{c-1}$

CSP Refinements

Notation

Given $\mathrm{stat}\colon X \to \mathbb{Z}_{\geq 0}$, write

$$X^{\mathrm{stat}}(q) := \sum_{x \in X} q^{\mathrm{stat}(x)} \in \mathbb{Z}_{\geq 0}[q].$$

CSP Refinements

Notation

Given $\mathrm{stat}\colon X \to \mathbb{Z}_{\geq 0}$, write

$$X^{\mathrm{stat}}(q) := \sum_{x \in X} q^{\mathrm{stat}(x)} \in \mathbb{Z}_{\geq 0}[q].$$

Note $X^{\mathrm{stat}}(1) = \#X$.

CSP Refinements

Notation

Given $\mathrm{stat}\colon X \to \mathbb{Z}_{\geq 0}$, write

$$X^{\mathrm{stat}}(q) := \sum_{x \in X} q^{\mathrm{stat}(x)} \in \mathbb{Z}_{\geq 0}[q].$$

Note $X^{\mathrm{stat}}(1) = \#X$.
In *many* CSP triples, $f(q) = X^{\mathrm{stat}}(q)$ for some stat.

Example

$\binom{n}{k}_q = \binom{[n]}{k}^{\mathrm{Sum}'}(q)$ where $\mathrm{Sum}'(A) = (\sum_{a \in A} a) - (1 + 2 + \cdots + k)$.

CSP Refinements

Definition (Ahlbach–S.)

Given a CSP triple $(X, C, X^{\text{stat}}(q))$ and $Y \subset X$ closed under the C-action

CSP Refinements

Definition (Ahlbach–S.)

Given a CSP triple $(X, C, X^{\text{stat}}(q))$ and $Y \subset X$ closed under the C-action, if $(Y, C, Y^{\text{stat}}(q))$ also exhibits the CSP, we say $(Y, C, Y^{\text{stat}}(q))$ *refines* the CSP triple $(X, C, X^{\text{stat}}(q))$.

CSP Refinements

Definition (Ahlbach–S.)

Given a CSP triple $(X, C, X^{\text{stat}}(q))$ and $Y \subset X$ closed under the C-action, if $(Y, C, Y^{\text{stat}}(q))$ also exhibits the CSP, we say $(Y, C, Y^{\text{stat}}(q))$ *refines* the CSP triple $(X, C, X^{\text{stat}}(q))$.

(In this case, $(X - Y, C, (X - Y)^{\text{stat}}(q))$ also exhibits the CSP.)

CSP Refinements

Definition (Ahlbach–S.)

Given a CSP triple $(X, C, X^{\text{stat}}(q))$ and $Y \subset X$ closed under the C-action, if $(Y, C, Y^{\text{stat}}(q))$ also exhibits the CSP, we say $(Y, C, Y^{\text{stat}}(q))$ *refines* the CSP triple $(X, C, X^{\text{stat}}(q))$.

(In this case, $(X - Y, C, (X - Y)^{\text{stat}}(q))$ also exhibits the CSP.)

Example

Take $X = \binom{[6]}{3}$, $Y = \mathbb{Z}/6 \cdot \{2, 3, 4\}$

CSP Refinements

Definition (Ahlbach–S.)

Given a CSP triple $(X, C, X^{\text{stat}}(q))$ and $Y \subset X$ closed under the C-action, if $(Y, C, Y^{\text{stat}}(q))$ also exhibits the CSP, we say $(Y, C, Y^{\text{stat}}(q))$ *refines* the CSP triple $(X, C, X^{\text{stat}}(q))$.

(In this case, $(X - Y, C, (X - Y)^{\text{stat}}(q))$ also exhibits the CSP.)

Example

Take $X = \binom{[6]}{3}$, $Y = \mathbb{Z}/6 \cdot \{2, 3, 4\}$. Then $Y^{\text{Sum}'}(q) = 1 + 2q^3 + 2q^6 + q^9$, and

$$Y^{\text{Sum}'}(1) = 6, \qquad Y^{\text{Sum}'}(-1) = 0,$$
$$Y^{\text{Sum}'}(\omega_3) = 6, \qquad Y^{\text{Sum}'}(\omega_6) = 0.$$

CSP Refinements

Definition (Ahlbach–S.)

