INTRO TO COMPLEX REFLECTION GROUPS

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1. Shephard-Todd Theorems

Notation 1.1. Let V be a finite-dimensional complex vector space and let G be a finite subgroup of $\mathrm{GL}(V)$. Let $S := \mathrm{Sym}\, V$ denote the symmetric algebra of V with the induced G-action. Write $S^G := \{f \in S : g \cdot f = f,\ \forall g \in G\}$ for the algebra of G-invariants.

Definition 1.2. A *pseudoreflection* is an element $\rho \in \mathrm{GL}(V)$ which fixes a hyperplane pointwise and where $\rho \neq \mathrm{id}$ has finite order. We say G is a *complex reflection group* if G is generated by pseudoreflections.

Remark 1.3. ρ is a pseudoreflection if and only if the matrix of ρ is $\mathrm{diag}(\zeta, 1, \ldots, 1)$ in some basis where $\zeta^k = 1$, $\zeta \neq 1$. Note that pseudoreflections coming from $\mathbb{R}^n \hookrightarrow \mathbb{C}^n$ are just reflections.

Theorem 1.4 ([ST54]; see also [Che55])**.** *G is generated by pseudoreflections if and only if S^G is a polynomial ring.*

Remark 1.5. Shephard-Todd proved this by classifying the complex reflection groups and verifying ($\Rightarrow$) case-by-case. Chevalley gave an independent, case-free proof of ($\Rightarrow$) over $\mathbb{R}$. Serre noted Chevalley's argument gives ($\Rightarrow$) for $\mathbb{C}$. Shephard-Todd's proof of ($\Leftarrow$) is essentially the one Sara presented using Molien's theorem.

Theorem 1.6 ([ST54])**.** *The complex reflection groups are direct products of groups of the form $G(m, p, n)$ and 34 exceptional groups.*

Remark 1.7. The result can be made more precise, but this version suffices for our purposes. We next describe the groups $G(m, p, n)$ which make up the bulk of the complex reflection groups. In particular, they must include types A_{n-1}, B_n, D_n.

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2. $G(m,p,n)$

Definition 2.1. Let $C_m := \{\zeta \in \mathbb{C} : \zeta^m = 1\}$. The group $G(m,1,n)$ consists of the $n \times n$ complex matrices whose non-zero entries are from C_m and with precisely one non-zero entry in each row and column.

Example 2.2. The matrix

$$\begin{pmatrix} & & & -1 \\ 1 & & & \\ & 1 & & \\ & & 1 & \end{pmatrix}$$

corresponds to the cycle $(1\,2\,3\,4\,\overline{1}\,\overline{2}\,\overline{3}\,\overline{4})$, which is a Coxeter element in type B_n.

Definition 2.3. Define a (surjective) group homomorphism

$$G(m,1,n) \twoheadrightarrow C_m$$

by

$$A \mapsto \text{the product of the non-zero entries of } A.$$

For $p \mid m$, we have a further (surjective) group homomorphism $C_m \twoheadrightarrow C_p$ given by $\zeta \mapsto \zeta^{m/p}$. Define the group

$$G(m,p,n) := \ker G(m,1,n) \twoheadrightarrow C_p.$$

Concretely, suppose $A \in G(m,1,p)$ has non-zero entries $\zeta^{a_1}, \zeta^{a_2}, \ldots, \zeta^{a_n}$ for a fixed primitive mth root of unity ζ. Then $A \in G(m,p,n)$ if and only if $a_1 + \cdots + a_n \equiv_p 0$.

Remark 2.4. We have

- $G(2,2,2)$ is the Klein 4-group.
- $|G(m,p,n)| = \frac{m^n n!}{p}$
- $G(m,1,1)$ is C_m, and $G(p,p,2)$ is the dihedral group of order $2p$.
- $G(1,1,n) \cong S_n$, $G(2,1,n)$ is the type B_n group, and $G(2,2,n)$ is the type D_n group.

3. Wreath Products

Notation 3.1. Let G and S be finite groups and let X be a finite set on which S acts.

