ON THE EXISTENCE OF TABLEAUX WITH GIVEN MODULAR MAJOR INDEX

JOSH SWANSON

Contents

(This talk was given at the University of San Diego on February 27th, 2017. arXiv: 1701.04963)

1. Statement of the Main Result

Definition 1.1. The *descent set* of a standard tableaux $T \in \mathrm{SYT}(\lambda)$, $\lambda \vdash n$ is the set

$$\mathrm{Des}\, T := \{i \in [n-1] : i+1 \text{ is in a lower row than } i\}$$

using English notation.

Example 1.2. We have:

- $\mathrm{Des}(1/2/3/4/5) = \{1, 2, 3, 4\}$;
- $\mathrm{Des}(125/346) = \{2, 5\}$.

Definition 1.3. The *major index* of T is

$$\mathrm{maj}\, T := \sum_{i \in \mathrm{Des}(T)} i.$$

Question 1.4. *Let* $a_{\lambda,r} := \#\{T \in \mathrm{SYT}(\lambda) : \mathrm{maj}\, T \equiv_n r\}$. *When is* $a_{\lambda,r} \neq 0$? *(Is there a good asymptotic for* $a_{\lambda,r}$*?)*

Theorem 1.5 (Klyachko74, Kraskiewicz-Weyman01)**.** *Pick* $\lambda \vdash n \geq 1$. *Then* $a_{\lambda,1} = 0$ *if and only if*

- $\lambda = (2, 2)$, *or* $\lambda = (2, 2, 2)$, *or*
- $\lambda = (n)$ *with* $n > 1$, *or* $\lambda = (1^n)$ *with* $n > 2$.

Date: February 28, 2017.

Remark 1.6. Sundaram '16 conjectured an answer for the $r = 0$ case. She was interested in the following question: let S_n act on permutations of cycle type λ by conjugation. For which λ does the resulting module contain all irreducible representations (λ is a *global class*)?

She proved a sufficient condition:

- λ is a global class if $n > 8$, λ has at least 2 parts, all parts are distinct, and all parts other than 1 are odd primes.

She conjectured a complete classification:

- (Say $n > 8$.) λ is a global class if and only if λ has at least 2 parts, all odd and distinct.

She was able to prove the classification subject to the determination of when $a_{\lambda,0} = 0$.

Theorem 1.7 (Swanson, 2017). *Pick $\lambda \vdash n \geq 1$. Then $a_{\lambda,r} = 0$ if and only if*

- $\lambda = (2,2)$, $r = 1,3$; *or* $\lambda = (2,2,2)$, $r = 1,5$; *or* $\lambda = (3,3)$, $r = 2,4$;
- $\lambda = (n-1,1)$ *and* $r = 0$;
- $\lambda = (2,1^{n-2})$, $r = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{n}{2} & \text{if } n \text{ is even;} \end{cases}$
- $\lambda = (n)$, $r \in \{1,\ldots,n-1\}$;
- $\lambda = (1^n)$, $r \in \begin{cases} \{1,\ldots,n-1\} & \text{if } n \text{ is odd} \\ \{0,\ldots,n-1\} - \{\frac{n}{2}\} & \text{if } n \text{ is even.} \end{cases}$

Remark 1.8. Marianne Johnson (2007) gave an alternate proof of Klyachko's theorem by constructing appropriate tableaux, though she used Kraskiewicz-Weyman's observation about $\gcd(n,r)$, and I don't find the argument sheds light on why the result should be true. Kov'acs-Stöhr (2006) gave a different proof of Klyachko's theorem using free Lie algebras and the Littlewood-Richardson rule. Their argument gives the statement $a_{\lambda,1} \neq 0 \Rightarrow a_{\lambda,1} \geq \frac{n}{6} - 1$.

The argument I'll sketch gives a more conceptual explanation using S_n-representation theory valid for all r while also giving a vastly stronger estimate.

2. Representation Theory Connection

Notation 2.1. $\sigma_n := (1\,2\,\cdots\,n) \in S_n$, $C_n := \langle \sigma_n \rangle$, $\lambda \vdash n$, S^λ is a Specht module. Further,

$$\chi^r : C_n \to \mathbb{C}^\times$$
$$\sigma^i \mapsto \omega_n^{ri}$$

where ω_n is any fixed primitive nth root of unity.

