THREE EXAMPLES OF ALGEBRAIC GEOMETRY IN ALGEBRAIC COMBINATORICS

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Outline:

(1) Bruhat stuff, Schubert varieties, Coxeter group generalizations
(2) Schubert multiplication, positivity
(3) Smoothness and pattern avoidance
(4) Gamma positivity

Definition 1. The (complete) flag manifold is

$$\mathrm{Fl}(\mathbb{C}^n) := \{F_\bullet = F_1 \subset \cdots \subset F_n \subset \mathbb{C}^n : \dim_{\mathbb{C}} F_i = i\}.$$

$G := \mathrm{GL}(\mathbb{C}^n)$ acts on $\mathrm{Fl}(\mathbb{C}^n)$ with stabilizer $B_+ :=$ upper triangular matrices in $\mathrm{GL}(\mathbb{C}^n)$. Hence

$$\mathrm{Fl}(\mathbb{C}^n) = G/B_+ = \{gB_+\}_{g \in G}.$$

(as a set, or as a homogeneous space, i.e. a quotient of a Lie group).

Here the leftmost k columns of a matrix in G span F_k.

Question 2. This is asymmetric; what if we considered the *double cosets* $B_-\backslash G/B_+ := \{B_- g B_+\}_{g \in G}$? At $g = \mathrm{id}$, this is the set of matrices with an LU-decomposition ("almost all").

Fact 3. Such double cosets are determined by discrete *rank conditions*. Let $g_{[q,p]}$ denote the upper left rectangle of width p and height q in the matrix for g. Define

$$\mathrm{rk}\colon G \to M_{n\times n}(\mathbb{N})$$
$$(\mathrm{rk}\, g)_{(q,p)} := \mathrm{rank}_{\mathbb{C}}\, g_{[q,p]}.$$

Then the fibers of rk are precisely the double cosets above. Moreover, each such double coset contains a unique permutation $w \in S_n$. That is,

$$\mathrm{GL}(\mathbb{C}^n) = \coprod_{w \in S_n} B_- w B_+,$$

which is called the *Bruhat decomposition* for G. (It exists in significantly more generality.) Here we're embedding $S_n \hookrightarrow G$ by, for instance,

$$w = [25134] \mapsto \begin{pmatrix} & & 1 & & \\ 1 & & & & \\ & & & 1 & \\ & & & & 1 \\ & 1 & & & \end{pmatrix}$$

Question 4. What do these cells "look like" in $\mathrm{Fl}(\mathbb{C}^n) = G/B_+$?

Definition 5. The *open Schubert cell* associated to $w \in S_n$ is

$$\begin{aligned} X_w^\circ &:= B_- \cdot (wB_+) \subset G/B_+ \\ &= \{g \in \mathrm{GL}(\mathbb{C}^n) : \mathrm{rk}\, g = \mathrm{rk}\, w\}. \end{aligned}$$

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Example 6. If $w = [25134]$, then $F_\bullet \in X_w^\circ$ can be represented by $A \in \mathrm{GL}(\mathbb{C}^n)$ such that

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ * & 0 & * & 1 & 0 \\ * & 0 & * & * & 1 \\ * & 1 & 0 & 0 & 0 \end{pmatrix}$$

Here $*$ can be any element of $\mathbb{C}$, and all such choices correspond to distinct flags. Hence $X_{[25134]}^\circ \cong \mathbb{A}^6$.

Remark 7. The above example generalizes to any $w \in S_n$—place zeros right and up from each 1, and the remaining entries are unconstrained. Hence $X_w^\circ \cong \mathbb{A}^{\dim X_w}$. Moreover, we find

$$\begin{aligned} \dim X_w = \#\text{stars} &= \sum_{i=1}^n \#\text{1's below, right of the 1 at } (w_i, i) \\ &= \sum_{i=1}^n \#\{j > i : w_j > w_i\} = \#\text{“non-inversions” } (i,j) \text{ s.t. } i < j,\, w_i < w_j \\ &= \binom{n}{2} - \#\text{“inversions” } (i,j) \text{ s.t. } i < j,\, w_i > w_j \end{aligned}$$

Aside 8. (As an example of $\operatorname{inv} w$, one can show $\#\operatorname{inv} uv \equiv_2 (\#\operatorname{inv} u)(\#\operatorname{inv} v)$. Deduce $\operatorname{sgn}\colon S_n \to \mathbb{Z}/2$ is well-defined, and use this to define $\det(x_{ij}) := \sum_{w \in S_n} (-1)^{\operatorname{sgn} w} \prod_i x_{w_i}\cdot$.)

