A ZOO OF COINVARIANT ALGEBRAS

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Abstract. These notes were for a lecture given in the informal "1, 2, 3" seminar at the University of Washington on October 12th, 2017.

Contents

1. Example 1: Classical Type A

Definition 1.1. *"The" coinvariant algebra* is

$$R_n := \frac{\mathbb{C}[x_1, \ldots, x_n]}{I_n}$$

where I_n is generated by the non-constant, homogeneous, S_n-invariant elements.

Example 1.2. $\dim_{\mathbb{C}} R_n < \infty$. A slick approach:

$$(t - x_1) \cdots (t - x_n) = t^n + (\text{lower order terms in } t \text{ whose coefficients in } \mathbb{C}[x_1, \ldots, x_n] \text{ are in } I_n).$$

Now let $t = x_i$, giving $x_i^n \in I_n$.

Remark 1.3. Here's some motivation for considering R_n.

- The symmetric polynomials $\mathbb{C}[x_1, \ldots, x_n]$ are a free $\mathbb{C}[x_1, \ldots, x_n]^{S_n}$-module of rank $\dim_{\mathbb{C}} R_n$. (In fact, a vector space basis corresponds to a module basis.)
- The cohomology of the complete flag manifold $H^*(G/B, \mathbb{C})$ is isomorphic to R_n. (Borel's isomorphism.)
- An an S_n-representation over $\mathbb{C}$, R_n is isomorphic to the the regular representation, but is graded, so it's a "graded regular representation." (Chevalley's theorem.)

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1.1. **Hilbert Series.**

Definition 1.4. Let $A = \oplus_{n \geq 0} A_n$ where each A_n is a finite-dimensional vector space. The *Hilbert series* of A is

$$\operatorname{Hilb}(A; q) := \sum_{n \geq 0} (\dim A_n) q^n \in \mathbb{Z}_{\geq 0}[[q]].$$

Question 1.5. *What is* $\operatorname{Hilb}(R_n; q)$*? So far we only know It's a polynomial and not a power series. [Check if audience is tracking by asking constant and linear coefficients, namely* 1 *and* $n-1$*.]*

Theorem 1.6. $\dim_{\mathbb{C}} R_n = n!$. *In fact,*

$$\operatorname{Hilb}(R_n; q) = [n]_q! := [n]_q \cdots [2]_q [1]_q$$

where $[k]_q := 1 + q + \cdots + q^{k-1}$.

Proof. The dimension count follows from the Hilbert series by taking $q = 1$. As it happens, $I_n = \langle e_1, \ldots, e_n \rangle$, and $e_1, \ldots, e_n$ is a regular sequence. Consequently, we have short exact sequences

$$0 \to \frac{\mathbb{C}[x_1, \ldots, x_n]}{\langle e_1, \ldots, e_{i-1} \rangle} \xrightarrow{\cdot e_i} \frac{\mathbb{C}[x_1, \ldots, x_n]}{\langle e_1, \ldots, e_{i-1} \rangle} \to \frac{\mathbb{C}[x_1, \ldots, x_n]}{\langle e_1, \ldots, e_{i-1}, e_i \rangle} \to 0.$$

It follows that $\operatorname{Hilb}(R_{n;i-1}; q) = q^{\deg e_i} \operatorname{Hilb}(R_{n;i-1}; q) + \operatorname{Hilb}(R_{n;i}; q)$, so that

$$\begin{aligned}
\operatorname{Hilb}(R_n; q) &= (1 - q^n) \operatorname{Hilb}(R_{n;n-1}; q) \\
&= \cdots \\
&= (1 - q^n) \cdots (1 - q^1) \operatorname{Hilb}(\mathbb{C}[x_1, \ldots, x_n]; q) \\
&= (1 - q^n) \cdots (1 - q^1)/(1 - q)^n \\
&= [n]_q!.
\end{aligned}$$

□

Theorem 1.7 (Artin's basis)**.** *The "sub-staircase" monomials*

$$\{x_1^{a_1} \cdots x_n^{a_n} : 0 \leq a_i \leq n - i\}$$

form a $\mathbb{C}$*-basis for* R_n.