Theorem 2.2 (KW-01). *We have*

$$\textit{multiplicity of } S^\lambda \textit{ in } \chi^r\uparrow_{C_n}^{S_n} = a_{\lambda,r} = \textit{multiplicity of } \chi^r \textit{ in } S^\lambda\downarrow_{C_n}^{S_n}.$$

Consequently, $a_{\lambda,r}$ depends only on λ and $\gcd(n,r)$.

Remark 2.3. Thus "almost all" irreps appear in each $\chi^r\uparrow_{C_n}^{S_n}$ or $S^\lambda\downarrow_{C_n}^{S_n}$. Klyachko '74 was actually motivated by the study of the nth homogeneous piece of a free Lie algebra. (Adriano wrote a nice long article on this topic in 1990.) Klyachko showed that such a module is Schur-Weyl dual to $\chi^1\uparrow_{C_n}^{S_n}$. His proof found faithful representations of C_n in $S^\lambda\downarrow_{C_n}^{S_n}$ and does not clearly generalize to cover Sundaram's $r = 0$ case.

We begin by crystallizing a simple observation that has appeared in numerous guises in the literature.

Theorem 2.4 (Foulkes '72). *Pick $\lambda \vdash n \geq 1$, $r \in \mathbb{Z}/n$. Then*

$$\mathrm{ch}\,\chi^r\uparrow_{C_n}^{S_n} = \frac{1}{n}\sum_{\ell | n} c_\ell(r) p_{(\ell^{n/\ell})}$$

where

$$\begin{aligned} c_\ell(r) &= \text{a Ramanujan sum} \\ &:= \text{sum of } r\text{th powers of primitive } \ell\text{th roots of unity} \\ &= \mu(\ell/(\ell,r))\phi(\ell)/\phi(\ell/(\ell,r)). \end{aligned}$$

Example 2.5. $c_4(2) = i^2 + (-i)^2 = -2$.

Example 2.6. We have

- $r = 1$: $\mathrm{ch}\,\chi^1\uparrow_{C_n}^{S_n} = \frac{1}{n}\sum_{\ell \vdash n} \mu(\ell) p_{(\ell^{n/\ell})}$
- $r = 0$: $\mathrm{ch}\,\chi^0\uparrow_{C_n}^{S_n} = \frac{1}{n}\sum_{\ell \vdash n} \phi(\ell) p_{(\ell^{n/\ell})}$.

Theorem 2.7 (Desarmenien '90). *Write χ^λ for the character of S^λ, $f^\lambda := \chi^\lambda(1)$. Pick $\lambda \vdash n \geq 1$, $r \in \mathbb{Z}/n$. Then*

$$\frac{a_{\lambda,r}}{f^\lambda} = \frac{1}{n} + \frac{1}{n}\sum_{\substack{\ell | n \\ \ell \neq 1}} \frac{\chi^\lambda(\ell^{n/\ell})}{f^\lambda} c_\ell(r).$$

More generally, we have the following simple result, which is essentially implicit in, for instance, EC2, p. 351.

Proposition 2.8. *Let $H \leq S_n$ and let M be an H-module with character χ^H. Then*

$$\mathrm{ch}\,M\uparrow_H^{S_n} = \frac{1}{|H|}\sum_{\mu \vdash n} c_\mu p_\mu$$

and, for all $\lambda \vdash n$,

$$\text{multiplicity of } S^\lambda \text{ in } M\uparrow_H^{S_n} = \frac{1}{|H|}\sum_{\mu \vdash n} c_\mu \chi^\lambda(\mu)$$

where

$$c_\mu := \sum_{\substack{h \in H \\ \text{cycle type of } h \text{ is } \mu}} \chi^M(h).$$

Remark 2.9. Letting $M = \chi^r$ gives Foulkes and Desarmenien's results immediately. More generally, one could for instance replace $(1\,2\,\cdots\,n)$ with $(1\,2\,\cdots\,k)$, or even with other cycle types. Analogues of my methods should generalize to these cases, so long as the order of the cyclic group is not large compared to n, and is a potential avenue for further research.

