INTRODUCTION TO HOPF MONOIDS

JOSH SWANSON

ABSTRACT. These notes were for a lecture given in the informal combinatorics seminar at the Unviersity of Washington on October 10th, 2017. The exposition closely follows Aguiar–Ardila's preprint, arXiv: 1709.07504 (version 1), section 2, with some (standard) additional context and background.

CONTENTS

The paper is 113 pages—what's the big idea? Very roughly, many combinatorial structures have natural notions of "merging" and "breaking apart" which can frequently be organized into an algebraic structure called a Hopf monoid. Hopf monoids come with a frequently non-obvious "antipode" map. Takeuchi's formula allows one to compute the antipode map in great generality, but it typically has huge amounts of cancellation. Aguiar–Ardila were able to give cancellation-free formulas for antipodes for certain Hopf monoids by analyzing "generalized permutahedra."

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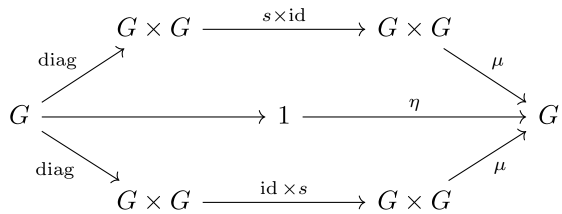
1. Context

1.1. **Groups and group objects.** Recall that a group can be thought of as a quadruple (G, μ, η, s) where G is a set and

$$\begin{aligned} &\mu\colon G \times G \to G \qquad \text{(multiplication)} \\ &s\colon G \to G \qquad \text{(inverse)} \\ &\eta\colon 1 \to G \qquad \text{(unit)} \end{aligned}$$

are maps of sets (where 1 is a singleton set) satisfying several axioms. These axioms may be encoded as diagrams in the category of sets. For instance, inverses correspond to



Following the three paths gives

$$g \mapsto (g, g) \mapsto (g^{-1}, g) \mapsto g^{-1}g, \qquad g \mapsto 1 \mapsto \mathrm{id}, \qquad g \mapsto (g, g) \mapsto (g, g^{-1}) \mapsto gg^{-1}$$

so that commutativity is equivalent to requiring $g^{-1}g = \mathrm{id} = gg^{-1}$.

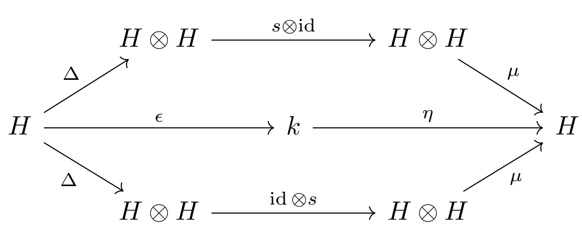
More generally, one may pick a category $\mathcal{C}$ (with a terminal object 1 and binary products) and define a "group object" in $\mathcal{C}$ in exactly the same manner, where the diagrams are now in $\mathcal{C}$.

1.2. **Hopf algebras.** Schemes are a certain algebraic generalization of the classical notion of algebraic varieties. Commutative rings can be turned into schemes by taking their "spectrum," which by definition results in an affine scheme. In fact, there is an (anti-)equivalence of categories between the category of affine schemes over a field k and the category of commutative k-algebras, and everybody loves commutative k-algebras.

Many people like to think about group schemes, which are group objects in a suitable category of schemes–elliptic curves are examples. So, one naturally asks: given an affine group scheme over k, what is the analogue as a commutative k-algebra? The answer is a Hopf algebra. More explicitly, a Hopf algebra is a tuple $(H, \mu, \eta, \Delta, \epsilon, s)$ where H is a set and

$$\begin{aligned} &\mu\colon H \otimes_k H \to H \qquad \text{(multiplication)} \\ &\eta\colon k \to H \qquad \text{(unit)} \\ &\Delta\colon H \to H \otimes_k H \qquad \text{(comultiplication)} \\ &\epsilon\colon H \to k \qquad \text{(counit)} \\ &s\colon H \to H \qquad \text{(antipode)} \end{aligned}$$

are k-linear maps subject to various compatibility conditions. The inverse axiom for instance corresponds to the condition



(Warning: the diagonal map $G \stackrel{\text{diag}}{\to} G \times G$ on the level of schemes corresponds to the multiplication map $H \otimes H \stackrel{\mu}{\to} H$ on the level of commutative algebras. Consequently Hopf algebras over k are not quite cogroup objects in the category of commutative k-algebras, since that would require using the codiagonal map $H \otimes H \to H$ instead.)

