REFINED CYCLIC SIEVING ON WORDS

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Outline.

(1) CSP on words, refinement
(2) Main result, CCSP
(3) Connections, higher Lie multiplicities

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1. Setup

Definition 1 (Reiner-Stanton-White '04)**.** Triple $(W, C_n, f(q))$ where W is a finite set with a C_n-action, $f(q) \in \mathbb{N}[q]$ exhibits cyclic sieving phenomenon (CSP) if for all $k \in \mathbb{Z}$,

$$f(\omega_n^k) = \chi^W(\sigma_n^k).$$

(ω_n primitive nth root of unity; $C_n = \langle \sigma_n \rangle$ cyclic of order n; χ^W character of C_n-action, say over $\mathbb{C}$.)

Example 2. Let $\sigma_n = (1\ 2\ \cdots\ n) \in S_n$; $W = [n]$; $f(q) = \frac{q^n-1}{q-1} = 1 + q + \cdots + q^{n-1}$. By L'Hopital,

$$f(\omega_n^k) = \begin{cases} 0 & k \not\equiv_n 0 \\ n & k \equiv_n 0 \end{cases}.$$

Matrix of σ_n is

$$\begin{pmatrix} & & & 1 \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{pmatrix}$$

so that

$$\chi^{C_n}(\sigma_n^k) = \mathrm{Tr}(\cdots^k) = \begin{cases} 0 & k \not\equiv_n 0 \\ n & k \equiv_n 0 \end{cases}.$$

Hence $([n], C_n, \frac{q^n-1}{q-1})$ exhibits the CSP.

Remark 3. Alternatively, we could replace the defining condition with

$$f(q) \equiv \sum_{\text{orbits } \mathcal{O} \subset W} \frac{q^n - 1}{q^{n/|\mathcal{O}|} - 1} \qquad (\mathrm{mod}\ q^n - 1).$$

Here $n/|\mathcal{O}|$ is the order of the stabilizer of any element of $\mathcal{O}$.

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Notation 4. Given stat: $W \to \mathbb{N}$, write

$$W^{\mathrm{stat}}(q) := \sum_{w \in W} q^{\mathrm{stat}(w)}.$$

Theorem 5 (RSW, Thm. 1.6)**.** *Let (W, S) be a finite Coxeter system, $J \subset S$, C a cyclic subgroup generated by a regular element of W, and W^J the set of minimal length representatives of cosets W/W_J. Then*

$$(W/W_J, C, (W^J)^{\mathrm{inv}}(q))$$

exhibits the CSP.

Notation 6. The *content* of a word:

$$\mathrm{cont}(132553) = (1, 1, 2, 0, 2) \vDash 6.$$

Set

$$W_\alpha := \{\text{words from } \mathbb{P} \text{ of content } \alpha\}$$

and

$$\mathrm{maj}(\text{a word}) = \text{sum of its descents}$$

e.g.

$$\mathrm{maj}(13.255.3) = 2 + 5 = 7.$$

Action on W_α for $\alpha \vDash n$:

$$\sigma_6 \cdot 132553 = 313255.$$

Corollary 7 (RSW)**.** *Let $\alpha \vDash n$. Then $(W_\alpha, C_n, W_\alpha^{\mathrm{maj}}(q))$ exhibits the CSP. (MacMahon: $W_\alpha^{\mathrm{maj}}(q) = \binom{n}{\alpha}_q = W_\alpha^{\mathrm{inv}}(q)$.)*

Remark 8. W_α is the S_n-orbit of a single word of content α. Consequently, $(W, C_n, W^{\mathrm{maj}}(q))$ exhibits the CSP whenever W is the union of S_n-orbits of words and C_n acts by $(1\ 2\ \cdots\ n)$.

Definition 9. A *refinement* of a CSP triple $(W, C_n, W^{\mathrm{stat}}(q))$ is a CSP triple $(V, C_n, V^{\mathrm{stat}}(q))$ where $V \subset W$ has the restricted C_n-action.