Given a CSP triple $(X, C, X^{\text{stat}}(q))$ and $Y \subset X$ closed under the C-action, if $(Y, C, Y^{\text{stat}}(q))$ also exhibits the CSP, we say $(Y, C, Y^{\text{stat}}(q))$ *refines* the CSP triple $(X, C, X^{\text{stat}}(q))$.

(In this case, $(X - Y, C, (X - Y)^{\text{stat}}(q))$ also exhibits the CSP.)

Example

Take $X = \binom{[6]}{3}$, $Y = \mathbb{Z}/6 \cdot \{2, 3, 4\}$. Then $Y^{\text{Sum}'}(q) = 1 + 2q^3 + 2q^6 + q^9$, and

$$Y^{\text{Sum}'}(1) = 6, \qquad Y^{\text{Sum}'}(-1) = 0,$$
$$Y^{\text{Sum}'}(\omega_3) = 6, \qquad Y^{\text{Sum}'}(\omega_6) = 0.$$

We would need $Y^{\text{Sum}'}(\omega_3) = 0$, not 6. So, $(Y, \mathbb{Z}/n, Y^{\text{Sum}'}(q))$ does NOT quite refine the RSW CSP $(X, \mathbb{Z}/n, X^{\text{Sum}'}(q))$.

A First Refinement Result

The *cyclic blocks* of a subset of $[n]$ are maximal sequences of adjacent elements in the subset, where 1 is considered adjacent to n. (Ex: $\{1, 2, 4, 6\} \subset [6]$ has two cyclic blocks, 612 and 4.)

A First Refinement Result

The *cyclic blocks* of a subset of $[n]$ are maximal sequences of adjacent elements in the subset, where 1 is considered adjacent to n. (Ex: $\{1, 2, 4, 6\} \subset [6]$ has two cyclic blocks, 612 and 4.) Let

$$S_k := \text{the } k\text{-element subsets of } [n]$$
$$S_{k,b} := \text{the } k\text{-element subsets of } [n] \text{ with } b \text{ cyclic blocks.}$$

A First Refinement Result

The *cyclic blocks* of a subset of $[n]$ are maximal sequences of adjacent elements in the subset, where 1 is considered adjacent to n. (Ex: $\{1, 2, 4, 6\} \subset [6]$ has two cyclic blocks, 612 and 4.) Let

$$S_k := \text{the } k\text{-element subsets of } [n]$$
$$S_{k,b} := \text{the } k\text{-element subsets of } [n] \text{ with } b \text{ cyclic blocks.}$$

Let mbs be the sum of the ends of the cyclic blocks of a subset of $[n]$. (Ex: $\text{mbs}(\{1, 2, 4, 6\} \subset [6]) = 2 + 4 = 6$.)

A First Refinement Result

The *cyclic blocks* of a subset of $[n]$ are maximal sequences of adjacent elements in the subset, where 1 is considered adjacent to n. (Ex: $\{1, 2, 4, 6\} \subset [6]$ has two cyclic blocks, 612 and 4.) Let

$$S_k := \text{the } k\text{-element subsets of } [n]$$
$$S_{k,b} := \text{the } k\text{-element subsets of } [n] \text{ with } b \text{ cyclic blocks.}$$

Let mbs be the sum of the ends of the cyclic blocks of a subset of $[n]$. (Ex: $\text{mbs}(\{1, 2, 4, 6\} \subset [6]) = 2 + 4 = 6$.)

Theorem (Ahlbach–S.)

$(S_{k,b}, \mathbb{Z}/n, S_{k,b}^{\text{mbs}}(q))$ refines the CSP triple $(S_k, \mathbb{Z}/n, S_k^{\text{mbs}}(q))$.

A First Refinement Result

The *cyclic blocks* of a subset of $[n]$ are maximal sequences of adjacent elements in the subset, where 1 is considered adjacent to n. (Ex: $\{1, 2, 4, 6\} \subset [6]$ has two cyclic blocks, 612 and 4.) Let

$$S_k := \text{the } k\text{-element subsets of } [n]$$
$$S_{k,b} := \text{the } k\text{-element subsets of } [n] \text{ with } b \text{ cyclic blocks.}$$

Let mbs be the sum of the ends of the cyclic blocks of a subset of $[n]$. (Ex: $\mathrm{mbs}(\{1, 2, 4, 6\} \subset [6]) = 2 + 4 = 6$.)

