Major Index Asymptotics

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based partly on joint work with
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Global conjugacy classes

Question (Sundaram)

Fix an S_n-conjugacy class μ. Let S_n act by conjugation on μ $\mathbb{C}$-linearly. For which μ does every S_n-irreducible appear in this representation?

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- When $\mu = (n)$, this representation is $1\uparrow_{C_n}^{S_n}$ where $C_n := \langle(\sigma_n)\rangle$ with $\sigma_n := (1\ 2\ \cdots\ n)$

Combinatorial representation theory

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Theorem (Kraskiewicz–Weyman)

Let $\lambda \vdash n$. Then

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- Related to work of Thrall, Klyachko, Stembridge, Lusztig, Stanley, ...

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has $\mathrm{Des}(T) = \{1,2,4,7\}$ and $\mathrm{maj}(T) = 1+2+4+7 = 14$

Sundaram's conjecture, Klyachko's theorem

Restatement of earlier conjecture:

Conjecture (Sundaram)

Let $\lambda \vdash n > 1$. *Then* $a_{\lambda,0} = 0$ *if and only if*

- $\lambda = (n-1, 1)$, *or*
- $\lambda = (2, 1^{n-2})$ *if* n *is odd, or*
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Related earlier work:

Theorem (Klyachko)

Let $\lambda \vdash n > 1$*. Then* $a_{\lambda,1} = 0$ *if and only if*

- $\lambda = (2, 2)$*, or* $\lambda = (2, 2, 2)$*, or*
- $\lambda = (n)$*, or*
- $\lambda = (1^n)$ *when* $n > 2$

Estimating $a_{\lambda,r}$

Theorem (S.)

For all $\lambda \vdash n \geq 1$ *and all* r*,*

$$\left| \frac{a_{\lambda,r}}{f^\lambda} - \frac{1}{n} \right| \leq \frac{2n^{3/2}}{\sqrt{f^\lambda}}$$

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Let $\lambda \vdash n \geq 81$ *with* $\lambda_1, \lambda'_1 < n - 7$*. Then* $f^\lambda \geq n^5$ *and*

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- Proof uses "opposite hook products" arising independently in recent work of Morales–Pak–Panova

Results for $a_{\lambda,r}$

Corollary (S.)

Let $\lambda \vdash n > 1$. Then $a_{\lambda,r} = 0$ if and only if

- $\lambda = (2,2)$, $r = 1,3$; *or* $\lambda = (2,2,2)$, $r = 1,5$; *or or* $\lambda = (3,3)$, $r = 2,4$; *or*
- $\lambda = (n-1,1)$ *and* $r = 0$; *or*
- $\lambda = (2,1^{n-2})$, $r = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{n}{2} & \text{if } n \text{ is even; or} \end{cases}$
- $\lambda = (n)$, $r \in \{1,\ldots,n-1\}$; *or*
- $\lambda = (1^n)$, $r \in \begin{cases} \{1,\ldots,n-1\} & \text{if } n \text{ is odd} \\ \{0,\ldots,n-1\} - \{\frac{n}{2}\} & \text{if } n \text{ is even.} \end{cases}$

Results for $a_{\lambda,r}$

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Sundaram's conjecture is true!

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Question

Can we remove "mod n"?

Asymptotic normality: definition

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Definition

$X_1, X_2, \ldots$ is *asymptotically normal* if for all $t \in \mathbb{R}$,

$$\lim_{N \to \infty} F_N(t) = F(t)$$

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- If X has a density function $f(t)$, the *characteristic function* $\mathbb{E}[e^{itX}]$ of X is the Fourier transform of $f(t)$

Asymptotic normality: examples

The "original" asymptotic normality result:

Theorem (de Moivre, Laplace)

Let X_N be the "cardinality" statistic on subsets of $[N]$. Then $X_1, X_2, \ldots$ is asymptotically normal.

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Let X_N be the “cardinality” statistic on subsets of $[N]$. Then $X_1, X_2, \ldots$ is asymptotically normal.

More generally:

Theorem (Central limit theorem)

Let X_N be the average of N i.i.d. random variables with finite variance. Then $X_1, X_2, \ldots$ is asymptotically normal.

Asymptotic normality: criteria

We can use characteristic functions:

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Theorem (Lévy's continuity theorem)

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There's a classic, straightforward proof of the CLT using characteristic functions.