Definition 9. Let $X_w := \overline{X_w^\circ}$ (the Zariski or Euclidean closure) be a *Schubert variety.*

Theorem 10. *We have*
(a) $\operatorname{codim} X_w^\circ = \#\operatorname{inv} w$.
(b) $X_w = \{\operatorname{rk} A \leq \operatorname{rk} w\}/B_+$ *(compare ranks pointwise)*
(c) $X_w = \coprod_{\operatorname{rk} v \leq \operatorname{rk} w} X_v^\circ$ *forms an affine stratification of* $\mathrm{Fl}(\mathbb{C}^n)$
(d) X_w *is an integral, Cohen-Macaulay variety.*
(e) $X_u \supseteq X_v \Leftrightarrow u \leq v$.

Warning: there are (a dihedral group of order) 8 “natural” ways to reindex the X_w, so there are many conflicting conventions and definitions in the literature.

Example 11. We find

$$\operatorname{rk}\mathrm{id} = \begin{pmatrix} 1 & 1 & 1 & \cdots \\ 1 & 2 & 2 & \cdots \\ 1 & 2 & 3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad \operatorname{rk}(w_0 \in S_3) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

where $w_0 = [n(n-1)\cdots 21] \in S_n$. One can check $\operatorname{rk}\mathrm{id}$ is the maximum of $\operatorname{rk} S_n$ and $\operatorname{rk} w_0$ is the minimum. Hence $X_{\mathrm{id}} = \mathrm{Fl}(\mathbb{C}^n)$. Indeed, since $\#\operatorname{inv} w_0 = \binom{n}{2} = \dim \mathrm{Fl}(\mathbb{C}^n)$, X_{w_0} is a single point (flag).

Definition 12. For $u, v \in S_n$, set $u \leq v$ if and only if $\operatorname{rk} u \geq \operatorname{rk} v$ pointwise. This is *Bruhat order* on S_n. (Switching the order is motivated by a desire to have minimum id, maximum w_0.)

We next compare Bruhat order and $\operatorname{codim} X_w$ to standard Coxeter group theory, which provides a significant generalization.

Definition 13. Let $\mathcal{G}$ be a graph with vertices $1, 2, \ldots, n$, and optional edge labels from $\{4, 5, 6, \ldots\} \cup \{\infty\}$. (Draw picture of such a graph; include two components, an unlabeled edge, edges labeled 4, 6, and ∞.) This is a *Coxeter diagram.*

Given such a $\mathcal{G}$, set

$$m_{\alpha\beta} := \begin{cases} 1 & \text{if } \alpha = \beta \\ 2 & \text{if } \alpha \neq \beta \text{ not connected} \\ 3 & \text{if } \alpha \neq \beta \text{ connected, unlabeled} \\ k & \text{if } \alpha \neq \beta \text{ labeled by } k \end{cases}$$

and define the *Coxeter group* of $\mathcal{G}$ by

$$W := \langle s_\alpha : (s_\alpha s_\beta)^{m_{\alpha\beta}} = 1 \rangle$$

(where $(s_\alpha s_\beta)^\infty = 1$ is interpreted as no constraint).

Example 14. If $m_{\alpha\beta} = 2$, then $s_\alpha s_\beta = s_\beta s_\alpha$. Hence W is the product of the Coxeter groups of the connected components of $\mathcal{G}$.

Theorem 15. *W is finite if and only if the connected components of $\mathcal{G}$ all come from one of the four infinite families A_n (linear), $B_n = C_n$ (linear, last edge labeled 4), D_n (linear with a fork on the end), $I_2(m)$ (two vertices connected by an edge labeled m), or one of six other exceptions.*

Homework 16. Show $I_2(m)$ gives the dihedral group of order $2m$.

Example 17. Type A_{n-1} gives

$$W = \langle s_1, \dots, s_{n-1} : s_i^2 = 1, s_i s_j = s_j s_i \ \forall |i-j| > 1, s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \rangle \xrightarrow{\sim} S_n$$
$$s_i \mapsto (i, i+1).$$

Definition 18. A *reduced expression* for $w \in W$ is one of the form $w = s_{i_1} \cdots s_{i_r}$ for r minimal. We say $\ell(w) := r$ is the *length* of w.

Example 19. For $w \in S_n$, we have $\ell(w) = \#\operatorname{inv} w$.

Fact 20. For general W, the "braid relations" (those coming from $m_{\alpha\beta} \geq 2$) connect all reduced words for a particular w.

