Tableaux posets and the fake degrees of coinvariant algebras

AMS Western Sectional Meeting at SFSU

Josh Swanson
University of California, San Diego

based on joint work with
Sara Billey and Matjaž Konvalinka

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slides: `http://www.math.ucsd.edu/~jswanson/`

October 27th, 2018

Outline

- Complex reflection groups and coinvariant algebras

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- Fake degrees and internal zeros

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- Type A: rotation and block rules

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- $G(m, 1, n)$ generalization

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- Type A: rotation and block rules
- $G(m, 1, n)$ generalization
- $G(m, d, n)$ further generalization

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- **(Type A)** Symmetric groups
- **(Type B)** Hyperoctahedral groups (signed permutations; symmetries of the hypercube $\mathrm{conv}\{\pm e_i\}$)
- Cyclic groups

Complex Reflection Groups

Definition Given $m, n \in \mathbb{Z}_{\geq 1}$, let $G(m, 1, n)$ be the group of $n \times n$ *pseudo-permutation matrices* whose non-zero entries are from $C_m \coloneqq \{\zeta \in \mathbb{C} : \zeta^m = 1\}$.

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Definition Given $d \mid m$, let $G(m,1,n) \twoheadrightarrow C_m$ be given by multiplying the non-zero elements, let $\phi\colon G(m,1,n) \twoheadrightarrow C_m \twoheadrightarrow C_d$, and set $G(m,d,n) \coloneqq \ker \phi$.

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Theorem (Shephard–Todd '53) *Up to isomorphism, the complex reflection groups are precisely the direct products of the groups $G(m, d, n)$ along with 34 exceptional groups.*

Coinvariant Algebras

Definition

Given $G \leq \mathrm{GL}(V)$, the *coinvariant algebra* of G is

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- **(Type B)** When $G = G(2, 1, n)$,

$$R_G = R_{2,1,n} = \mathbb{C}[x_1, \ldots, x_n]/(e_i(x_1^2, \ldots, x_n^2) : 1 \leq i \leq n).$$

Fake Degrees

Theorem (Chevalley '55) *R_G as an ungraded module is isomorphic to the regular representation of the complex reflection group G.*

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Question What is the graded irreducible decomposition of R_G for $G = G(m,d,n)$? **Equivalently, what are the $f^S(q)$'s?**

Partitions

Definition

A *partition* λ of n is a sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots$ such that $\sum_i \lambda_i = n$.

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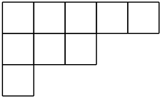
$$\lambda = (5, 3, 1) \leftrightarrow$$



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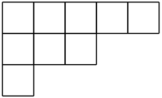
$\lambda = (5, 3, 1) \leftrightarrow$ 

Theorem (Young, early 1900's) *The complex inequivalent irreducible representations S^λ of S_n are* **canonically** *indexed by partitions of n.*

Remark By contrast, the irreps of C_m are most naturally indexed by $\mathbb{Z}/m$ only up to $\phi(m)$ additive automorphisms.

Standard Tableaux

Definition

A *standard Young tableau* (*SYT*) of shape $\lambda \vdash n$ is a filling of the cells of the Ferrers diagram of λ with $1, 2, \ldots, n$ which **increases along rows** and **decreases down columns**.

$$T = \begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 6 & 7 & 9 \\ \hline 2 & 5 & 8 \\ \cline{1-3} 4 \\ \cline{1-1}\end{array} \in \mathrm{SYT}(\lambda)$$

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Descent set: $\{1, 3, 7\}$.

Definition The *descent set* of $T \in \mathrm{SYT}(\lambda)$ is the set

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Definition The *major index* of $T \in \mathrm{SYT}(\lambda)$ is the sum of the descents.

Type A Fake Degrees

Theorem (Lusztig–Stanley '70's) *The type A fake degrees are*

$$f^{S^\lambda}(q) = f^\lambda(q) = \sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(T)}.$$

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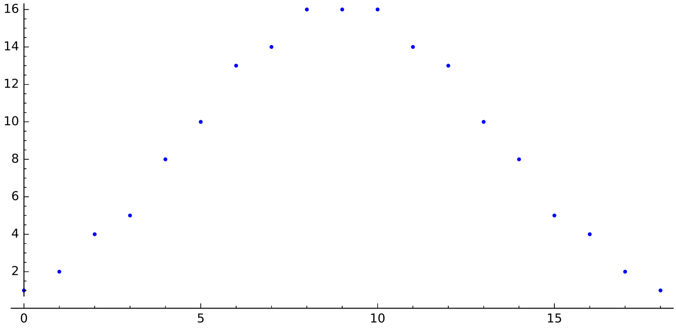
Example $f^{(5,3,1)}(q) = q^5(q^{18} + 2q^{17} + 4q^{16} + 5q^{15} + 8q^{14} + 10q^{13} + 13q^{12} + 14q^{11} + 16q^{10} + 16q^9 + 16q^8 + 14q^7 + 13q^6 + 10q^5 + 8q^4 + 5q^3 + 4q^2 + 2q + 1)$.