A Hopf algebra over k can be thought of as a "bimonoid object with antipode" in the category of k-vector spaces. *Hopf monoids* are likewise such objects in a general "lax braided monoidal category," which explains the name. They have been extensively studied in Aguiar–Mahajan's 800 page book "Monoidal Functors, Species and Hopf Algebras," which is firmly categorical and highly abstract. Ardila–Aguiar seem to go out of their way to avoid categorical language, and I'll now largely follow their lead.

2. Species

2.1. **Set species.** Ardila–Aguiar generally use Joyal's category of set species as the underlying category for their Hopf monoids. We now introduce this category explicitly.

Definition 2.1. A *set species* is a functor from the category of finite sets with morphisms given by bijections to the category of sets.

Explicitly, a *set species* P consists of the following data.

- For each finite set I, a set $P[I]$.
- For each bijection $\sigma \colon I \to J$, a map $P[\sigma] \colon P[I] \to P[J]$. These should be such that $P[\sigma \circ \tau] = P[\sigma] \circ P[\tau]$ and $P[\text{id}] = \text{id}$.

Example 2.2. The species L of linear orders is defined as follows. $L[I]$ is the set of all linear orders on I. For instance $L[\{a, b, c\}]$ has six elements, namely the linear orders $abc, acb, bac, bca, cab, cba$. The maps $P[\sigma] \colon P[I] \to P[J]$ are defined

$$L[\sigma](a_1 a_2 \cdots a_k) := \sigma(a_1)\sigma(a_2) \cdots \sigma(a_k).$$

Definition 2.3. A *morphism of set species* is a natural transformation between the underlying functors.

Explicitly, a *morphism* $f \colon P \to Q$ between set species P and Q is a collection of maps $f_I \colon P[I] \to Q[I]$ which satisfy the following *naturality axiom*: for each bijection $\sigma \colon I \to J$, $f_J \circ P[\sigma] = Q[\sigma] \circ f_I$, i.e.

$$\begin{array}{ccc} P[I] & \xrightarrow{P[\sigma]} & P[J] \\ {\scriptstyle f_I}\downarrow & & \downarrow{\scriptstyle f_J} \\ Q[I] & \xrightarrow[Q[\sigma]]{} & Q[J] \end{array}$$

Example 2.4. An automorphism of the species of linear orders L is given by the reversal maps $\text{rev}_I \colon L[I] \to L[I]$ defined by $\text{rev}_I(a_1 a_2 \cdots a_k) := a_k \cdots a_2 a_1$.

2.2. **Vector species.** Set species are easy to define and natural to think about. Later we'll want to consider formal linear combinations of combinatorial objects, which requires imposing some additional vector space structure on set species.

Definition 2.5. A *vector species* is a functor from the category of finite sets with morphisms given by bijections to the category of vector spaces over a fixed field $\Bbbk$. We will always assume $\Bbbk$ has characteristic 0.

Explicitly, a vector species $\mathbf{P}$ is a collection of $\Bbbk$-vector spaces $\mathbf{P}[I]$ and $\Bbbk$-linear maps between them exactly as for set species. Morphisms of vector species are exactly like morphisms of set species as well.

Definition 2.6. The *linearization functor*

$$\text{Set} \to \text{Vec}$$

sends a set to the vector space with that set as its basis. Given a set species P, we can compose it with the linearization functor to get a vector species $\mathbf{P}$.

Example 2.7. $\mathbf{L}[\{a,b,c\}]$ is the 6-dimensional $\Bbbk$-vector space with basis $\{abc, acb, bac, bca, cab, cba\}$. The "formal sum of all linear orders on $\{a,b,c\}$" thus lives in $\mathbf{L}[\{a,b,c\}]$.

3. Hopf monoids

3.1. Hopf monoids in set species.

Definition 3.1. A *decomposition* of a finite set I is a finite sequence $(S_1, \ldots, S_k)$ of pairwise disjoint subsets of I whose union is I. This is notated as

$$I = S_1 \sqcup \cdots \sqcup S_k.$$

A *composition* is a decomposition whose parts are all non-empty.

It should be emphasized that order matters for decompositions, i.e. $S \sqcup T \neq T \sqcup S$ (unless $S = T = \varnothing$).

Definition 3.2. A *connected Hopf monoid in set species* consists of the following data.

- A set species H such that the set $H[\varnothing]$ is a singleton.
- For each finite set I and each decomposition $I = S \sqcup T$, *product* and *coproduct* maps
  $$H[S] \times H[T] \overset{\mu_{S,T}}{\to} H[I] \qquad \text{and} \qquad H[I] \overset{\Delta_{S,T}}{\to} H[S] \times H[T]$$
  satisfying the naturality, unitality, associativity, and compatibility axioms below.

