LEHRER–SOLOMON COHOMOLOGY DECOMPOSITION

JOSHUA P. SWANSON

Abstract. These are lecture notes for a 25-minute talk given in the University of Washington Hyperplane Arrangements class on June 1st, 2018. It is essentially a summary of Lehrer–Solomon's *On the Action of the Symmetric Group on the Cohomology of the Complement of Its Reflecting Hyperplanes*

Contents

1. The cohomology ring

Let $\mathcal{A}$ be a central hyperplane arrangement in $V = \mathbb{C}^n$. Let

$$M := V - \bigcup_{H \in \mathcal{A}} H$$

be the complement of the hyperplanes. View M as a (real) manifold of dimension $2n$.

Example 1.1. When $\mathcal{A}$ is the braid arrangement,

$$M = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : z_i \neq z_j, 1 \leq i \neq j \leq n\}.$$

Note that this M is *connected*, in contrast to what would happen if we used real coefficients. Sean will have more to say on this.

Let $H^*(M) = H^*(M, \mathbb{C})$ denote the de Rham cohomology of M with complex coefficients. This is a graded-commutative $\mathbb{C}$-algebra. (Recall that this means $xy = (-1)^{\deg(x)\deg(y)} yx$; think the exterior algebra of a vector space.) The underlying topological constructions are unimportant to us and will just motivate the combinatorics.

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2. G-module structures

Suppose $G \subset \mathrm{GL}(V)$ is a group which preserves M, i.e. $g\colon M \to M$ for each $g \in G$. Cohomology is contravariantly functorial, i.e. a smooth map $f\colon M \to N$ induces a morphism of graded-commutative $\mathbb{C}$-algebras $H^*(N) \to H^*(M)$ which respects function composition. Consequently, G acts by automorphisms on $H^*(M)$. Since $H^*(M)$ is a $\mathbb{C}$-algebra already, this turns $H^*(M)$ into a (graded) $\mathbb{C}G$-module.

Question 2.1. *How can we usefully describe the $\mathbb{C}G$-module structure on $H^*(M)$? For instance, how many copies of the trivial representation are there? In which degrees?*

Example 2.2. When $\mathcal{A}$ is the braid arrangement, $G = S_n$ embedded in $\mathrm{GL}(V)$ as permutation matrices preserves M, so we get a $\mathbb{C}S_n$-module. $\mathbb{C}S_n$-modules are extremely well-studied in algebraic combinatorics.

3. The Orlik–Solomon algebra

Orlik and Solomon (1980) described $H^*(M)$ by generators and relations as follows.

Definition 3.1. Let $\mathcal{A} \subset V = \mathbb{C}^n$ be a central hyperplane arrangement. Let $E(\mathcal{A})$ be the exterior algebra over $\mathbb{C}$ generated by e_H for $H \in \mathcal{A}$. Suppose $H_1, \ldots, H_p$ are dependent hyperplanes (i.e. $\operatorname{codim} H_1 \cap \cdots \cap H_p < p$). Let $I(\mathcal{A})$ be the ideal generated by relations

$$\sum_{k=1}^{p} (-1)^{k-1} e_{H_1} \cdots \widehat{e_{H_k}} \cdots e_{H_p}$$

for all dependent $H_1, \ldots, H_p$. The *Orlik–Solomon algebra* of $\mathcal{A}$ is

$$A(\mathcal{A}) := E(\mathcal{A})/I(\mathcal{A}).$$

Write a_H for the image of e_H in $A(\mathcal{A})$.

Example 3.2. For instance, multiplying the relation by a_{H_1} and using the fact that $a_{H_1}^2 = 0$ gives

$$a_{H_1} \cdots a_{H_p} = 0$$

for dependent $H_1, \ldots, H_p$.

Definition 3.3. Continuing the previous definition, if $G \subset \mathrm{GL}(V)$ preserves $\mathcal{A}$, then $\mathcal{A}$ is a $\mathbb{C}G$-module where

$$g \cdot a_H := a_{g \cdot H}$$

for all $g \in G$, $H \in \mathcal{A}$.

Theorem 3.4 (Orlik–Solomon (1980))**.** *Let $\mathcal{A} \subset V = \mathbb{C}^n$ be a central hyperplane arrangement. Let $M = V - \cup_{H \in \mathcal{A}} H$ be the complement of the hyperplanes. Then*

$$H^*(M) \cong A(\mathcal{A})$$

as graded-commutative $\mathbb{C}$-algebras. If $G \subset \mathrm{GL}(V)$ preserves $\mathcal{A}$, then this isomorphism is also an isomorphism of graded $\mathbb{C}G$-modules.