Definition 21. For $u, v \in W$, set $u \leq v$ iff there is a reduced expression for u which is a subexpression for a reduced expression for v. This is *Bruhat order* on W.

Example 22. Draw the Bruhat graph of S_3; label the vertices by e.g. $s_1 s_2$.

Fact 23. We have

- This agrees with the earlier Bruhat order for S_n
- The poset $(W, \leq)$ is ranked by ℓ, i.e. every poset-theoretic covering relation increases length by 1.
- There are reasonably efficient tests for determining if $u \leq v$ in S_n.

Corollary 24. *Every X_u is a codimension-1 subvariety of some X_v (for $u \neq \mathrm{id}$).*

We now turn to Schubert multiplication. By the affine stratification of $\mathrm{Fl}(\mathbb{C}^n)$ above, it follows that the Chow ring $A^\cdot(\mathrm{Fl}(\mathbb{C}^n))$ has a $\mathbb{Z}$-basis given by $[X_w]$'s.

Question 25. What is the Poincare polynomial of $A^\cdot(\mathrm{Fl}(\mathbb{C}^n))$? Since $\deg[X_w] = \operatorname{codim} X_w = \ell(w)$, it's

$$\sum_{i \geq 0} q^i \dim A^i(\mathrm{Fl}(\mathbb{C}^n)) = \sum_{w \in S_n} q^{\deg[X_w]} = \sum_{w \in S_n} q^{\ell(w)}$$
$$= [n]_q! = [n]_q [n-1]_q \cdots [1]_q$$

where $[k]_q = 1 + q + \cdots + q^{k-1} = \frac{q^k - 1}{q - 1}$. The second-to-last equality is a classical result (set $x_i = q$ in the natural Inv generating function). MacMahon famously showed the same statment holds if $\ell(w)$ is replaced by the "major index" of w.

Question 26. What is the ring structure of $A^\cdot(\mathrm{Fl}(\mathbb{C}^n))$? More precisely, let $s_w := [X_w]$. We have

$$s_u s_v = \sum_{w \in S_n} c_{uv}^w s_w.$$

What are the structure constants $c_{uv}^w \in \mathbb{Z}$?

Example 27. We have $s_{\mathrm{id}} s_w = [X_{\mathrm{id}} \cap X_w] = [X_w] = s_w$, so $s_{\mathrm{id}} = 1$ and $c^w_{\mathrm{id},v} = \delta_{vw}$.

Theorem 28. $c^w_{uv} \geq 0$. *Moreover, c^w_{uv} counts the number of points in a certain "generic" intersection of three Schubert varieties.*

Proof. If $\ell(u) = \ell(v)$, then one may check $s_u s_{w_0 v} = \delta_{uv} s_{w_0}$ geometrically using "opposite" flags. By grading, $c^w_{uv} = 0$ unless $\ell(u) + \ell(v) = \ell(w)$. Hence

$$s_u s_v s_{w_0 w} = \sum_{\substack{w' \in S_n \\ \ell(w') = \ell(u) + \ell(v)}} c^w_{uv} s_{w'} s_{w_0 w} = c^w_{uv} s_{w_0}.$$

□

Remark 29. For certain u, v, w, we have

$$\begin{aligned} c^w_{uv} &= c^\lambda_{\mu\nu} \\ &= \text{the classical Littlewood-Richardson coefficients} \\ &= \#\text{semi-standard Young tableaux of shape } \lambda/\mu, \text{ weight } \nu, \text{ Yamanouchi.} \end{aligned}$$

We next define the Schubert polynomials $\mathfrak{S}_w \in \mathbb{Z}[x_1, x_2, \ldots]$ for $w \in S_\infty := \cup_{i \geq 1} S_i$ where $S_i \hookrightarrow S_{i+1}$ is given by fixing $i + 1$.

Definition 30. Define $\partial_i \in \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}[x_1, x_2, \ldots])$ by $\partial_i f := \frac{f - s_i \cdot f}{x_i - x_{i+1}}$ where $s_i \cdot f$ means to interchange x_i, x_{i+1}.

Fact 31. $\partial_i^2 = 0$, $\partial_i \partial_j = \partial_j \partial_i$ if $|i - j| > 1$, and $\partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1}$.

