IRREDUCIBLE DECOMPOSITIONS

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Contents

(This lecture was given in the Applied Algebraic Geometry topics course at the University of Washington on May 1st, 2017. It essentially follows [CLO15, §4.6].)

1. Irreducible Decompositions

Example 1.1. We have $V(xy) = V(x) \cup V(y)$. Can we break $V(x)$ down further, usefully? (Draw.) We give a formal generalization of this example today.

Remark 1.2. Recall the following version of Hilbert's Basis Theorem. Every ascending chain of ideals

$$I_1 \subset I_2 \subset \cdots \subset k[x_1, \ldots, x_n]$$

eventually stabilizes, i.e. $\exists N$ such that $I_N = I_{N+1} = \cdots$.

Corollary 1.3. *Any descending chain of affine varieties*

$$V_1 \supset V_2 \supset \cdots \supset k^n$$

eventually stabilizes, i.e. $\exists N$ such that $V_N = V_{N+1} = \cdots$.

Proof. Apply $\mathcal{I}$, then $\mathcal{V}$, noting $\mathcal{V}(\mathcal{I}(V_i)) = \overline{V_i} = V_i$. □

Theorem 1.4. *Let $V \subset k^n$ be an affine variety. Then V can be written as a finite union*

$$V = V_1 \cup \cdots \cup V_m$$

where each V_i is irreducible.

Proof. If V is irreducible, then we're done, so suppose V is reducible. Write $V = V_1 \cup V_1'$ where V_1, V_1' are varieties such that $V_1 \neq V \neq V_1'$. If V_1, V_1' are each irreducible, then we're done, so suppose V_1 is reducible. Hence write $V_1 = V_2 \cup V_2'$ with $V_2 \neq V_1 \neq V_2'$. Repeating this, we either terminate eventually or we get an infinite chain

$$V \supsetneq V_1 \supsetneq V_2 \supsetneq \cdots,$$

a contradiction. □

Example 1.5. We have $V(xy) = V(x) \cup V(y) \cup V(x, y)$, but $V(x, y)$ is "unnecessary."

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Theorem 1.6. *Let $V \subset k^n$ be an affine variety. Then V has a minimal decomposition*

$$V = V_1 \cup \cdots \cup V_m$$

where each V_i is an irreducible variety with $V_i \subsetneq V_j$ for all $I \neq j$. Up to reordering, this decomposition is unique.

Proof. Existence follows from the previous theorem after removing parts contained in others. For uniqueness, suppose we have another minimal decomposition

$$V = V_1' \cup \cdots \cup V_l'.$$

Fix i and note

$$V_i = V_i \cap V = V \cap (V_1' \cup \cdots \cup V_l') = (V \cap V_1') \cup \cdots \cup (V \cap V_l').$$

Since V_i is irreducible, $V_i = V_i \cap V_j'$ for some j, so $V_i \subset V_j'$. By the symmetry of this argument, we have some k such that $V_j' \subset V_k$. Hence $V_i \subset V_k$, so $i = k$ and $V_i = V_j'$. This procedure results in a bijection

$$\{V_1, \ldots, V_m\} \leftrightarrow \{V_1', \ldots, V_l'\}.$$

In particular, $l = m$ and the result follows. □

Corollary 1.7. *If $k = \overline{k}$, every $I = \sqrt{I}$ has a minimal decomposition*

$$I = P_1 \cap \cdots \cap P_r$$

for P_i prime with $P_i \subsetneq P_j$ for all $i \neq j$.

Proof. Apply the $\mathcal{I}$ and $\mathcal{V}$ bijections. □

Remark 1.8. In fact, the corollary holds even if $k \neq \overline{k}$.

Theorem 1.9. *If we have a minimal decomposition*

$$I = \sqrt{I} = P_1 \cap \cdots \cap P_r \subset k[x_1, \ldots, x_n]$$

with P_i prime, $P_i \subsetneq P_j$ for all $i \neq j$. Then

$$\{P_1, \ldots, P_r\} = \{I : f \text{ proper, prime } | f \in k[x_1, \ldots, x_n]\}.$$

Proof. First, some ingredients.

Fact	Intuition
$(\cap_{i=1}^n I_i) : J = \cap_{i=1}^n (I_i : J)$	$(\cup_i V_i) - V = \cup_i (V_i - V)$
$f \in P \Rightarrow P : f = \langle 1 \rangle$	$W := V(f) \supset V(P)$ implies $V(P) - V(f) = \varnothing$
$f \notin P \Rightarrow P : f = P$	$W := V(f) \not\supset V(P)$ implies $\overline{V(P) - W} = V(P)$
If $P = \cap_{i=1}^n I_i$, then $P = I_i$ for some i	$V = V_1 \cup \cdots \cup V_n$ implies $V = V_i$ for some i

Now, for $(\supset)$, suppose $I : f$ is proper and prime. First by (1),

$$I : f = (\cap_i P_i) : f = \cap_i (P_i : f).$$

By (4) since $I : f$ is prime, $I : f = P_i : f$ for some i. By (2) and (3) since $I : f$ is proper, $P_i : f = P_i$. Hence $I : f = P_i : f = P_i$.

For $(\supset)$, fix i and pick $f \in (\cap_{j \neq i} P_j) - P_i$, which is $\neq \varnothing$ by minimality. Hence by (3), $P_i : f = P_i$, and by (2), $P_j : f = \langle 1 \rangle$ for all $j \neq i$. Thus $I : f = P_i$. □

Example 1.10. One may verify the following irreducible decomposition.

$$\begin{aligned} I &= \langle xz - y^2, x^3 - yz \rangle \\ &= \langle x, y \rangle \cap \langle xz - y^2, x^3 - yz, x^2 y - z^2 \rangle \\ &= (I : x^2 y - z^2) \cap (I : x). \end{aligned}$$

Remark 1.11. There are algorithms for

- deciding if an ideal is prime, or if an affine variety is irreducible;
- finding the irreducible decomposition of a variety or radical ideal.

They're not discussed in [CLO15]; see the references at the end of [CLO15, §4.6]

References

[CLO15] David A. Cox, John Little, and Donal O'Shea. *Ideals, varieties, and algorithms*. Undergraduate Texts in Mathematics. Springer, Cham, fourth edition, 2015. An introduction to computational algebraic geometry and commutative algebra.