Before getting to the axioms, we introduce some notation surrouding the *product* map μ and the *coproduct* map Δ. Given a decomposition $I = S \sqcup T$ with $x \in S$, $y \in T$, and $z \in I$, we write

$$(x, y) \overset{\mu_{S,T}}{\to} x \cdot y \qquad \text{and} \qquad z \overset{\Delta_{S,T}}{\to} (z|_S, z/_S).$$

We call $x \cdot y$ the *product* of x and y, $z|_S \in H[S]$ the *restriction* of z to S, and $z/_S \in H[T]$ the *contraction* of S from z. We call the unique element $1 \in H[\varnothing]$ the *unit* of H.

Example 3.3. For the set species L of linear orders, define:

- A product given by concatenation: $s_1 \cdots s_i \cdot t_1 \cdots t_j := s_1 \cdots s_i t_1 \cdots t_j$.
- A coproduct given by restriction,
  $$\ell \overset{\Delta_{S,T}}{\to} (\ell|_S, \ell|_T)$$
  where $\ell|_S$ denotes ℓ but with elements not in S removed. Consequently, $\ell/_S = \ell|_T$ by definition for this example.

The compatibility axioms are as follows.

Axiom 3.4 (Naturality)**.** For each decomposition $I = S \sqcup T$, each bijection $\sigma\colon I \to J$, and any choice of $x \in H[S]$, $y \in H[T]$, and $z \in H[I]$, we have

$$H[\sigma](x \cdot y) = H[\sigma|_S](x) \cdot H[\sigma|_T](y),$$
$$H[\sigma](z)|_S = H[\sigma|_S](z|_S), \qquad H[\sigma](z)/_S = H[\sigma|_T](z/_S).$$

For instance, the first line is encoded by

$$\begin{array}{ccc} H[S] \times H[T] & \xrightarrow{\sigma|_S \times \sigma|_T} & H[\sigma(S)] \times H[\sigma(T)] \\ \mu_{S,T}\downarrow & & \downarrow \mu_{\sigma(S),\sigma(T)} \\ H[I] & \xrightarrow[H[\sigma]]{} & H[J] \end{array}$$

This axiom encodes the intuitive notion that "relabeling maps respect the merging and breaking operations." It's a good exercise to convince yourself of each equality when $H = L$.

Axiom 3.5 (Unitality)**.** For each I and $x \in H[I]$, we must have

$$x \cdot 1 = x = 1 \cdot x, \qquad x|_I = x = x/_\varnothing .$$

Axiom 3.6 (Associativity)**.** For each decomposition $I = R \sqcup S \sqcup T$, and any $x \in H[R]$, $y \in H[S]$, $z \in H[T]$, and $w \in H[I]$, we must have

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z,$$
$$(w|_{R \sqcup S})|_R = w|_R, \qquad (w|_{R \sqcup S})/_R = (w/_R)|_S, \qquad w/_{R \sqcup S} = (w/_R)/_S.$$

The last three encode the intuitive notion that "breaking up a single structure on I into three structures on R, S, T is well-defined." These may be extended to arbitrarily many factors by induction.

Axiom 3.7 (Compatibility)**.** Fix decompositions $S \sqcup T = I = S' \sqcup T'$, and consider the pairwise intersections $A := S \cap S'$, $B := S \cap T'$, $C := T \cap S'$, $D := T \cap T'$. In this situation, for any $x \in H[S]$ and $y \in H[T]$, we must have

$$(x \cdot y)_{S'} = x|_A \cdot y|_C \qquad \text{and} \qquad (x \cdot y)/_{S'} = x/_A \cdot y/_C.$$

See picture (1) in Aguiar–Ardila. [Verbal description: top half is S, bottom half is T; left half is S', right half is T'; upper left is A, upper right is B, lower left is C, lower right is D.] This encodes the intuitive notion that "merging then breaking is the same as breaking them merging."

Remark 3.8. No mention has yet been made of the antipode map s. This is related to the "connected" assumption. It turns out a unique antipode exists for connected Hopf monoids in vector species. We'll get there.

3.2. **Commutativity, cocommutativity, and duality.**

Definition 3.9. A connected Hopf monoid H in set species is *commutative* if $x \cdot y = y \cdot x$ for any $I = S \sqcup T$, $x \in H[S]$, and $y \in H[T]$. It is *cocommutative* if $(z|_S, z/_S) = (z/_T, z|_T)$ for any $I = S \sqcup T$ and $z \in H[I]$; it is enough to check that $z/_S = z|_T$ for any $I = S \sqcup T$ and $z \in H[I]$.

