INTRODUCTION TO λ-RINGS AND PLETHYSMS

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λ-Rings and Plethysms

The purpose of this talk is to define and motivate the often-mysterious notion of "plethysm" of symmetric functions.

We begin with λ-rings. Abstractly, λ rings are decategorifications of symmetric monoidal abelian categories. More concretely, let $R(G)$ be the representation ring over $\mathbb{C}$ of a finite group G, or more generally a compact Lie group, whose elements can be thought of as virtual characters. We have correspondences $+ \leftrightarrow \oplus$ and $\cdot \leftrightarrow \otimes$. Given a G-module V, we can form a new G-module $\Lambda^k V$, the kth exterior power of V. On the level of the representation ring, this corresponds to the map (of sets!)

$$\begin{aligned} \lambda^k \colon R(G) &\to R(G) \\ \lambda^k([V]) &:= [\Lambda^k V]. \end{aligned}$$

for $k \in \mathbb{Z}_{\geq 0}$.

Definition 1. A *"non-special" λ-ring* is a commutative, unital ring R together with maps of sets $\lambda^k \colon R \to R$ for $k \in \mathbb{Z}_{\geq 0}$ such that for all $x, y \in R$

(i) $\lambda^0(x) = 1$
(ii) $\lambda^1(x) = x$
(iii) $\lambda^k(x+y) = \sum_{i+j=k} \lambda^i(x)\lambda^j(y)$.

(We'll shortly remove the awkward phrase "non-special" by adding three more conditions.)

Example 2. The ring $R(G)$ satisfies (i) and (ii) trivially. Identity (iii) is a consequence of the natural isomorphism

$$\bigoplus_{i+j=k} (\Lambda^i U) \otimes (\Lambda^j V) \cong \Lambda^k(U \oplus V).$$

Example 3. When G is the trivial group, $R(G) \cong \mathbb{Z}$ via $[V] \mapsto \dim V$. Hence we have a λ-ring structure on $\mathbb{Z}$ given by

$$\lambda^k(m) = \dim \Lambda^k(\mathbb{C}^m) = \binom{m}{k}.$$

Identity (iii) corresponds to the Vandermonde convolution identity

$$\binom{x+y}{k} = \sum_{i+j=k} \binom{x}{i}\binom{y}{j}.$$

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Example 4. Let $G = \mathrm{GL}(\mathbb{C}^m)$ and let R be the corresponding ring of polynomial representations. Let Sym_m be the ring of symmetric polynomials in m variables over $\mathbb{Z}$. We have the usual ring isomorphism

$$R \cong \mathrm{Sym}_m$$
$$[V] \mapsto p_V(x_1, \ldots, x_m) \text{ where}$$
$$p_V(x_1, \ldots, x_m) := \mathrm{Tr}_V(\mathrm{diag}(x_1, \ldots, x_m)).$$

For instance, $\mathbb{C}^m \mapsto x_1 + \cdots + x_m = e_1$. More generally,

$$\Lambda^k \mathbb{C}^m \mapsto e_k(x_1, \ldots, x_m)$$

since $\Lambda^k \mathbb{C}^m$ has a basis indexed by k-element subsets of any fixed basis of $\mathbb{C}^m$. The corresponding λ-ring structure on Sym_m hence satisfies

$$\lambda^k(e_1) = e_k. \tag{1}$$

Example 5. Let $\mathrm{Sym} := \lim_{m\to\infty} \mathrm{Sym}_m$ be the usual ring of symmetric functions, which inherits maps λ^k from the Sym_m satisfying (1).

Question 6. *Is λ^k a ring homomorphism?*

- Condition (iii) shows that while λ^k is not (typically) additive, $\lambda^k(x+y)$ is determined by $\lambda^i(x)$, $\lambda^j(y)$ for $i + j = k$.
- Similarly, while λ^k is not (typically) multiplicative, when $R = \mathbb{Z}$ we may clearly write $\lambda^k(xy)$ in terms of $\lambda^1(x), \ldots, \lambda^k(x), \lambda^1(y), \ldots, \lambda^k(y)$. Similarly, $\lambda^h(\lambda^k(x))$ can be written in terms of $\lambda^1(x), \ldots, \lambda^{hk}(x)$.