Type A Fake Degrees

Example

Visualizing the coefficients of $q^{-5} f^{(5,3,1)}(q)$:

$$(1, 2, 4, 5, 8, 10, 13, 14, 16, 16, 16, 14, 13, 10, 8, 5, 4, 2, 1)$$



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- Are the fake degree coefficients log-concave?
- Are they unimodal?
- **When are they zero?** (Adin–Elizalde–Roichman recently and independently asked this question about the number of descents rather than maj.)

Type A Internal Zeros Classification

Lemma (BKS 18+) *The $b(\lambda)+1$ coefficient of $f^{\lambda}(q)$ is zero if and only if λ is a rectangle (not a row or column).*

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Theorem (BKS 18+) *The fake degree* $f^{\lambda}(q)$ *has internal zeros if and only if* λ *is a rectangle (not a row or column).*

Corollary (Best Primality Test!) $n > 1$ *is prime if and only if* $f^{\lambda}(q)$ *has no internal zeros for any* $\lambda \vdash n$.

Proof Strategy

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- Iterate ϕ starting at $\mathrm{minmaj}(\lambda)$, ending at $\mathrm{maxmaj}(\lambda)$.

Rotations

Definition

A *positive rotation* for $T \in \mathrm{SYT}(\lambda)$ is an interval $[i, k] \subset [n]$ such that if $T' \coloneqq (i, i+1, \ldots, k-1, k) \cdot T$, then $T' \in \mathrm{SYT}(\lambda)$ and there is some j for which

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Key Fact

Applying rotations increases maj by 1!

Rotations

Definition

$T' := (i, i+1, \ldots, k-1, k) \cdot T$ or
$T' := (k, k-1, \ldots, i+1, i) \cdot T$, $T' \in \mathrm{SYT}(\lambda)$, $\exists j$ s.t. the descent at $j-1$ in T turned into a descent at j in T'.

Example

1	**4**
2	**5**
3	9
6	
7	
8	

$\xrightarrow[(543)]{}$

1	**3**
2	**4**
5	9
6	
7	
8	

$$\mathrm{Des}(T) = \{1, \underline{2}, 4, 5, 6, 7\}$$
$$\longrightarrow \mathrm{Des}(T') = \{1, \underline{3}, 4, 5, 6, 7\}$$

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Lemma *Every (non-exceptional) tableau which avoids the pattern*

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admits a rotation.

- Rotations preserve the number of descents, but $\mathrm{minmaj}(\lambda)$ and $\mathrm{maxmaj}(\lambda)$ typically have different numbers of descents.

Example

Block Rules

- We have 5 additional "block rules" which add a descent while incrementing maj by 1.

B2:

1	2	3	4	5
6	7	8	9	10
11	12	13	~~14~~	

$\circlearrowright$

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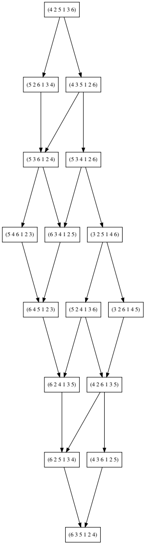
Definition The *strong SYT poset* $P(\lambda)$ on $\mathrm{SYT}(\lambda)$ is obtained by defining the cover relations to be rotations, block rules, and their inverse-transposes.

Corollary *If λ is not a rectangle, $P(\lambda)$ is ranked (up to a shift) by* maj *and has unique minimal and maximal elements.*
Indeed, $P(\lambda)$ is ranked by $(\mathrm{des}, \mathrm{maj} - \mathrm{des})$ in the sense that rotation rules increase this by $(0, 1)$ and block rules increase this by $(1, 0)$.

Strong Poset

Example

For $SYT(3, 2, 1)$:



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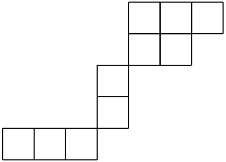
- maj $-\mathrm{des}$ internal zeros classification for free

$G(m, 1, n)$ Fake Degrees

Theorem (Specht, '35) *The irreps of $G(m, 1, n)$ are indexed (more-or-less canonically) by block diagonal skew partitions $\underline{\lambda}$ with m blocks and n total cells.*

Example $n = 10, m = 3$:

$$\underline{\lambda} = ((3, 2), (1, 1), (3)) =$$



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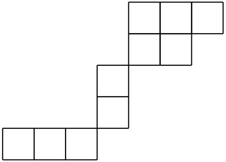

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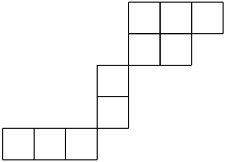