Definition 3.2. Give $\operatorname{Hom}(X,G)$, the set of functions from X to G, a group structure by pointwise multiplication. Now S acts on X, and hence on $\operatorname{Hom}(X,G)$ by

$$(s \cdot \alpha)(x) := \alpha(s^{-1} \cdot x).$$

That is, we have a group homomorphism $S \to \operatorname{Aut}(\operatorname{Hom}(X,G))$. Thus we may form the semi-direct product of $\operatorname{Hom}(X,G)$ and S, giving the *wreath product*

$$G \operatorname{wr}_X S := \operatorname{Hom}(X,G) \rtimes S.$$

Explicitly, $(\alpha, s), (\beta, t) \in G \operatorname{wr}_X S$ multiply as

$$(\alpha, s)(\beta, t) := (\alpha(s \cdot \beta), st).$$

Remark 3.3. We have $|G \operatorname{wr}_X S| = |G|^{|X|}|S|$.

Remark 3.4. Let $S = S_n$ and $X = \{1, 2, \ldots, n\} =: [n]$ with S acting naturally. Then

$$C_m \operatorname{wr}_{[n]} S_n \cong G(m,1,n)$$

by sending the pair $(\alpha \colon [n] \to C_m, w \in S_n)$ to the matrix whose ith column is $\alpha(w(i))$. (Or is it $\alpha(w^{-1}(i))$? Exercise for the audience!)

In this context, $\operatorname{wr}_{[n]}$ is universally abbreviated by wr. One often sees the symbol $\wr$ in place of wr. For instance, the type B_n group is sometimes described as $\mathbb{Z}/2 \wr S_n$.

Question 3.5. *What are the conjugacy classes of $G(m,p,n)$?*

Definition 3.6. Let G be any finite group. Write $C(G)$ for the set of conjugacy classes of G.

Proposition 3.7. *$C(G \operatorname{wr} S_n)$ is in bijection with functions*

$$\underline{\mu}\colon C(G) \to \{\textit{integer partitions}\}$$

where $\sum_{c \in C(G)} |\underline{\mu}(c)| = n$.

Example 3.8. Consider

$$\begin{pmatrix} & i & & & & \\ -i & & & & & \\ & & & i & & \\ & & 1 & & & \\ & & & & & 1 \\ & & & & 1 & \\ & & & & & & 1 \end{pmatrix} \in C_4 \operatorname{wr} S_6.$$

The corresponding "permutation" has three 2-cycles and one 1-cycle. Group these as

$$\begin{pmatrix} & i \\ -i & \end{pmatrix}, \quad \begin{pmatrix} & i \\ 1 & \end{pmatrix}, \quad \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad (1)\,.$$

The corresponding products of non-zero entries are

$$(-i)(i) = 1, \quad (1)(i) = i, \quad (1)(1) = 1, \quad 1.$$

Since C_4 is abelian, we identify $C(C_4)$ and C_4. Form $\underline{\mu}(\zeta)$ as the partition recording the lengths of cycles whose product is ζ:

$$\begin{aligned} \underline{\mu}(1) &= (2,2,1) \\ \underline{\mu}(-1) &= \varnothing \\ \underline{\mu}(i) &= (2) \\ \underline{\mu}(-i) &= \varnothing. \end{aligned}$$

Remark 3.9. See [Ste89, §4] for a nice summary of the conjugacy classes and irreducible representations of $G \wr S_n$ in terms of the corresponding data for G and S_n. See [JK81, Chapter 4] for a textbook treatment. Also see [Ste89, §6] for extensions to $G(m,p,n)$.

4. Molien Series

We now continue the notation of Notation 1.1. In particular G is a finite subgroup of $\mathrm{GL}(V)$.

Definition 4.1. $S = \operatorname{Sym} V$ is a $\mathbb{C}G$-module. Hence it has an irreducible decomposition

$$S = \oplus_i T_i$$

for irreducible $\mathbb{C}G$-modules T_i. Recall that the irreducible $\mathbb{C}G$-modules are classified by their characters. For each irreducible character χ of G, let S^G_χ denote the direct sum of those T_i with character χ. Hence

$$S = \oplus_\chi S^G_\chi,$$

which is called the *isotypic decomposition* of S.

Remark 4.2. Let $\epsilon\colon G \to \mathbb{C}^\times$ by $\epsilon(g) := 1$ be the character of the trivial representation. Then

$$f \in S^G_\epsilon \Rightarrow g \cdot f = \epsilon(g) f = f, \forall g \in G \Rightarrow f \in S^G.$$

The converse also holds, so

$$S^G = S^G_\epsilon.$$

By similar arguments,

$$S^G \cdot S^G_\chi \subset S^G_\chi.$$

Hence the S^G_χ are S^G-modules.

Moreover, $S = \oplus_{n=0}^\infty S_n$ is graded, and the isotypic components S_χ^G inherit the grading with $(S_\chi^G)_n := S_\chi^G \cap S_n$.

