MAJOR INDEX ASYMPTOTICS

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Abstract. These notes were for a lecture given in the University of Michigan combinatorics seminar on January 19th, 2018.

Contents

1. Modular Major Index Estimates

For the corresponding paper, see arXiv:1701.04963; accepted to ALCO.

Question 1.1 (Sundaram)**.** *Let S_n act by conjugation $\mathbb{C}$-linearly on permutations of cycle type μ. For which μ does every S_n-irreducible appear? (When is μ a global class?)*

Conjecture 1.2 (Sundaram)**.** *Take $n \geq 8$. μ is a global class if and only if μ has at least 2 parts and all parts are odd and distinct.*

Remark 1.3. She proved the conjecture contingent on a classification of which irreducibles appear when $\mu = (n)$ (all but $(n-1,1)$ and $(2, 2^{n-2})$) when n is odd, and all but $(n-1,1)$ and (1^n) when n is even). The first part of the talk describes asymptotics strong enough to answer this question and, hence, prove Sundaram's conjecture. The second part describes related work on major index statistic asymptotics.

Remark 1.4. When $\mu = (n)$, the module in question is $1\uparrow_{C_n}^{S_n}$ where $C_n = \langle (1\ 2\ \cdots\ n)\rangle$. We consider more generally

$$\chi^r : C_n \to \mathbb{C}^\times \text{ by } \chi^r((1\ 2 \cdots\ n)^k) := \omega_n^{rk}$$

where ω_n is any primitive nth root of unity.

Definition 1.5. Set

$$a_{\lambda,r} := \langle S^\lambda, \chi^r\uparrow_{C_n}^{S_n}\rangle = \langle S^\lambda\downarrow_{C_n}^{S_n}, \chi^r\rangle.$$

Sundaram was interested in the $r = 0$ case.

Definition 1.6. For $T \in \mathrm{SYT}(\lambda/\nu)$, set

$$\mathrm{Des}(T) := \{i : i+1 \text{ is in a lower row than } i\}$$

and

$$\mathrm{maj}(T) := \sum_{i \in \mathrm{Des}(T)} i.$$

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For instance, if $\lambda/\nu = (4,3,2)/(1)$,

$$T = \begin{array}{|c|c|c|c|}\hline \cdot & 1 & 6 & 7 \\ \hline 2 & 4 & 8 & \\ \hline 3 & 5 & & \\ \hline \end{array}$$

has $\mathrm{Des}(T) = \{1,2,4,7\}$ and $\mathrm{maj}(T) = 1+2+4+7 = 14$.

Theorem 1.7 (Kraskiewicz–Weyman)**.** *Let $\lambda \vdash n$. Then*

$$a_{\lambda,r} = \#\{T \in \mathrm{SYT}(\lambda) : \mathrm{maj}(T) \equiv_n r\}.$$

Remark 1.8. Klyachko classified when $a_{\lambda,1} = 0$ by finding faithful representations of C_n in S^λ, though this argument doesn't generalize to other r in any obvious way. Marianne Johnson gave a combinatorial argument re-proving Klyachko's result from the K-W theorem, though it relied on the representation-theoretic result that $a_{\lambda,r}$ depends only on λ and $\gcd(r,n)$ and was relatively ad-hoc.

We give the following stronger result, answering Sundaram's conjecture in the affirmative and hence completing her classification of the global conjugacy classes of S_n.

Theorem 1.9 (S.)**.** *Let $\lambda \vdash n$ and $r \in \mathbb{Z}/n$. Then $a_{\lambda,r} \neq 0$ except for six particular pairs (λ, r) and four infinite families of λ, namely $(1^n), (n), (2,1^{n-1}), (n-1,1)$.*

Moreover, the argument is more general, more conceptual, and offers vastly more precise estimates of each $a_{\lambda,r}$ than in earlier work. The key idea is obtaining the following type of bound.

Theorem 1.10 (S.)**.** *Let $\lambda \vdash n$. Independent of r, we have*

$$\left| \frac{a_{\lambda,r}}{f^\lambda} - \frac{1}{n} \right| \leq \frac{2n^{3/2}}{\sqrt{f^\lambda}}.$$

Remark 1.11. Intuitively, since f^λ is typically enormous compared to $n^{3/2}$, this says that the "maj mod n" statistic on $\mathrm{SYT}(\lambda)$ is approximately uniformly distributed, with $a_{\lambda,r} \approx f^\lambda/n$ independent of r. Some ingredients in the proof are as follows.

Theorem 1.12 (Foulkes)**.** *We have*

$$\mathrm{ch}\, \chi^r\!\uparrow_{C_n}^{S_n} = \frac{1}{n} \sum_{\lambda \vdash n} c_\ell(r) p_{(\ell^{n/\ell})}$$

where

$$\begin{aligned} c_\ell(r) &:= \textit{sum of } r\textit{th powers of primitive } \ell\textit{th roots of unity} \\ &\quad (\, = \mu(\ell/(\ell,r))\phi(\ell)/\phi(\ell/(\ell,r))\ ) \end{aligned}$$

is a so-called Ramanujan sum.