Theorem (Ahlbach–S.)

$(S_{k,b}, \mathbb{Z}/n, S_{k,b}^{\mathrm{mbs}}(q))$ *refines the CSP triple* $(S_k, \mathbb{Z}/n, S_k^{\mathrm{mbs}}(q))$.

Here $X^{\mathrm{Sum}'}(q)$ is "equivalent" to $\binom{n}{k}_q$, so the unrefined triple is essentially RSW's.

Word Combinatorics

Definition

Given a word $w = w_1 \cdots w_n$ with letters $w_i \in \mathbb{Z}_{\geq 1}$, the *descent set* of w is

$$\mathrm{Des}(w) := \{i \in [n-1] : w_i > w_{i+1}\}.$$

Word Combinatorics

Definition

Given a word $w = w_1 \cdots w_n$ with letters $w_i \in \mathbb{Z}_{\geq 1}$, the *descent set* of w is

$$\mathrm{Des}(w) := \{i \in [n-1] : w_i > w_{i+1}\}.$$

The *major index* of w is

$$\mathrm{maj}(w) := \sum_{i \in \mathrm{Des}(w)} i.$$

Word Combinatorics

Definition

Given a word $w = w_1 \cdots w_n$ with letters $w_i \in \mathbb{Z}_{\geq 1}$, the *descent set* of w is

$$\mathrm{Des}(w) := \{i \in [n-1] : w_i > w_{i+1}\}.$$

The *major index* of w is

$$\mathrm{maj}(w) := \sum_{i \in \mathrm{Des}(w)} i.$$

The *content* of w is the weak composition $\alpha = (\alpha_1, \alpha_2, \ldots) \vDash n$ where

$$\alpha_i := \#i\text{'s in } \alpha.$$

Word Combinatorics

Definition

Given a word $w = w_1 \cdots w_n$ with letters $w_i \in \mathbb{Z}_{\geq 1}$, the *descent set* of w is

$$\mathrm{Des}(w) := \{i \in [n-1] : w_i > w_{i+1}\}.$$

The *major index* of w is

$$\mathrm{maj}(w) := \sum_{i \in \mathrm{Des}(w)} i.$$

The *content* of w is the weak composition $\alpha = (\alpha_1, \alpha_2, \ldots) \vDash n$ where

$$\alpha_i := \#i\text{'s in } \alpha.$$

(Ex: If $w = 323314$, then $\mathrm{Des}(w) = \{1, 4\}$, $\mathrm{maj}(w) = 1 + 4 = 5$, and $\alpha = (1, 1, 3, 1)$.)

A CSP on Words

Notation

Let

$$W_\alpha := \text{words of content } \alpha.$$

A CSP on Words

Notation

Let

$$W_\alpha := \text{words of content } \alpha.$$

$\mathbb{Z}/n$ acts on W_α by rotation:

$$\overline{2} \cdot 011010 = 100110.$$

A CSP on Words

Notation

Let

$$W_\alpha := \text{words of content } \alpha.$$

$\mathbb{Z}/n$ acts on W_α by rotation:

$$\overline{2} \cdot 011010 = 100110.$$

Theorem (RSW)

The triple $(W_\alpha, \mathbb{Z}/n, W_\alpha^{\text{maj}}(q))$ *exhibits the CSP.*

A CSP on Words

Notation

Let

$$W_\alpha := \text{words of content } \alpha.$$

$\mathbb{Z}/n$ acts on W_α by rotation:

$$\overline{2} \cdot 011010 = 100110.$$

Theorem (RSW)

The triple $(W_\alpha, \mathbb{Z}/n, W_\alpha^{\mathrm{maj}}(q))$ *exhibits the CSP.*

Remark

They actually proved a generalization valid for all finite Coxeter groups using Springer's regular elements, representation theory, coinvariant algebras, and len instead of maj.