Definition 4.3. The *Molien series* of S_χ^G is the normalized Hilbert series

$$F_{G,\chi}(t) := \frac{1}{\chi(1)} \sum_{n=0}^\infty \dim_{\mathbb{C}} (S_\chi^G)_n t^n$$

whose nth term is the multiplicity of χ in the nth graded piece of S.

Theorem 4.4 (Molien; see [Sta79, Theorem 2.1])**.** *The Molien series for S_χ^G is*

$$F_{G,\chi}(t) = \frac{1}{|G|} \sum_{g \in G} \frac{\overline{\chi(g)}}{\det(1 - gt)}.$$

Remark 4.5. Note that when $\chi = \epsilon$, $F_{G,\epsilon}(t)$ is the Hilbert series of S^G. If additionally G comes from a real reflection group, you get precisely the statement of Molien's theorem from class.

Proof. The argument is virtually identical to the one Sara presented in class. Two changes of note: the orthogonality relations for characters are used; and one must justify why $\dim_{\mathbb{C}}(S_\chi^G)_n$ is finite. See Stanley for details. $\square$

Corollary 4.6. $S_\chi^G \neq 0$.

Proof. We have $S_\chi^G = 0$ if and only if $F_{G,\chi}(t) = 0$, which occurs if and only if $F_{G,\chi}(1) = 0$. Consider letting $t \to 1$ in Molien's theorem. At $g = 1$ we get a term with a pole at $t = 1$ of order $\dim V$, and all other terms have lower order poles. Hence they can't possibly cancel, and $F_{G,\chi}(1) \neq 0$. $\square$

Remark 4.7. From Molien's theorem we can also for instance read off the Krull dimension of S_χ^G, since it's the order of the pole at $t = 1$, namely $\dim V$.

5. Coinvariants and Chevalley's Theorem

Definition 5.1. For simplicity, let G now be a complex reflection group. Consider the ideal $S \cdot S_+^G$ in S generated by invariants with 0 constant term. The quotient ring

$$R := \frac{S}{S \cdot S_+^G}$$

is the *algebra of coinvariants*.

Remark 5.2. R is a graded $\mathbb{C}$-algebra with graded G-action. Hence the isotypic components R_χ^G make sense just as they did for S. It is not immediately obvious that R is finite-dimensional, though we will shortly see this.

Theorem 5.3 (See [Sta79, Proposition 4.9])**.** *Let $f_1, \ldots, f_r$ be a basic set of invariants for G with degrees $d_1, \ldots, d_r$. Then the Molien series for R_χ^G is*

$$\frac{1}{\chi(1)} \sum_{n=0}^\infty \dim_{\mathbb{C}} (R_\chi^G)_n t^n = \frac{1}{|G|} (1 - t^{d_1}) \cdots (1 - t^{d_r}) \sum_{g \in G} \frac{\overline{\chi}(g)}{\det(1 - gt)}.$$

Proof. One can check that

$$R_\chi^G = S_\chi^G / (f_1 S_\chi^G + \cdots + f_r S_\chi^G).$$

Suppose $r = 1$. Then clearly

$$\mathrm{Hilb}(R_\chi^G; t) = \mathrm{Hilb}(S_\chi^G; t) - \mathrm{Hilb}(f_1 S_\chi^G; t) = (1 - t^{d_1}) \, \mathrm{Hilb}(S_\chi^G; t).$$

Now use Molien's theorem and the result follows. The general case requires more work, namely it uses the fact that R_χ^G is a "Cohen-Macaulay module." See [Sta79, Theorem 3.10, Proposition 4.9] for more. $\square$

Corollary 5.4 (Chevalley)**.** *The G-action on R is isomorphic to the regular representation.*

Proof. Equivalently, we must show the multiplicity of χ in R is $\chi(1)$. Now the multiplicity of χ in R is the $t = 1$ specialization of the Molien series for R^G_χ. Using the preceding theorem, the only term without more zeroes than poles arises from $g = 1$. By L'Hopital's rule, the requested multiplicity is then

$$\lim_{t\to 1} \frac{1}{|G|}(1 - t^{d_1}) \cdots (1 - t^{d_r}) \frac{\overline{\chi(1)}}{(1 - t)^r} = \frac{\chi(1)}{|G|} d_1 \cdots d_r = \chi(1).$$

(The equality $d_1 \cdots d_r = |G|$ can be deduced from Molien's theorem and the Shephard-Todd theorem.) □

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