For a reduced expression $w = s_{i_1} \cdots s_{i_{\ell(w)}}$, define

$$\partial_w := \partial_{i_1} \cdots \partial_{i_{\ell(w)}}.$$

This is well-defined since reduced words are connected by braid moves. Now define the *Schubert polynomials*

$$\begin{aligned} \mathfrak{S}_{w_0} &:= x_1^{n-1} x_2^{n-2} \cdots x_{n-1}^1 \qquad (w_0 \in S_n) \\ \mathfrak{S}_w &:= \partial_{w^{-1} w_0} \mathfrak{S}_{w_0} \qquad (w \in S_n) \end{aligned}$$

(This indeed respects the inclusions $S_n \hookrightarrow S_{n+1}$.)

Theorem 32. *(Lascoux-Schutzenberger, Bernstein-Gelfand-Gelfand, Demazure) Let "$A^\cdot(\mathrm{Fl}(\mathbb{C}^\infty))$" denote the "limiting ring" of the $A^\cdot(\mathrm{Fl}(\mathbb{C}^n))$'s. Then*

$$\begin{aligned} A^\cdot(\mathrm{Fl}(\mathbb{C}^\infty)) &\xrightarrow{\sim} \mathbb{Z}[x_1, x_2, \ldots] \\ s_w &\mapsto \mathfrak{S}_w \end{aligned}$$

is a ring isomorphism. (The x_i's in the above are essentially Chern roots. For many more details and a significant generalization in a readable account, see Fulton's "Flags, ..., Degeneracy Loci" (1991).)

Open Problem 33. By the theorem, $\mathfrak{S}_u \mathfrak{S}_v = \sum_{w \in S_\infty} c^w_{uv} \mathfrak{S}_w$ where $c^w_{uv} \geq 0$. A large open problem is to show $c^w_{uv} \geq 0$ "combinatorially", so for instance to interpret c^w_{uv} as counting combinatorially defined objects like $c^\lambda_{\mu\nu}$ does.

We turn to our third example:

Question 34. When is X_w singular?

Theorem 35. *(Lakshmibai-Sandhya, 1990) X_w is smooth if and only if w avoids* 3412 *and* 4231. *(This occurs if and only if the Poincare polynomial of X_w, $\sum_{v \leq w} q^{\ell(v)}$, is palindromic. For this and more, see Billey-Abe (2013).)*

Proof. (Sketch.) (See Manivel, "Symmetric ..., Degeneracy Loci", Theorem 3.7.5.)
(1) The singular locus of X_w is a union of some X_u's with $u \leq w$.
(2) One can describe $I(X_w)$ sufficiently well to apply the Jacobian criterion.

(3) By combinatorially computing tangent space dimensions, one gets

$$X_w \text{ is singular on } X_v \ (v \leq w) \Leftrightarrow \ell(w) < \#\{i < j : vt_{ij} \leq w\}$$
$$X_w \text{ is smooth} \Leftrightarrow \ell(w) = \#\{i < j : t_{ij} \leq w\}$$

(where t_{ij} interchanges i and j).

(4) Do a combinatorial analysis using the last condition to deduce the result.

□

We end with an oddity of a much different flavor.

Definition 36. Define the *two-sided Eulerian polynomials*

$$A_n(s,t) := \sum_{w \in S_n} s^{\#\operatorname{Des}(w^{-1})+1} t^{\#\operatorname{Des}(w)+1}$$

where the (one-sided) Eulerian polynomials are $A_n(t) := A_n(1,t)$.

Fact 37. (1) $A_n(t) = \sum_{i=1}^{\lceil n/2 \rceil} \gamma_{i;n} t^i (1+t)^{n+1-2i}$ for unique $\gamma_{i;n} \in \mathbb{Z}$;
(2) $\gamma_{i;n} \geq 0$
(3) There exists a *very nice* combinatorial proof of (2);
(4) $A_n(s,t) = \sum_{i,j} \gamma_{i,j;n} (st)^i (s+t)^j (1+st)^{n+1-j-2i}$ for unique $\gamma_{i,j;n} \in \mathbb{Z}$;
(5) $\gamma_{i,j;n} \geq 0$
(6) There currently only exists a recursive, in some sense unenlightening proof of (5) (see recent work of Zhicong Lin).

((5) is often called Gessel's conjecture.)

Open Problem 38. Is there a combinatorial proof of (5)? Is there a *geometric* proof?

Note to self: this was a bit long (approximately 70 minutes) and generally was too full of content. Go at a slower pace; probably cut out the pure Coxeter group theory section and the precise Schubert polynomial definition (though keep positivity) and go more leisurely through everything, especially the examples. Also, add a few more words on the Jacobian criterion and what $I(X_w)$ really means in this context.