Fact 7. *There are "universal", unique polynomials $P_k(x_1, \ldots, x_k, y_1, \ldots, y_k)$ and $P_{h,k}(x_1, \ldots, x_{hk})$ with integer coefficients such that in the rings $R(G)$, we have*

$$[\Lambda^k(U \oplus V)] = P_k([\Lambda^1 U], \ldots, [\Lambda^k U], [\Lambda^1 V], \ldots, [\Lambda^k V])$$

and

$$[\Lambda^h(\Lambda^k U)] = P_{h,k}([\Lambda^1 U], \ldots, [\Lambda^{hk} U]).$$

Example 8. In $R(G)$, we have

$$\lambda^2(xy) = x^2(\lambda^2 x) + (\lambda^2 x)y^2 - 2(\lambda^2 x)(\lambda^2 y), \qquad \lambda^2(\lambda^2(x)) = x(\lambda^3 x) - \lambda^4 x.$$

For instance, the first of these corresponds to the (not-at-all-obvious) isomorphism

$$\Lambda^2(U \otimes V) \oplus 2(\Lambda^2 U \otimes \Lambda^2 V) \cong (U^{\otimes 2} \otimes \Lambda^2 U) \oplus (\Lambda^2 V \otimes V^{\otimes 2}).$$

Definition 9. A *λ-ring* is a commutative, unital ring R together with maps of sets $\lambda^k \colon R \to R$ for $k \in \mathbb{Z}_{\geq 0}$ such that for all $x, y \in R$

(i) $\lambda^0(x) = 1$
(ii) $\lambda^1(x) = x$
(iii) $\lambda^k(x+y) = \sum_{i+j=k} \lambda^i(x)\lambda^j(y)$
(iv) $\lambda^k(xy) = P_k(\lambda^1(x), \ldots, \lambda^k(x), \lambda^1(y), \ldots, \lambda^k(y))$
(v) $\lambda^h(\lambda^k(x)) = P_{h,k}(\lambda^1(x), \ldots, \lambda^{hk}(x))$
(vi) $\lambda^k(1) = 0$ for $k > 1$.

Example 10. $R(G)$, Sym_m, and Sym are all λ-rings.

The following fact connects us to plethysms. Note how checking that e_1 is a λ-ring generator of Sym entails extensive use of (1) and conditions (iii)-(v).

Fact 11. Sym *is the free λ-ring on one generator.*

A homomorphism of λ-rings is a ring morphism commuting with the λ^k maps. The above fact can be rephrased as follows.

Proposition 12. *Given a λ-ring R, for every $x \in R$ there is a unique homomorphism of λ-rings $\phi_x \colon \mathrm{Sym} \to R$ such that $\phi_x(e_1) = x$.*

Definition 13. For $g \in \mathrm{Sym}$, let $\phi_g\colon \mathrm{Sym} \to \mathrm{Sym}$ be the λ-ring homomorphism defined by $\phi_g(e_1) = g$. For any $f \in \mathrm{Sym}$, the *plethysm of g and f* is

$$f[g] := \phi_g(f).$$

Equivalently, $f[g]$ is can be defined by requiring

$$\phi_{f[g]} = \phi_g \circ \phi_f. \tag{2}$$

Eq. (2) says that plethysm of symmetric functions is (up to contravariance) precisely composition of λ-ring endomorphisms of Sym. The explains the quite rare alternate notation $g \circ f$ for $f[g]$; a much more common alternate notation is $f \circ g$, as discussed below. In any case, (2) shows that plethysm is associative in that

$$f[g[h]] = (f[g])[h].$$

Since $\phi_{e_1} = \mathrm{id}$, (2) also shows that e_1 is a two-sided identity for plethysm, namely that

$$f[e_1] = e_1[f] = f.$$

Example 14. We may now use (1) to identify the operators λ^k in terms of plethysms:

$$\lambda^k(g) = \lambda^k(\phi_g(e_1)) = \phi_g(\lambda^k(e_1)) = \phi_g(e_k) = e_k[g]. \tag{3}$$

Example 15. For $k \in \mathbb{Z}_{\geq 0}$, define maps $\psi^k\colon R(G) \to R(G)$ by declaring that

$$\chi_{\psi^k[V]}(g) = \chi_{[V]}(g^k)$$

for all $g \in G$ and all G-modules V. It is straightforward to verify that ψ^k is a well-defined ring homomorphism. Likewise we have ring endomorphisms $\psi^k\colon \mathrm{Sym}_m \to \mathrm{Sym}_m$. Since

$$e_1(x_1, \ldots, x_m) = p_{\mathbb{C}^m}(x_1, \ldots, x_m) = \mathrm{Tr}_{\mathbb{C}^m}(\mathrm{diag}(x_1, \ldots, x_m)),$$

we have

$$\psi^k(e_1) = \mathrm{Tr}_{\mathbb{C}^m}(\mathrm{diag}(x_1, \ldots, x_m)^k) = p_{\mathbb{C}^m}(x_1^k, \ldots, x_m^k) = e_1(x_1^k, \ldots, x_m^k).$$

The corresponding ring endomorphisms ψ^k on Sym thus satisfy

$$\psi^k(e_1) = p_k.$$

Fact 16. *The ring endomorphisms $\psi^k\colon \mathrm{Sym} \to \mathrm{Sym}$ just defined are λ-ring endomorphisms, i.e. $\psi^k \circ \lambda^h = \lambda^h \circ \psi^k$.*

Consequently, $\psi^k = \phi_{\psi^k(e_1)} = \phi_{p_k}$. That is,

$$\psi^k(g) = g[p_k].$$

(In fact, $p_k[g] = g[p_k]$.)