3. Proof Outline

Remark 3.1. The idea is simple: use Desarmenien's formula and show $\frac{\chi^\lambda(\ell^{n/\ell})}{f^\lambda}$ is "small."

Theorem 3.2 (Fomin-Lulov '95). *Pick $\lambda \vdash n = \ell s$. Then*

$$|\chi^\lambda(\ell^s)| \leq \frac{s!\ell^s}{(n!)^{1/\ell}}(f^\lambda)^{1/\ell}.$$

Example 3.3. When $\ell = 1$, $s = n$, and the LHS and RHS are each f^λ.

Their inequality is based on the following generalization of the hook length formula.

Theorem 3.4 (Fomin-Lulov '95). *Pick $\lambda \vdash n = \ell s$. Then*

$$|\chi^\lambda(\ell^s)| = \frac{\prod_{\substack{i \in [n] \\ i \equiv_\ell 0}} i}{\prod_{\substack{c \in \lambda \\ h_c \equiv_\ell 0}} h_c}$$

if λ can be written as s successive ribbons each of length ℓ, and 0 otherwise.

Example 3.5. When $\ell = 1$, we recover the hook length formula.

Remark 3.6. Consequently,

$$\frac{|\chi^\lambda(\ell^s)|}{f^\lambda} = \frac{\prod_{\substack{i \in [n] \\ i \not\equiv_\ell 0}} i}{\prod_{\substack{c \in \lambda \\ h_c \not\equiv_\ell 0}} h_c}$$

or 0.

I have an alternate proof of this formula in the tradition of Stanley rather than James and Kerber (found before unearthing Fomin-Lulov). The following lemma is key.

Lemma 3.7. *Pick $\lambda \vdash n \geq 1$ with $f^\lambda \geq n^3$. Then $a_{\lambda,r} \neq 0$.*

Proof. Combine Desarmenien's formula, the Fomin-Lulov bound, Stirling's approximation, and a little careful bookkeeping. □

4. When is $f^\lambda < n^3$?

Definition 4.1. For $c \in \lambda$, the *opposite hook length* h_c^{op} is (draw picture).

Example 4.2. If $\lambda = (3, 2)$ the hook length tableau is 431/21 while the opposite hook length tableau is 123/23. It's easy to see that $\sum_{c \in \lambda} h_c = \sum_{c \in \lambda} h_c^{\mathrm{op}}$.

Proposition 4.3. $\prod_{c \in \lambda} h_c^{\mathrm{op}} \geq \prod_{c \in \lambda} h_c$. *Moreover, equality occurs if and only if λ is a rectangle.*

Example 4.4. For $\lambda = (3, 2)$, the products are $4 \cdot 3 \cdot 1 \cdot 2 \cdot 1 < 1 \cdot 2 \cdot 3 \cdot 2 \cdot 3$.

Proof. Algebraic. Would be interesting to have a combinatorial or representation-theoretic explanation. □

Definition 4.5. The *diagonal preorder* is defined by

$$\lambda \lesssim^{\mathrm{diag}} \mu \Leftrightarrow \forall i \in \mathbb{Z}_{>0}, \#\{c \in \lambda : h_c^{\mathrm{op}} \geq i\} \leq \#\{d \in \mu : h_d^{\mathrm{op}} \geq i\}.$$

Remark 4.6. The diagonal preorder is reflexive and transitive, but not anti-symmetric, so it is a preorder and not a partial order. Its key property is

$$\lambda \lesssim^{\mathrm{diag}} \mu \Rightarrow \prod_{c \in \lambda} h_c^{\mathrm{op}} \leq \prod_{d \in \mu} h_d^{\mathrm{op}}.$$

Definition 4.7. The *diagonal excess* of λ is

$$N(\lambda) := |\lambda| - \#\{h_c^{\mathrm{op}} : c \in \lambda\}.$$

For instance, $N((3, 2)) = 5 - 3 = 2$.

Theorem 4.8. *Suppose $2N(\lambda) + 1 \leq n$. Then the hook $(n - N(\lambda), 1^{N(\lambda)})$ is maximal for the diagonal preorder on partitions of n with diagonal excess $N(\lambda)$. Furthermore,*

$$f^\lambda \geq \frac{1}{N(\lambda) + 1} \binom{n}{N(\lambda)}.$$

Remark 4.9. Consequently, those λ with $f^\lambda < n^3$ are essentially those with $N(\lambda) \leq 3$, say, which are very easy to classify. To finish off the proof of the main result, some inequalities are used which kick in at $n \geq 34$, so the result is brute-forced (with help from the key lemma!) for $n \leq 33$. The seven families where $N(\lambda) \leq 3$ are handled case-by-case.