Example 3.10. The species of linear orders L is a connected Hopf monoid under the concatenation and restriction product and coproduct defined above. Since $z/_S = z|_T$ by definition, it is cocommutative. However, it is obviously not commutative.

Definition 3.11. Given a connected Hopf monoid H in set species, the *co-opposite* Hopf monoid H^{cop} is defined by preserving the product and reversing the coproduct: if $\Delta_{S,T}(z) = (z|_S, z/_S)$ in H, then $\Delta_{S,T}(z) = (z/_T, z|_T)$ in H^{cop}. The co-opposite Hopf monoid is in fact also a connected Hopf monoid in set species.

3.3. **Hopf monoids in vector species.**

Definition 3.12. A *connected Hopf monoid in vector species* is a vector species $\mathbf{H}$ with $\mathbf{H}[\varnothing] = \Bbbk$ that is equipped with linear maps

$$\mathbf{H}[S] \otimes \mathbf{H}[T] \overset{\mu_{S,T}}{\to} \mathbf{H}[I] \qquad \text{and} \qquad \mathbf{H}[I] \overset{\Delta_{S,T}}{\to} \mathbf{H}[S] \otimes \mathbf{H}[T]$$

for each decomposition $I = S \sqcup T$, subject to exactly the same sort of axioms as above.

The coproduct becomes more subtle for vector species. We can no longer generally decompose $\Delta_{S,T}(z)$ into just two pieces $z|_S$ and $z/_S$. However, writing out the above axioms categorically rather than element-by-element makes the issue disappear. Moreover, most of our coproducts will result in pure tensors, allowing us to think of $z|_S$ and $z/_S$ separately as before. Indeed, we may use the variant of Sweedler's notation

$$\Delta_{S,T}(z) = \sum z|_S \otimes z/_S.$$

Given a connected Hopf monoid in set species, we can apply the linearization functor, which results in a connected Hopf monoid in vector species.

Example 3.13. We can think of linear orders as a Hopf monoid L in set species or a Hopf monoid $\mathbf{L}$ in vector species. We'll generally use the latter.

4. Antipodes

4.1. **Takeuchi's formula.** We now restrict attention to Hopf monoids in vector species. A consequence of the associativity axiom is that we may iterate products or coproducts to get uniquely defined maps

$$\mathbf{H}[S_1] \otimes \cdots \otimes \mathbf{H}[S_k] \overset{\mu_{S_1,\ldots,S_k}}{\to} \mathbf{H}[I], \qquad \mathbf{H}[I] \overset{\Delta_{S_1,\ldots,S_k}}{\to} \mathbf{H}[S_1] \otimes \cdots \otimes \mathbf{H}[S_k].$$

For $k = 1$, we set μ_I and Δ_I to be the identity map. For $k = 0$, we set $\mu_{()} \colon \Bbbk \to \mathbf{H}[\varnothing]$ and $\Delta_{()} \colon \mathbf{H}[\varnothing] \to \Bbbk$ to be the linear maps that send 1 to 1.

Recall that a composition of a finite set $I \neq \varnothing$ is a decomposition where none of the parts are empty. We use the shorthand

$$(S_1, \ldots, S_k) \vDash I.$$

Definition 4.1 (or Theorem)**.** Let $\mathbf{H}$ be a connected Hopf monoid in vector species. The *antipode* of $\mathbf{H}$ is the collection of maps $s_I \colon \mathbf{H}[I] \to \mathbf{H}[I]$, one for each finite set I, given by $s_\varnothing = \mathrm{id}$ and

$$s_I = \sum_{\substack{(S_1,\ldots,S_k)\vDash I \\ k\geq 1}} (-1)^k \mu_{S_1,\ldots,S_k} \circ \Delta_{S_1,\ldots,S_k}$$

for $I \neq \varnothing$.

Note that the sum has finitely many terms since each part is non-empty and I is a finite set. Indeed, the number of terms is the ordered Bell numbers $\omega(n) \approx n!/2(\log 2)^{n+1}$; the first few terms are $1, 1, 3, 13, 75, 541, 4683, 47293, 545835$.

Remark 4.2. The preceding definition is referred to as *Takeuchi's formula*. The usual version is given for connected Hopf algebras, but this is a natural generalization. One may instead define the antipode categorically, where the most important axiom is the analogue of the inverse axiom written above for Hopf algebras. In any case, in examples we will encounter, antipodes are canonical but non-obvious maps arising from each connected Hopf monoids in vector species. This leads to the following fundamental question for this paper.