Refined CSP on Words

Definition

Cyclic descent type (CDT) of a word:

Refined CSP on Words

Definition

Cyclic descent type (CDT) of a word: if $w = 143124114223$, then

$$w^{(1)} = 1111 \qquad \mathrm{cdes}(w^{(1)}) = 0,$$
$$w^{(2)} = 112.1122. \qquad \mathrm{cdes}(w^{(2)}) = 2,$$
$$w^{(3)} = 13.12.11223. \qquad \mathrm{cdes}(w^{(3)}) = 3,$$
$$w^{(4)} = 14.3.124.114.223. \qquad \mathrm{cdes}(w^{(4)}) = 5.$$

Refined CSP on Words

Definition

Cyclic descent type (CDT) of a word: if $w = 143124114223$, then

$$\begin{aligned} w^{(1)} &= 1111 & \text{cdes}(w^{(1)}) &= 0, \\ w^{(2)} &= 112.1122. & \text{cdes}(w^{(2)}) &= 2, \\ w^{(3)} &= 13.12.11223. & \text{cdes}(w^{(3)}) &= 3, \\ w^{(4)} &= 14.3.124.114.223. & \text{cdes}(w^{(4)}) &= 5. \end{aligned}$$

We set $\text{CDT}(143124114223) = (0, 2-0, 3-2, 5-3) = (0, 2, 1, 2)$.

Refined CSP on Words

Definition

Cyclic descent type (CDT) of a word: if $w = 143124114223$, then

$$\begin{aligned} w^{(1)} &= 1111 & \text{cdes}(w^{(1)}) &= 0, \\ w^{(2)} &= 112.1122. & \text{cdes}(w^{(2)}) &= 2, \\ w^{(3)} &= 13.12.11223. & \text{cdes}(w^{(3)}) &= 3, \\ w^{(4)} &= 14.3.124.114.223. & \text{cdes}(w^{(4)}) &= 5. \end{aligned}$$

We set $\text{CDT}(143124114223) = (0, 2-0, 3-2, 5-3) = (0, 2, 1, 2)$.

Notation

Let

$$W_{\alpha,\delta} := \text{words } w \text{ with content } \alpha \text{ and } \text{CDT}(w) = \delta.$$

Refined CSP on Words

Theorem (Ahlbach-S.)

$(W_{\alpha,\delta}, \mathbb{Z}/n, W^{\mathsf{maj}}_{\alpha,\delta}(q))$ refines the CSP triple $(W_\alpha, \mathbb{Z}/n, W^{\mathsf{maj}}_\alpha(q))$.

Refined CSP on Words

Theorem (Ahlbach-S.)

$(W_{\alpha,\delta}, \mathbb{Z}/n, W_{\alpha,\delta}^{\mathrm{maj}}(q))$ *refines the CSP triple* $(W_\alpha, \mathbb{Z}/n, W_\alpha^{\mathrm{maj}}(q))$.

Remark

Completely different proof than RSW. Combinatorial and largely recursive. Involves Carlitz-style decomposition, (more or less new) notion of "modular periodicity," a CSP extension lemma, a non-equivariant-but-fixed-point-preserving bijection, products of CSP's on sets and multisets.

Refined CSP on Words

One key step:

Refined CSP on Words

One key step:

Theorem (Ahlbach-S.)

Let $\alpha \vDash n$ be a strong composition with m parts, $\delta \vDash k$, $n_i := |w^{(i)}|$, $k_i := \mathrm{cdes}(w^{(i)})$, $d := \gcd(n, k)$. Then, modulo $q^n - 1$,

$$W^{\mathrm{maj}}_{\alpha,\delta}(q) \equiv \frac{d}{\alpha_1}[n/d]_{q^d} \prod_{\ell=2}^{m} q^{k_\ell \alpha_\ell} \binom{n_{\ell-1} - k_{\ell-1}}{\delta_\ell}_q \left(\!\!\binom{k_\ell}{\alpha_\ell - \delta_\ell}\!\!\right)_{q^{-1}}$$

$$\equiv \frac{d}{\alpha_1}[n/d]_{q^d} q^{\eta} \prod_{\ell=2}^{m} \binom{n_{\ell-1} - k_{\ell-1}}{\delta_\ell}_q \left(\!\!\binom{k_\ell}{\alpha_\ell - \delta_\ell}\!\!\right)_q$$

where $\eta := \binom{k}{2} + \sum_{\ell=2}^{m} \binom{\delta_\ell}{2} - \alpha_1$.