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Theorem (Frechét–Shohat theorem)

A sequence $X_1, X_2, \ldots$ of random variables (with density functions that decay at least exponentially in the tails) is asymptotically normal if and only if for all $d \in \mathbb{Z}_{\geq 1}$ we have

$$\lim_{N \to \infty} \mathbb{E}[(X_N^*)^d] = \begin{cases} 0 & \text{if } d \text{ is odd} \\ (d-1)!! & \text{if } d \text{ is even} \end{cases}$$

Asymptotic normality and standard tableaux

Definition

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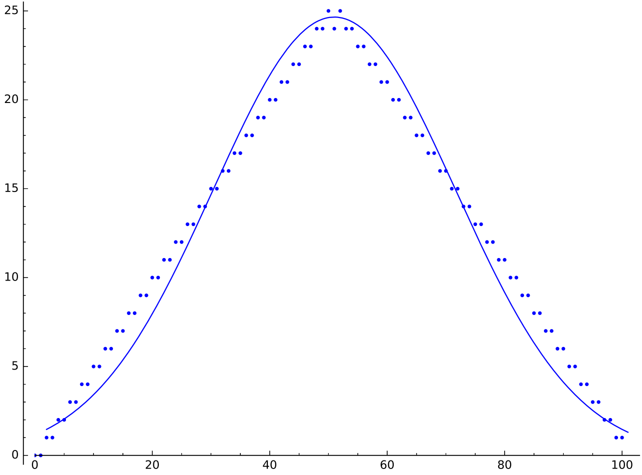
Theorem (Billey–Konvalinka–S.)

Suppose $\lambda^{(1)}, \lambda^{(2)}, \ldots$ *is a sequence of partitions. Let* X_N *be the random variable corresponding to the major index statistic on* $\mathrm{SYT}(\lambda^{(N)})$. *Then, the sequence* $X_1, X_2, \ldots$ *is asymptotically normal if and only if* $\mathrm{aft}(\lambda^{(N)}) \to \infty$ *as* $N \to \infty$.

Asymptotic normality and standard tableaux

Example

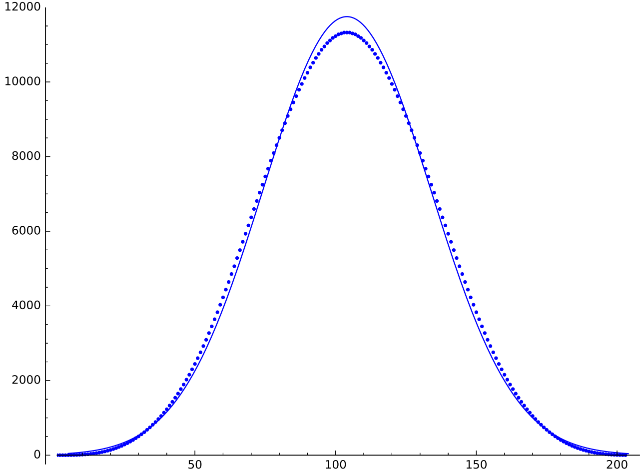
$\lambda^{(1)} = (50, 2)$, $\mathrm{aft}(\lambda^{(1)}) = 2$



Asymptotic normality and standard tableaux

Example

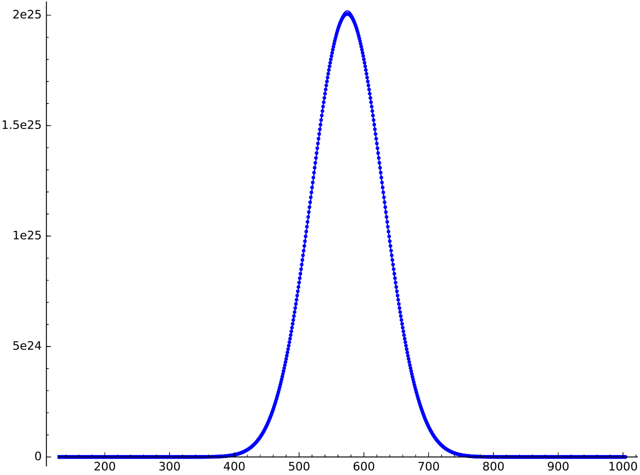
$\lambda^{(2)} = (50, 3, 1)$, aft$(\lambda^{(2)}) = 4$



Asymptotic normality and standard tableaux

Example

$\lambda^{(3)} = (8, 8, 7, 6, 5, 5, 5, 2, 2)$, $\mathrm{aft}(\lambda^{(3)}) = 39$



Asymptotic normality and standard tableaux

Corollary (Chen–Wang–Wang)

Using $\lambda^{(N)} = (N, N)$, the coefficients of the q-Catalan numbers $\frac{1}{[N+1]_q}\binom{2N}{N}_q$ are asymptotically normal.

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- Proof of theorem uses cumulants, Stanley's formula for $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q)$, hook length estimates, method of moments
- $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q)$ connects to principal specializations of s_λ; type A coinvariant algebra and Lusztig–Stanley theorem; $\mathrm{GL}_n(\mathbb{F}_q)$-representation theory by work of Green, Steinberg

Further work

- diag($\underline{\lambda}$) case settled; generalizes earlier work of Canfield–Janson–Zeilberger, Diaconis, Mann–Whitney, . . .

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- General skew shapes?

Thanks!

$\mathcal{FIN}.$

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