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Theorem (Stembridge '89) *For $\underline{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(m)}) \vdash n$,*

$$f^{S^{\underline{\lambda}}(q)=}f^{\underline{\lambda}}(q) = q^{b(\alpha(\underline{\lambda}))}\binom{n}{\alpha(\underline{\lambda})}_{q^m}\prod_{i=1}^{m} f^{\lambda^{(i)}}(q^m).$$

$G(m,1,n)$ Internal Zeros

Theorem (BKS 18+) *Let $\underline{\lambda}$ be a sequence of m partitions with $|\underline{\lambda}| = n$, and assume $f^{\underline{\lambda}}(q) = \sum_k b_{\underline{\lambda},k} q^k$. Then for $k \in \mathbb{Z}$, $b_{\underline{\lambda},k} \neq 0$ if and only if*

$$\frac{k - b(\alpha(\underline{\lambda}))}{m} - b(\underline{\lambda}) \in \left\{0, 1, \ldots, \binom{n+1}{2} - \sum_{c \in \underline{\lambda}} h_c \right\} \smallsetminus \mathcal{D}_{\underline{\lambda}},$$

where $\mathcal{D}_{\underline{\lambda}}$ is empty unless $\underline{\lambda}$ has a single non-empty partition $\lambda^{(i)}$ which is a rectangle with at least two rows and columns, in which case

$$\mathcal{D}_{\underline{\lambda}} = \left\{1, \binom{n+1}{2} - \sum_{c \in \lambda^{(i)}} h_c - 1 \right\}.$$

$G(m, d, n)$ Fake Degrees

Theorem

(Clifford Theory) *The irreps of $G(m, d, n)$ are (more-or-less canonically) indexed by pairs $(\{\underline{\lambda}\}^d, c)$ where $\underline{\lambda}$ has m parts and n cells, $\{\underline{\lambda}\}^d$ is its orbit under the size d group of cyclic rotations, and c is an element of the stabilizer of this orbit.*

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Example **(Type D)** For $G(2,2,n)$, one can index by sets $\{\lambda, \mu\}$ with $|\lambda| + |\mu| = n$, at least when $\lambda \neq \mu$.

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(In fact, $f^{\{\underline{\lambda}\}^d, c}(q)$ does not depend on c.)

$G(m, d, n)$ Fake Degrees

Theorem (Clifford Theory) *The irreps of $G(m, d, n)$ are (more-or-less canonically) indexed by pairs $(\{\underline{\lambda}\}^d, c)$ where $\underline{\lambda}$ has m parts and n cells, $\{\underline{\lambda}\}^d$ is its orbit under the size d group of cyclic rotations, and c is an element of the stabilizer of this orbit.*

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Theorem (Stembridge '89, BKS 18+)

$$f^{S^{\{\underline{\lambda}\},c}}(q) = f^{\{\underline{\lambda}\}^d}(q)$$
$$= \frac{\#\{\underline{\lambda}\}^d}{d} \cdot \begin{bmatrix} n \\ \alpha(\underline{\lambda}) \end{bmatrix}_{q;d} \cdot \prod_{i=1}^{m} f^{\lambda^{(i)}}(q^m).$$

$G(m, d, n)$ Internal Zeros

Theorem (BKS 18+) *Let $\underline{\lambda}$ be a sequence of m partitions with $|\underline{\lambda}| = n \geq 1$, let $d \mid m$, and let $\{\underline{\lambda}\}^d$ be the orbit of $\underline{\lambda}$ under the group C_d of (m/d)-fold cyclic rotations. Then $b_{\{\underline{\lambda}\}^d,k} \neq 0$ if and only if for some $\underline{\mu} \in \{\underline{\lambda}\}$ we have $|\mu^{(1)}| + \cdots + |\mu^{(m/d)}| > 0$ and*

$$\frac{k - b(\alpha(\underline{\mu}))}{m} - b(\underline{\mu}) \in \{0, 1, \ldots, |\mu^{(1)}| + \cdots + |\mu^{(m/d)}| + \binom{n}{2} - \sum_{c \in \underline{\mu}} h_c\} \setminus \mathcal{D}_{\underline{\mu};d}.$$

$G(m, d, n)$ Internal Zeros

Theorem (Continued.) *Here $\mathcal{D}_{\underline{\mu};d}$ is empty unless either*

1. *$\underline{\mu}$ has a partition μ of size n; or*
2. *$\underline{\mu}$ has a partition μ of size $n-1$ and $|\mu^{(1)}| + \cdots + |\mu^{(m/d)}| = 1$,*

where in both cases μ must be a rectangle with at least two rows and columns. In case (1), we have

$$\mathcal{D}_{\underline{\mu};d} \coloneqq \left\{1, \binom{n+1}{2} - \sum_{c \in \mu} h_c - 1\right\},$$

and in case (2) we have

$$\mathcal{D}_{\underline{\mu};d} \coloneqq \left\{1, \binom{n}{2} - \sum_{c \in \mu} h_c\right\}.$$

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- Conceptual explanation for primality corollary/why are rectangles special?

Thanks!

THANKS!