Tableaux combinatorics

Definition

Given $T \in \mathrm{SYT}(\lambda)$,

$$\mathrm{Des}(T) := \{i : i + 1 \text{ is in a lower row than } i\}.$$

Tableaux combinatorics

Definition

Given $T \in \mathrm{SYT}(\lambda)$,

$$\mathrm{Des}(T) := \{i : i+1 \text{ is in a lower row than } i\}.$$

Ex:

$$T = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 6 \\ \cline{1-2} 5 & 7 \\ \cline{1-2}\end{array} \Rightarrow \mathrm{Des}(T) = \{2, 4, 6\}.$$

Tableaux combinatorics

Definition

Given $T \in \mathrm{SYT}(\lambda)$,

$$\mathrm{Des}(T) := \{i : i+1 \text{ is in a lower row than } i\}.$$

Ex:

$$T = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 6 \\ \cline{1-2} 5 & 7 \\ \cline{1-2}\end{array} \Rightarrow \mathrm{Des}(T) = \{2, 4, 6\}.$$

As before, $\mathrm{maj}(T) := \sum_{i \in \mathrm{Des}(T)} i$.

Tableaux combinatorics

Definition

Given $T \in \mathrm{SYT}(\lambda)$,

$$\mathrm{Des}(T) := \{i : i+1 \text{ is in a lower row than } i\}.$$

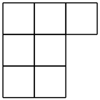
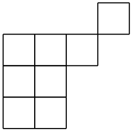
Ex:

$$T = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 6 \\ \cline{1-2} 5 & 7 \\ \cline{1-2}\end{array} \Rightarrow \mathrm{Des}(T) = \{2, 4, 6\}.$$

As before, $\mathrm{maj}(T) := \sum_{i \in \mathrm{Des}(T)} i$.

Definition

Given $\lambda \vdash n-1$, let $\lambda^{\square} \vdash n$ be the following "slightly skew partition":

$\lambda =$  $\Rightarrow \lambda^{\square} =$ 

Refined Sieving on Tableaux

Remark

Elizalde–Roichman (2017) defined a bijection
$\sigma\colon \mathsf{SYT}(\lambda^{\Box}) \to \mathsf{SYT}(\lambda^{\Box})$ whose orbits are size n.

Refined Sieving on Tableaux

Remark

Elizalde–Roichman (2017) defined a bijection $\sigma\colon \mathrm{SYT}(\lambda^\Box) \to \mathrm{SYT}(\lambda^\Box)$ whose orbits are size n. They also defined $\mathrm{cDes}\colon \mathrm{SYT}(\lambda^\Box) \to 2^{[n]}$ such that

(i) $\mathrm{cDes}(T) \cap [n-1] = \mathrm{Des}(T)$

(ii) $\mathrm{cDes}(\sigma^k \cdot T) = \overline{k} \cdot \mathrm{cDes}(T)$.

Refined Sieving on Tableaux

Remark

Elizalde–Roichman (2017) defined a bijection $\sigma\colon \mathrm{SYT}(\lambda^{\square}) \to \mathrm{SYT}(\lambda^{\square})$ whose orbits are size n. They also defined $\mathrm{cDes}\colon \mathrm{SYT}(\lambda^{\square}) \to 2^{[n]}$ such that

(i) $\mathrm{cDes}(T) \cap [n-1] = \mathrm{Des}(T)$

(ii) $\mathrm{cDes}(\sigma^k \cdot T) = \overline{k} \cdot \mathrm{cDes}(T)$.

Theorem (Ahlbach–Rhoades–S.)

The triple $(\mathrm{SYT}(\lambda^{\square}), \langle\sigma\rangle, \mathrm{SYT}(\lambda^{\square})^{\mathrm{maj}}(q))$ *exhibits the CSP.*

Refined Sieving on Tableaux

Remark

Elizalde–Roichman (2017) defined a bijection $\sigma\colon \mathrm{SYT}(\lambda^{\square}) \to \mathrm{SYT}(\lambda^{\square})$ whose orbits are size n. They also defined $\mathrm{cDes}\colon \mathrm{SYT}(\lambda^{\square}) \to 2^{[n]}$ such that

(i) $\mathrm{cDes}(T) \cap [n-1] = \mathrm{Des}(T)$

(ii) $\mathrm{cDes}(\sigma^k \cdot T) = \overline{k} \cdot \mathrm{cDes}(T)$.

Theorem (Ahlbach–Rhoades–S.)

The triple $(\mathrm{SYT}(\lambda^{\square}), \langle\sigma\rangle, \mathrm{SYT}(\lambda^{\square})^{\mathrm{maj}}(q))$ *exhibits the CSP.*

Remark

Proof reduces to showing $[n]_q$ divides $\mathrm{SYT}(\lambda^{\square})^{\mathrm{maj}}(q)$.