Problem 4.3 (Antipode problem)**.** Given a connected Hopf monoid in vector species, find a cancellation-free description for the antipode.

Example 4.4. For $\mathbf{L}$ and $I = \{a, b\}$, we have

$$s_{\{a,b\}} = (-1)^1\mu_{\{a,b\}} \circ \Delta_{\{a,b\}} + (-1)^2\mu_{\{a\},\{b\}} \circ \Delta_{\{a\},\{b\}} + (-1)^2\mu_{\{b\},\{a\}} \circ \Delta_{\{b\},\{a\}}.$$

Applied to $2ba$ this gives

$$s_{\{a,b\}}(2ba) = -2(ba) + 2ab + 2ba = 2ab$$

since

$$\Delta_{\{a\},\{b\}}(ba) = a \otimes b, \qquad \Delta_{\{b\},\{a\}}(ba) = b \otimes a.$$

4.2. **Basic properties.**

Proposition 4.5 (The antipode reverses products and coproducts)**.** *Let $\mathbf{H}$ be a connected Hopf monoid in vector species and $I = S \sqcup T$ a decomposition. Then for $x \in \mathbf{H}[S]$, $y \in \mathbf{H}[T]$, and $z \in \mathbf{H}[I]$, we have*

$$s_I(x \cdot y) = s_T(y) \cdot s_S(x)$$

and

$$\Delta_{S,T}(s_I(z)) = \sum s_S(z/_T) \otimes s_T(z|_T).$$

The right-hand side of the second equation should be interpreted as first applying $\Delta_{T,S}$, then applying the swap map, and then applying $s_S \otimes s_T$. This is a nice example of Sweedler notation being helpful.

Example 4.6. Consider again **L**. Takeuchi's formula says directly that $s_{\{a\}}(\lambda a) = -\lambda a$. Iterating the product formula and using the singleton base case shows that

$$s_I(a_1 \cdots a_n) = (-1)^n a_n \cdots a_1,$$

which solves the antipode problem for **L**. This shows that Takeuchi's formula can have utterly enormous amounts of cancellation.

5. Leftovers

5.1. Morphisms of Hopf monoids.

Definition 5.1. A *morphism* $f\colon H \to K$ between connected Hopf monoids in set species is a morphism of species which additionally preserves products, restrictions, and contractions; that is, we have

$$f_J(H[\sigma](x)) = K[\sigma](f_I(x)) \qquad \text{for all bijections } \sigma\colon I \to J \text{ and all } x \in H[I]$$
$$f_I(x \cdot y) = f_S(x) \cdot f_T(y) \qquad \text{for all } I = S \sqcup T \text{ and all } x \in H[S],\ y \in H[T]$$
$$f_S(z|_S) = f_I(z)|_S, \quad f_T(z/_S) = f_I(z)/_S \qquad \text{for all } I = S \sqcup T \text{ and all } z \in H[T].$$

(Units are preserved because of connectedness.)

A morphism in vector species is exactly analogous.

5.2. **A Fock functor.** Given a connected Hopf monoid H in set species, there is an associated (graded) Hopf algebra A which we next describe. [The current exposition on this is pretty rough, so I've modified it.] Let $H[n]$ denote $H[\{1, \ldots, n\}]$. Declare two elements $x, y \in H[n]$ to be "isomorphic" if there is a bijection $\sigma \in S_n$ such that $H[\sigma](x) = y$. First define the graded $\Bbbk$-vector space structure by $A := \oplus_{n \geq 0} A_n$ where A_n is the vector space with basis given by isomorphism classes of $H[n]$. Define a product and coproduct on A as follows. For $h_1 \in H[k_1]$, $h_2 \in H[k_2]$, and $h \in H[n]$, define

$$[h_1] \cdot [h_2] := [h_1 \cdot h_2'], \qquad \Delta([h]) := \sum_{[n] = S \sqcup T} [h|_S] \otimes [h/_S]$$

where h_2' is the image of h_2 under $H[\tau]$ where τ is the order-preserving bijection from $[k_2]$ to $k_1 + [k_2]$. Since H is connected, A comes with a natural unit $\eta\colon \Bbbk \to A$ given by $1 \mapsto 1 \in A_0$. It also has a natural counit $\epsilon\colon A \to \Bbbk$ given by $\epsilon(1) = 1$ and $\epsilon(A_n) = 0$ for $n \geq 1$. These operations give A the structure of a connected, graded Hopf algebra. The antipode of H is sent in a natural way to the antipode of A. Aguiar–Mahajan define four "Fock functors" accomplishing this sort of translation in great generality; see their book for more.