Remark 1.8. The claimed Hilbert series is a quick corollary of Artin's basis. Given a homogeneous basis B for a graded vector space A, we have

$$\operatorname{Hilb}(A; q) = \sum_{b \in B} q^{\deg b}.$$

Consequently

$$\operatorname{Hilb}(R_n; q) = \sum_{(a_1, \ldots, a_n) : 0 \leq a_i \leq n-i} q^{a_1 + \cdots + a_n} = \prod i = 1^n \sum_{a_i = 0}^{n-i} q^{a_i} = [n]_q!.$$

The tuples $(a_1, \ldots, a_n)$ can be thought of as the "Lehmer codes" of permutations as follows. Given a permutation $w_1 \cdots w_n \in S_n$, an *inversion* is a pair (i, j) where $i < j$ and $w_i > w_j$. To compute the Lehmer code, let a_i be the number of inversions with left endpoint i. The total number of inversions of a permutation is called its *length*. So, computing $\operatorname{Hilb}(R_n; q)$ using Artin's basis amounts to finding a nice formula for the length generating function of S_n. This computation is also intimately related to the Bruhat decomposition and Borel's isomorphism.

Theorem 1.9 (Garsia–Stanton's descent basis)**.** *The monomials*

$$\left\{ \prod_{i : w_i > w_{i+1}} x_{w(1)} \cdots x_{w(i)} : w \in S_n \right\}$$

form a $\mathbb{C}$*-basis for* R_n.

Remark 1.10. The degree of such a monomial is $\sum_{i:w_i>w_{i+1}} i$, which is called the *major index* of w. This proves a celebrated result of MacMahon:

$$\sum_{w\in S_n} q^{\operatorname{maj} w} = [n]_q!.$$

1.2. **Frobenius Series.**

Remark 1.11. Since R_n is a graded version of the regular representation of S_n, we can ask for a description of its graded isomorphism type.

Definition 1.12. An (integer) *partition* of n is a weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of integers such that $\lambda_1 + \lambda_2 + \cdots = n$. Abbreviate this as $\lambda \vdash n$.

Fact 1.13. *The complex irreducible representations of S_n are canonically indexed by partitions of n; call them S^λ.*

Definition 1.14. Let $A = \oplus_{d\geq 0} A_d$ where each A_d is a finite-dimensional S_n-representation (over $\mathbb{C}$). The *graded Frobenius series* of A is

$$\operatorname{Frob}(A; q) := \sum_{\substack{d\geq 0\\ \lambda\vdash n}} \langle A_d, S^\lambda\rangle q^d s_\lambda$$

where the s_λ are formal indeterminates (really Schur functions).

Question 1.15. *What is the Frobenius series for R_n?*

Theorem 1.16 (Lusztig–Stanley)**.** *We have*

$$\operatorname{Frob}(R_n; q) = \sum_{T\in \operatorname{SYT}(n)} q^{\operatorname{maj} T} s_{\operatorname{sh} T}$$

where $\operatorname{SYT}(n)$ *denotes the set of standard Young tableaux on shapes of size n,* $\operatorname{sh} T$ *denotes the shape of the tableau T, and* $\operatorname{maj} T$ *is the "major index" of T.*

Remark 1.17. That is, the number of copies of S^λ in the dth graded piece of R_n is the number of standard tableaux of shape λ with major index d. Haglund–Rhoades–Shimozono remark that this result may be deduced from the fact that the Kozsul complex of R_n is exact; since I haven't seen it written out or taken the time to try it myself, it would be interesting to see the argument.

2. Example 2: Reflection Group Generalizations

Definition 2.1. Let G be a finite subgroup of $\operatorname{GL}(V)$ where $V = \mathbb{C}\{x_1, \ldots, x_n\}$. G acts on V, by definition, and hence on the polynomial algebra $\mathbb{C}[x_1, \ldots, x_n]$. Consequently, we may define the *coinvariant algebra* of G by

$$R_G := \frac{\mathbb{C}[x_1, \ldots, x_n]}{I_G}$$

where I_G is the ideal generated by homogeneous, non-constant G-invariants. As usual, R_G has a graded G-action.