Remark 17. The operators ψ^k just defined are typically called Adams operations. They arise in K-theory, which involves an analogue of the ring $R(G)$ where G-modules are replaced by vector bundles over a compact Hausdorff space. Indeed, Grothendieck originally defined λ-rings in the K-theory context.

We next give the first of two alternate definitions of plethysm.

Definition 18. Let $\beta\colon \mathrm{GL}(\mathbb{C}^p) \to \mathrm{GL}(\mathbb{C}^n)$ and $\alpha\colon \mathrm{GL}(\mathbb{C}^n) \to \mathrm{GL}(\mathbb{C}^m)$ be polynomial representations with characters

$$g(x_1, \ldots, x_p) = \mathrm{Tr}(\beta(\mathrm{diag}(x_1, \ldots, x_p))) \in \mathrm{Sym}_p$$
$$f(x_1, \ldots, x_n) = \mathrm{Tr}(\alpha(\mathrm{diag}(x_1, \ldots, x_n))) \in \mathrm{Sym}_n\,.$$

Then $\alpha\beta\colon \mathrm{GL}(\mathbb{C}^p) \to \mathrm{GL}(\mathbb{C}^m)$ is a polynomial representation with character

$$f[g] := \mathrm{Tr}(\alpha\beta(\mathrm{diag}(x_1, \ldots, x_p)) \in \mathrm{Sym}_p\,. \tag{4}$$

Remark 19. Definition 18 explains the relatively common alternate notation $f \circ g$ for $f[g]$. To connect Definition 13 and Definition 18, fix $g \in \mathrm{Sym}_p$ and consider the map $\mathrm{Sym}_n \to \mathrm{Sym}_p$ given by $f \mapsto f[g]$ using (4). This is easily seen to be a ring homomorphism, and in fact it commutes with the λ^k, so is a λ-ring homomorphism. Finish off by taking limits and computing $e_1 \mapsto g$.

Definition 20. The following is the usual combinatorial definition of plethysm. Write $g \in \mathrm{Sym}$ as $g = \sum_{u \in I} u$ where I is a multiset of monomials. Then for any $f \in \mathrm{Sym}$,

$$f[g] := f(u : u \in I).$$

Remark 21. To connect Definition 13 and Definition 20, as before, note that $f \mapsto f(u : u \in I)$ is a well-defined ring endomorphism of Sym. One must check that it commutes with the λ^k maps. Since we clearly have $e_1 \mapsto \sum_{u \in I} u = g$, the equivalence of the definitions follows.

Remark 22. Further sources:

- `https://ncatlab.org/nlab/show/Lambda-ring` for abstract overview of λ-rings
- `http://www-math.mit.edu/~rstan/transparencies/plethysm.pdf` for some slides from Richard Stanley discussing Definition 18 and Definition 20 above
- `http://www.math.jhu.edu/~jmb/note/adamrept.pdf` for a lovely little note on Adams operations for representations of compact Lie groups
- `http://www.cip.ifi.lmu.de/~grinberg/algebra/lambda.pdf` for many gory details on λ-rings, e.g. constructing P_k and $P_{h,k}$; highly unpolished and incomplete
- [Ful97][Part II] for a nice summary of representation theory over $\mathrm{GL}(\mathbb{C}^n)$
- [Mac95][I.8 and I.A] for Macdonald's concise account of plethysm and its connection to symmetric group representation theory
- [Knu73] for a book on λ rings and symmetric functions; somewhat old, apparently not the best regarded, I didn't actually take the time to acquire a copy; written by Donald Knutson, apparently father of Allen Knutson
- [Yau10] for a more modern book on λ rings, though I didn't actually take the time to acquire a copy

References

[Ful97] William Fulton. *Young tableaux*, volume 35 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1997. With applications to representation theory and geometry.

[Knu73] Donald Knutson. *λ-rings and the representation theory of the symmetric group*. Lecture Notes in Mathematics, Vol. 308. Springer-Verlag, Berlin-New York, 1973.

[Mac95] I. G. Macdonald. *Symmetric functions and Hall polynomials*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, second edition, 1995. With contributions by A. Zelevinsky, Oxford Science Publications.

[Yau10] Donald Yau. *Lambda-rings*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2010.