4. A first decomposition

We can quickly use the Orlik–Solomon algebra to decompose $H^*(M)$ into a sum of induced $\mathbb{C}G$-modules as follows.

Definition 4.1. Let $X \in L(\mathcal{A})$ be an element of the intersection lattice of $\mathcal{A}$. Let $A_X(\mathcal{A})$ be the span of all $a_{H_1} \cdots a_{H_p}$ where $H_1 \cap \cdots \cap H_p = X$.

For $g \in G$, we have

$$g \cdot a_{H_1} \cdots a_{H_p} = a_{g \cdot H_1} \cdots a_{g \cdot H_p}.$$

Consequently, $g \cdot A_X(\mathcal{A}) = A_{g \cdot X}(\mathcal{A})$ where we've used the natural G-action on $L(\mathcal{A})$. Letting $G_X := \mathrm{Stab}_G(X) := \{g \in G : g \cdot X = X\}$, we also see that $A_X(\mathcal{A})$ is a $\mathbb{C}G_X$-module.

Theorem 4.2 (Orlik–Solomon)**.** *We have*

$$A(\mathcal{A}) = \bigoplus_{X \in L(\mathcal{A})} A_X(\mathcal{A}).$$

Definition 4.3. For a G-orbit $\mathcal{O}$ of $L(\mathcal{A})$, set

$$A_{\mathcal{O}}(\mathcal{A}) := \bigoplus_{X \in \mathcal{O}} A_X(\mathcal{A}).$$

Proposition 4.4. *Let $X \in \mathcal{O}$. Then as $\mathbb{C}G$-modules,*

$$A_{\mathcal{O}}(\mathcal{A}) \cong A_X(\mathcal{A}){\uparrow}_{G_X}^{G}$$

where $G_X = \mathrm{Stab}_G(X) = \{g \in G : g \cdot X = X\}$. (Recall that induced modules are defined by $M{\uparrow}_H^G := \mathbb{C}G \otimes_{\mathbb{C}H} M$.)

Proof. (Sketch.) There is a standard construction of the induced module $M{\uparrow}_H^G$ as a sum of copies of H indexed by cosets in G/H with a natural G-action. One can check this coincides with the above. This construction can be realized on the level of the tensor product by decomposing $\mathbb{C}G$ as a free $\mathbb{C}H$-module of rank $[G : H]$. □

Theorem 4.5 (Lehrer–Solomon)**.** *In the above situation,*

$$H^*(M) \cong A(\mathcal{A}) \cong \bigoplus_X A_X(\mathcal{A}){\uparrow}_{G_X}^{G}$$

as graded $\mathbb{C}G$-modules, where X runs over a complete set of representatives of orbits of the G-action on $L(\mathcal{A})$.

5. A refined type A decomposition

For the braid arrangement, the constituent G_X-modules $A_X(\mathcal{A})$ themselves can be described as certain induced linear characters. Lehrer–Solomon's argument is complicated but fundamentally comes from a combinatorial analysis of the Orlik–Solomon algebra's monomials and the S_n-action on the lattice of set partitions. The main result is the following.

Theorem 5.1 (Lehrer–Solomon)**.** *There is a $\mathbb{C}S_n$-module isomorphism*

$$H^p(M) \cong \bigoplus_c \xi_c{\uparrow}_{Z(c)}^{S_n}$$

for $p = 0, \ldots, n-1$ where c runs over a full set of representatives for the conjugacy classes of permutations with $n-p$ cycles and ξ_c is an explicitly defined linear character of the centralizer $Z(c)$ of c in S_n.

See Definition 4.3(ii) and (iii) for the explicit, somewhat involved definition of ξ_c.

6. A consequence

Proposition 6.1. *Let $\mathcal{A}$ be the braid arrangement in $\mathbb{C}^n$. Then $H^*(M)$ as an $\mathbb{C}S_n$-module has exactly two copies of the trivial representation.*

Proof. We compute by Frobenius reciprocity

$$\langle \xi_c{\uparrow}_{Z(c)}^{S_n}, 1\rangle_{S_n} = \langle \xi_c, 1{\downarrow}_{Z(c)}^{S_n}\rangle_{Z(c)},$$

i.e. we need to count the number of times ξ_c is the trivial representation. Using the explicit description one finds this happens exactly twice. □