A *pseudoreflection* is an element $g \in \operatorname{GL}(V)$ of finite order whose fixed point set has codimension 1.

Example 2.2. When G is the set of $n \times n$ permutation matrices, we recover the coinvariant algebra above. This is the "type A_{n+1}" case.

As is traditional, one next considers the "type B_n" case, namely letting G be the group of automorphisms of the n-cube in V. This group has order $2^n n!$ since we can reflect each axis independently and permute the axes amongst themselves arbitrarily. What is the coinvariant algebra in this case? We see that $x_1 + x_2 + \cdots + x_n$ is an invariant in type A_{n+1} but not in type B_n. In fact, the ability to toggle negatives forces invariants to exist only in even degrees. Indeed,

$$\mathbb{C}[x_1, \ldots, x_n]^{B_n} = \mathbb{C}[x_1^2, \ldots, x_n^2]^{S_n}.$$

Consequently, the type B_n coinvariant algebra is

$$\frac{\mathbb{C}[x_1, \ldots, x_n]}{\langle e_i(x_1^2, \ldots, x_n^2) : 1 \leq i \leq n \rangle}.$$

Remark 2.3. For reasons analogous to the type A computation above, the Hilbert series is

$$\mathrm{Hilb}(R_{B_n}; q) = [n]_{q^2}!.$$

Theorem 2.4. *Let G be a finite subgroup of $\mathrm{GL}(V)$ where $V = \mathbb{C}\{x_1, \ldots, x_n\}$. Then $\mathbb{C}[x_1, \ldots, x_n]^G$ is a Cohen–Macaulay algebra, i.e. it is a free module of finite rank over some $\mathbb{C}[\theta_1, \ldots, \theta_n]$ where the θ_i are algebraically independent G-invariants. This rank is 1 if and only if G is generated by pseudoreflections.*

Furthermore, $\mathbb{C}[x_1, \ldots, x_n]/\langle \theta_1, \ldots, \theta_n \rangle$ is isomorphic to the regular representation of G

Remark 2.5. There is a famous classification of G which are generated by pseudoreflections, which includes the type A and B cases in one large infinite family $G(m, p, n)$, along with finitely many explicitly described exceptional groups. Consequently, we've identified the ungraded Frobenius series of the type B coinvariant algebra. What is its graded Frobenius series?

Fact 2.6. *The complex irreducible representations of type B_n are canonically indexed by tuples of partitions (λ, μ) such that $|\lambda| + |\mu| = n$.*

Theorem 2.7. *We have*

$$\mathrm{Frob}(R_{B_n}; q) = \sum_{(\lambda,\mu)} \sum_{T \in \mathrm{SYT}(\lambda \oplus \mu)} q^{|\mu| + 2 \operatorname{maj} T} s_\lambda \otimes s_\mu$$

where $\lambda \oplus \mu$ is a "maximally disjoint" skew diagram obtained from λ and μ.

Remark 2.8. The above theorem generalizes immediately to all $G(m, 1, n)$ (Stembridge, Theorem 5.3), and (with an extra factor) to all $G(m, r, n)$ (Stembridge, Corollary 6.3). It turns out the invariant subalgebra in type D_n is generated by $e_i(x_1^2, \ldots, x_n^2)$ with $1 \leq i < n$ together with $x_1 \cdots x_n$, and so the type D_n coinvariant algebra has Hilbert series $[n-1]_{q^2}![n]_q$. This description generalizes directly to all $G(m, r, n)$–see e.g. Stembridge, proof of proposition 6.3.

Thus our guiding questions have already been answered for essentially all interesting examples in this direction.

3. Example 3: New Directions: Generalized Coinvariant Algebras

We next summarize some much more recent work. We'll use natural bigraded analogues of the Hilbert and Frobenius series.