Refined Sieving on Tableaux

Remark

Elizalde–Roichman (2017) defined a bijection $\sigma \colon \mathrm{SYT}(\lambda^{\square}) \to \mathrm{SYT}(\lambda^{\square})$ whose orbits are size n. They also defined $\mathrm{cDes} \colon \mathrm{SYT}(\lambda^{\square}) \to 2^{[n]}$ such that

(i) $\mathrm{cDes}(T) \cap [n-1] = \mathrm{Des}(T)$

(ii) $\mathrm{cDes}(\sigma^k \cdot T) = \overline{k} \cdot \mathrm{cDes}(T)$.

Theorem (Ahlbach–Rhoades–S.)

The triple $(\mathrm{SYT}(\lambda^{\square}), \langle \sigma \rangle, \mathrm{SYT}(\lambda^{\square})^{\mathrm{maj}}(q))$ *exhibits the CSP.*

Remark

Proof reduces to showing $[n]_q$ divides $\mathrm{SYT}(\lambda^{\square})^{\mathrm{maj}}(q)$. Follows from

$$\begin{aligned}\mathrm{SYT}(\lambda^{\square})^{\mathrm{maj}}(q) &= \binom{n}{n-1, 1}_q \mathrm{SYT}(\lambda)^{\mathrm{maj}}(q)\, \mathrm{SYT}(\square)^{\mathrm{maj}}(q) \\ &= [n]_q\, \mathrm{SYT}(\lambda)^{\mathrm{maj}}(q).\end{aligned}$$

Refined Sieving on Tableaux

Notation

Write

$$\mathrm{SYT}(\lambda^{\square}; k) := \{T \in \mathrm{SYT}(\lambda^{\square}) : \mathrm{cdes}(T) = k\}.$$

Refined Sieving on Tableaux

Notation

Write

$$\mathsf{SYT}(\lambda^{\square}; k) := \{T \in \mathsf{SYT}(\lambda^{\square}) : \mathrm{cdes}(T) = k\}.$$

Theorem (Ahlbach–Rhoades–S.)

$(\mathsf{SYT}(\lambda^{\square}; k), \langle\sigma\rangle, \mathsf{SYT}(\lambda^{\square}; k)^{\mathrm{maj}}(q))$ *refines the CSP triple* $(\mathsf{SYT}(\lambda^{\square}), \langle\sigma\rangle, \mathsf{SYT}(\lambda^{\square})^{\mathrm{maj}}(q))$.

Refined Sieving on Tableaux

Notation

Write

$$\mathrm{SYT}(\lambda^{\square}; k) := \{T \in \mathrm{SYT}(\lambda^{\square}) : \mathrm{cdes}(T) = k\}.$$

Theorem (Ahlbach–Rhoades–S.)

$(\mathrm{SYT}(\lambda^{\square}; k), \langle\sigma\rangle, \mathrm{SYT}(\lambda^{\square}; k)^{\mathrm{maj}}(q))$ *refines the CSP triple* $(\mathrm{SYT}(\lambda^{\square}), \langle\sigma\rangle, \mathrm{SYT}(\lambda^{\square})^{\mathrm{maj}}(q))$.

Remark

Showing $[n]_q \mid \mathrm{SYT}(\lambda^{\square}; k)^{\mathrm{maj}}(q)$ is significantly more involved. Uses an inner product formula of Adin–Reiner–Roichman (2017) for Elizalde–Roichman's cyclic descent extensions, a "change of basis," and the $W_{\alpha,\delta}^{\mathrm{maj}}(q)$ product formula above.

Further work

- In progress: refine Rhoades' sieving result on rectangular tableaux.

Further work

- In progress: refine Rhoades' sieving result on rectangular tableaux. (Catalan case done.)

Further work

- In progress: refine Rhoades' sieving result on rectangular tableaux. (Catalan case done.)
- In progress: further explore the CSP and Roichman et al's other cyclic descent extensions

Further work

- In progress: refine Rhoades' sieving result on rectangular tableaux. (Catalan case done.)
- In progress: further explore the CSP and Roichman et al's other cyclic descent extensions
- Give a representation-theoretic proof of $W_{\alpha,\delta}$ result

Thanks!

THANKS!