Corollary 1.13. *For $\lambda \vdash n$, let $f^\lambda := \#\,\mathrm{SYT}(\lambda) = \chi^\lambda(1^n)$. Then*

$$\frac{a_{\lambda,r}}{f^\lambda} = \frac{1}{n} + \frac{1}{n} \sum_{\substack{\ell \mid n \\ \ell \neq 1}} \frac{\chi^\lambda(\ell^{n/\ell})}{f^\lambda} c_\ell(r).$$

Theorem 1.14 (Fomin–Lulov)**.** *Let $\lambda \vdash n = \ell s$. Then*

$$|\chi^\lambda(\ell^s)| \leq \frac{s!\ell^s}{(n!)^{1/\ell}} (f^\lambda)^{1/\ell}.$$

Remark 1.15. The theorem follows from combining the corollary, the Fomin–Lulov bound, and Stirling's approximation (carefully). To actually show $a_{\lambda,r} \neq 0$ using this sort of estimate requires lower bounds of the form $f^\lambda \geq n^d$ for fixed d. This is accomplished by introducing an "opposite hook product" inequality (recently discovered independently by Morales–Pak–Panova) and using a certain recursive procedure to reduce to the case of hook shapes. The relevant estimates are strong enough when $n \geq 34$, with the remainder being brute-forced on computer.

2. Non-Modular Major Index Estimates

This work is joint with Sara Billey and Matjaž Konvalinka. The corresponding paper is in preparation; you can hopefully see it at FPSAC 2018. Let

$$b_{\lambda,i} := \#\{T \in \mathrm{SYT}(\lambda) : \mathrm{maj}(T) = i\}.$$

These constants appear in a number of contexts: the graded Frobenius series of the type A coinvariant algebra; stable principal specializations of Schur functions; and certain degree polynomials for $\mathrm{GL}_n(\mathbb{F}_q)$-representations. We won't describe these connections further here.

Question 2.1.

(1) What does the distribution of maj *on* $\mathrm{SYT}(\lambda)$ *look like?*
(2) When is $b_{\lambda,i} = 0$*?*

We'll next describe an answer to the first question.

Definition 2.2. Given a random variable X with mean μ and standard deviation σ, define the corresponding normalized random variable by

$$X^* := \frac{X - \mu}{\sigma}.$$

X^* has mean 0 and variance 1.

Definition 2.3. Let $X_1, X_2, \ldots$ be a sequence of real-valued random variables. Suppose X_N^* has cumulative distribution function $F_N(t) := \mathbb{P}[X_N^* \leq t]$. We say the sequence $X_1, X_2, \ldots$ is *asymptotically normal* if for all $t \in \mathbb{R}$,

$$\lim_{N\to\infty} F_N(t) = F(t)$$

where $F(t)$ is the CDF of the standard normal distribution.

Definition 2.4. Define a statistic

$$\mathrm{aft}(\lambda) := |\lambda| - \max\{\lambda_1, \tilde{\lambda}_1\}.$$

Theorem 2.5 (Billey–Konvalinka–S.)**.** *Suppose* $\lambda^{(1)}, \lambda^{(2)}, \ldots$ *is a sequence of partitions. Let* X_N *be the major index statistic on* $\mathrm{SYT}(\lambda^{(N)})$*. Then, the sequence* $X_1, X_2, \ldots$ *is asymptotically normal if and only if*

$$\lim_{N\to\infty} \mathrm{aft}(\lambda^{(N)}) = \infty.$$

The theorem for instance recovers the following earlier result by letting $\lambda^{(N)} := (N, N)$, since then $\mathrm{aft}(\lambda^{(N)}) = 2N - N = N \to \infty$.

Corollary 2.6 (Chen–Wang–Wang)**.** *The coefficients of the* q*-Catalan numbers* $\frac{1}{[N+1]_q}\binom{2N}{N}_q$ *are asymptotically normal.*

The proof uses Stanley's q-hook length formula, the method of moments, a nice explicit cumulant formula, and direct growth rate estimates of normalized cumulants.

Remark 2.7. We have an analogous result when the $\lambda^{(N)}$ are replaced by skew partitions "$\mathrm{diag}(\mu^{(1)}, \mu^{(2)}, \ldots)$." Letting the $\mu^{(i)} = (k_i)$, the standard tableaux are in bijection with words where the letter i appears k_i times, and in fact the two maj statistics are equidistributed. Our classification in this case reduces to an earlier result of Canfield–Janson–Zeilberger classifying when the major index statistic on words of fixed content is asymptotically normal.

Remark 2.8. One may ask what happens when $\mathrm{aft}(\lambda^{(N)})$ does not tend to ∞. We have the following result which, together with the above result, completely classifies all possible limiting distributions of maj on $\mathrm{SYT}(\lambda)$ for any sequence of λ's.

Theorem 2.9 (Billey–Konvalinka–S.)**.** *In the notation above, suppose* $\mathrm{aft}(\lambda^{(N)}) = k$ *for all* N *and* $|\lambda^{(N)}| \to \infty$*. Then* $X_1, X_2, \ldots$ *is asymptotically distributed according to* X^* *where* $X = \sum_{i=1}^{k} U[0, 1]$.

These are reasonably satisfying answers to question (1). As for (2), we have the following.

Theorem 2.10 (Billey–Konvalinka–S.)**.** *The generating function* $\sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(T)}$ *has "no internal zeros," except for two particular exceptions when* λ *is a rectangle with more than* 1 *row and column.*

Note: the proof of the theorem involves a somewhat delicate combinatorial argument whose proof is still being carefully written up, so take "Theorem" with a grain of salt for now. We're very confident in the statement.

Remark 2.11. We have further ongoing work on a "local limit theorem," attempting to give an estimate for each b_{λ_i} akin to the estimate $a_{\lambda,r} \approx f_{\lambda,r}/n$. The arguments are significantly harder.