The Hilbert Scheme of Points in the Plane is Connected

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- Slogan: the geometry of moduli space reflects the structure of the underlying objects.

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- Ex: affine and projective varieties; "the intersection" of $y = x$ and $y = x^2$ (it "remembers multiplicity 2").

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- Such ideals form an open, dense subset of H_n. Can you think of any other points?

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- Idea 2: $V(\mathrm{LT}(I)) = \{(0,0)\}$; "spread out" vanishing locus to n distinct points.
- Idea 3: Can connect arbitrary $I(\{p_1, \ldots, p_n\})$ and $I(\{q_1, \ldots, q_n\})$ by "making the points coincide."

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- Original construction due to Grothendieck: [Gro62].

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