3.1. Diagonal Coinvariants.

Definition 3.1. The *diagonal coinvariant algebra* (in type A) is

$$\mathcal{D}_n := \frac{\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]}{J_n}$$

where J_n is the ideal generated by the non-constant, homogeneous S_n-invariants where S_n acts diagonally on the two sets of variables simultaneously.

Theorem 3.2 (Haiman)**.** *We have* $\dim \mathcal{D}_n = (n+1)^{n-1}$. *In fact,*

$$\mathrm{Hilb}(\mathcal{D}_n; q, q^{-1}) = q^{-\binom{n}{2}} [n+1]_q^{n-1}.$$

Remark 3.3. Haglund–Loehr have a conjectured combinatorial generating function for $\mathrm{Hilb}(\mathcal{D}_n; q, t)$ using parking function statistics "area" and "dinv." Setting $t = 0$ essentially reduces $\mathcal{D}_n$ to R_n.

Theorem 3.4 (Haiman)**.** *We have*

$$\mathrm{Frob}(\mathcal{D}_n; q, t) = \nabla e_n$$

where ∇ is a certain eigen-operator (with coefficients in $\mathbb{Q}(q, t)$) on the basis of modified Macdonald polynomials.

Theorem 3.5 (Shuffle Conjecture; proved by Carlsson–Mellit)**.** *There is an explicit combinatorial generating function expression for* $\mathrm{Frob}(\mathcal{D}_n; q, t)$ *expanded in the monomial basis.*

Remark 3.6. Finding a direct analogue of the Stanley–Lusztig theorem for $\mathcal{D}_n$ is a major open problem which (to my knowledge) has only been solved for hook shapes.

3.2. **Generalized Coinvariant Algebras, Type** A.

Remark 3.7. The *Delta conjecture* of Haglund–Remmel–Wilson gives a certain explicit formula for $\Delta'_{e_{k-1}} e_n$ which in the $k = n$ case reduces to the shuffle conjecture. The bigraded S_n-module in the $k = n$ case is thus $\mathcal{D}_n$. The general Delta conjecture has no known or conjectured (natural) bigraded S_n-module. However, Haglund–Rhoades–Shimozono were able to define singly graded variants of R_n which (up to minor modification) have graded Frobenius series which agree with the specialization of the Delta conjecture at $t = 0$.

Definition 3.8. Given two positive integers $k \leq n$, let

$$R_{n,k} := \frac{\mathbb{Q}[x_1, \ldots, x_n]}{I_{n,k}}$$

where

$$I_{n,k} := \langle x_1^k, \ldots, x_n^k, e_n(x_1, \ldots, x_n), \ldots, e_{n-k+1}(x_1, \ldots, x_n) \rangle.$$

$R_{n,k}$ is a doubly-graded S_n-module.

Remark 3.9. When $k = n$, we get R_n (since $x_i^n \in I_n$ as noted above). When $k = 1$, we get $R_{n,1} \cong \mathbb{Q}$.

Theorem 3.10. *We have* $\dim R_{n,k} = k! \cdot \mathrm{Stir}(n, k)$ *where* $\mathrm{Stir}(n, k)$ *is the number of set partitions of* $[n] := \{1, \ldots, n\}$ *into* k *non-empty subsets. In fact,*

$$\mathrm{Hilb}(R_{n,k}; q) = \mathrm{rev}_q([k]_q! \cdot \mathrm{Stir}_q(n, k)) = \sum_{\sigma \in \mathrm{OP}_{n,k}} q^{\mathrm{comaj}\,\sigma}$$

where rev_q *reverses polynomials,*

$$\mathrm{Stir}_q(n, k) := \mathrm{Stir}_q(n-1, k-1) + [k]_q \cdot \mathrm{Stir}_q(n-1, k), \qquad \mathrm{Stir}_q(1, k) := \begin{cases} 1 & k = 1 \\ 0 & k > 1 \end{cases},$$

$\mathrm{OP}_{n,k}$ *is the set of ordered set partitions of* $[n]$ *into* k *blocks, and* comaj *is the co-major index of Remmel–Wilson.*

Remark 3.11. Unfortunately $\mathrm{Stir}(n, k)$ does not have a nice product formula. Consequently, there can be no quick regular sequence argument for computing $\dim R_{n,k}$.