5. Asymptotic Uniformity

These techniques yield the following promised, vastly stronger result.

Theorem 5.1. *Pick $\lambda \vdash n$, $f^\lambda \geq n^5 \geq 1$. Then for all r,*

$$\left| \frac{a_{\lambda,r}}{f^\lambda} - \frac{1}{n} \right| < \frac{1}{n^2}.$$

In particular, if $n \geq 81$, $\lambda_1 < n - 7$, and $\lambda_1' < n - 7$, then $f^\lambda \geq n^5$.

Remark 5.2. Since f^λ typically is extremely large compared to n, the result says that in absolute terms the $a_{\lambda,r}$ are essentially constant as r varies. Indeed, the theorem is really just a sample and can be improved in various ways depending on the need.

- One can use Roichman's exponential estimate of normalized symmetric group characters to get exponential decay in many cases at the cost of less explicit assumptions and bounds. Or, one may use Larsen-Shalev's bounds to get a decay rate involving a power of f^λ.
- The same techniques can be applied to many cycle types beyond a single long cycle.

I don't have a particular application for such tight bounds and so have not pursued them further.

6. Unimodality Remark

The argument classifying λ with $f^\lambda < n^3$ essentially replaces the hook length with the opposite hook length, since the opposite hook product is order-preserving with respect to the diagonal preorder, and the diagonal preorder is quite flexible. It might be nice to work directly with the symmetric group characters themselves, though this seems difficult in general.

Remark 6.1. Note that if $a + b + 1 = n$, then

$$\chi^{(a+1,1^b)}(1^n) = \binom{n-1}{a}$$

which is unimodal in a. By the Fomin-Lulov formula, if $\ell \mid n$, we have

$$|\chi^{(a+1,1^b)}(\ell^{n/\ell})| = \binom{\frac{n}{\ell} - 1}{\lfloor \frac{a}{\ell} \rfloor}$$

which is also unimodal in a. Recall that $K_{\lambda,(1^n)} = \chi^\lambda(1^n)$.

We have a very general sequence of inequalities:

Theorem 6.2 (Snapper '71, Liebler-Vitale '73, Lam '78; Garsia-Procesi '92)**.** *$K_{\lambda\nu} \leq K_{\lambda\mu}$ for all λ if and only if $\nu \geq \mu$ in dominance order. Indeed, $\nu \geq \mu$ implies $K_{\lambda\nu}(t) \leq K_{\lambda\mu}(t)$ coefficient-wise.*

Can anything approaching this level of beauty be said about symmetric group characters?

Question 6.3. *Are there any "nice" infinite families besides hooks and rectangles for which $|\chi^\lambda(\mu)|$ is monotonic, unimodal, or suitably order-preserving as λ varies? What about as μ varies?*

7. Extra Time

My argument proving the Fomin-Lulov hook formula for evaluations of symmetric group characters at rectangles uses Stanley's q-analogue of the hook length formula, a result of Stembridge concerning cyclic exponents, and the following combinatorial lemma which appears to be new.

Lemma 7.1. *Pick $\lambda \vdash n = \ell s$ where λ can be written as a sequence of s successive ribbons, each of length ℓ. Then for all $a \in \mathbb{Z}$,*

$$\#\{c \in \lambda : h_c \equiv_\ell \pm a\} = s \cdot \#\{a, -a (\text{mod } \ell)\}.$$

It is actually a special case of a somewhat more general result.

Lemma 7.2. *Suppose λ/μ is a ribbon of length ℓ. For any $a \in \mathbb{Z}$,*

$$\#\{c \in \mu : h_c \equiv_\ell \pm a\} + \#\{a, -a \pmod{\ell}\} = \#\{d \in \lambda : h_d \equiv_\ell \pm a\}.$$

Proof. The argument yields a very explicit description of the movement of hook lengths mod ℓ as a ribbon is added. $\square$