Theorem 3.12. *As an ungraded* S_n *representation,* $R_{n,k}$ *is isomorphic to the action of* S_n *on* $\mathbb{C}\{\mathrm{OP}_{n,k}\}$.

Theorem 3.13. *We have*

$$\mathrm{Frob}(R_{n,k}; q) = \mathrm{rev}_q \left[\sum_{T \in \mathrm{SYT}(n)} q^{\mathrm{maj}\,T + \binom{n-k}{2} - (n-k) \cdot \mathrm{Des}\,T} \binom{\mathrm{Des}\,T}{n-k}_q s_{\mathrm{sh}(T)'} \right].$$

Remark 3.14. At $k = n$ we recover the Lusztig–Stanley theorem. In any case, we have again answered the guiding questions for $R_{n,k}$.

3.3. **Generalized Coinvariant Algebras for Wreath Products.**

Remark 3.15. Chan–Rhoades found *two* generalizations of the $R_{n,k}$ above when replacing S_n with the wreath product $C_r \wr S_n = G(r, 1, n)$, which we next summarize.

Definition 3.16. Let $n \geq k \geq 1$ and $r \geq 1$.

(1) Suppose $r \geq 2$. Let

$$R^r_{n,k} := \frac{\mathbb{C}[x_1, \ldots, x_n]}{I^r_{n,k}}$$

where

$$I^r_{n,k} := \langle x_1^{kr+1}, \ldots, x_n^{kr+1}, e_n(x_1^r, \ldots, x_n^r), \ldots, e_{n-k+1}(x_1^r, \ldots, x_n^r) \rangle.$$

(2) Let

$$S^r_{n,k} := \frac{\mathbb{C}[x_1, \ldots, x_n]}{J^r_{n,k}}$$

where

$$J^r_{n,k} := \langle x_1^{kr}, \ldots, x_n^{kr}, e_n(x_1^r, \ldots, x_n^r), \ldots, e_{n-k+1}(x_1^r, \ldots, x_n^r) \rangle.$$

Definition 3.17. Let $\mathcal{F}^r_{n,k}$ denote the set of k-dimensional faces in the Coxeter complex of $G(r,1,n)$, which has a natural $G(r,1,n)$-action. Let $\mathrm{OP}^r_{n,k}$ denote the set of r-colored set partitions of $[n]$ into k non-empty blocks.

Remark 3.18. There is a natural bijection

$$\mathcal{F}^r_{n,k} \overset{\sim}{\to} \coprod_{z=0}^{n-k} \binom{[n]}{z} \times \mathrm{OP}^r_{n-z,k}\,.$$

We have

$$|\,\mathrm{OP}^r_{n,k}\,| = r^n \cdot k! \cdot \mathrm{Stir}(n,k).$$

Theorem 3.19. *As ungraded $G(r,1,n)$-modules,*

$$R^r_{n,k} \cong \mathbb{C}\{\mathcal{F}^r_{n,k}\}$$
$$S^r_{n,k} \cong \mathbb{C}\{\mathrm{OP}^r_{n,k}\}$$

Theorem 3.20. *We have*

$$\mathrm{Hilb}(R^r_{n,k}; q) = \sum_{z=0}^{n-k} \binom{n}{z} \cdot q^{krz} \cdot \mathrm{rev}_q([r]_q^{n-z} \cdot [k]_{q^r}! \cdot \mathrm{Stir}_{q^r}(n-z,k))$$
$$\mathrm{Hilb}(S^r_{n,k}; q) = \mathrm{rev}_q([r]_q^{n} \cdot [k]_{q}! \cdot \mathrm{Stir}_{q^r}(n,k)).$$

Remark 3.21. They have an explicit formula for $\mathrm{Frob}(R^r_{n,k}; q)$ and $\mathrm{Frob}(S^r_{n,k}; q)$ generalizing the $r = 1$ formula above; see their Theorem 6.14 and equation (3.21).