Notation 10. Write

$$W_{\alpha,\delta} := \{\text{words from } \mathbb{P} \text{ with content } \alpha \text{ and “cyclic descent type” } \delta\}.$$

Theorem 11 (Ahlbach-S. 2016)**.** *$(W_{\alpha,\delta}, C_n, W_{\alpha,\delta}^{\mathrm{maj}}(q))$ refines $(W_\alpha, C_n, W_\alpha^{\mathrm{maj}}(q))$.*

Remark 12. Switching from inv to maj is essential. For instance, when $\alpha = (2, 2)$, $\delta = (0, 2)$, we get $W_{\alpha,\delta} = \{1212, 2121\}$, so that

$$W_{\alpha,\delta}^{\mathrm{maj}}(q) = q^2 + q^4,$$

and

$$W_{\alpha,\delta}^{\mathrm{inv}}(q) = q^1 + q^3.$$

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2. Main result

Definition 13. *Cyclic descent type* (CDT) by example: say $w = 143124114223$, so $\mathrm{cont}(w) = (4, 3, 2, 3) \vDash 12$. Compute:

$$\begin{aligned} w^{(1)} &= 1111 & \mathrm{cdes}\, w^{(1)} &= 0, \\ w^{(2)} &= 112.1122. & \mathrm{cdes}\, w^{(2)} &= 2, \\ w^{(3)} &= 13.12.11223. & \mathrm{cdes}\, w^{(3)} &= 3, \\ w^{(4)} &= 14.3.124.114.223. & \mathrm{cdes}\, w^{(4)} &= 5. \end{aligned}$$

Hence, $\mathrm{CDT}(143124114223) = (0, 2 - 0, 3 - 2, 5 - 3) = (0, 2, 1, 2)$.

Theorem 14 (Ahlbach-S. 2016)**.** *Let $\alpha = (\alpha_1, \ldots, \alpha_m) \vDash n$ be a strong composition, $\delta = (0, \delta_2, \ldots, \delta_m) \vDash k$. Set $n_i := \alpha_1 + \cdots + \alpha_i$, $k_i := \delta_1 + \cdots + \delta_i$. Then:*

$$W^{\mathrm{maj}}_{\alpha,\delta}(q) \equiv \frac{d}{\alpha_1} \frac{q^n - 1}{q^d - 1} q^{\binom{k}{2} + \sum_{i=2}^m \binom{\delta_i}{2} - \alpha_1} \prod_{i=2}^m \binom{n_{i-1} - k_{i-1}}{\delta_i}_q \binom{k_i + (\alpha_i - \delta_i) - 1}{\alpha_i - \delta_i}_q$$

modulo $q^n - 1$, where $d := \gcd(n, k)$.

Remark 15. At $q = 1$, we get $|W_{\alpha,\delta}|$. The modulo $q^n - 1$ is essential; it doesn't factor nicely without it. (Though it does split up as the sum of things that factor nicely.)

Corollary 16. *$W_{\alpha,\delta} \neq \varnothing$ if and only if*

$$\begin{aligned} 0 \le \delta_i &\le \alpha_i && \text{for } 1 \le i \le m \\ \delta_1 + \cdots + \delta_{i+1} &\le \alpha_1 + \cdots + \alpha_i && \text{for } 1 \le i < m. \end{aligned}$$

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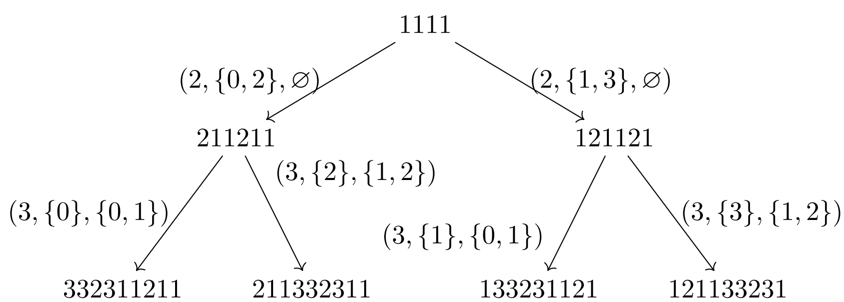
3. Proof Highlights

Remark 17. Tree decomposition of $W_{\alpha,\delta}$, by example. Let $\alpha = (4, 2, 3)$ and $\delta = (0, 2, 1)$. The following is the subgraph of $T_{\alpha,\delta}$ consisting of paths from the root to leaves that are rotations of 311211332:



The full tree $T_{\alpha,\delta}$ has edges (i, F, R) with

$$F \in \binom{[0, n_i - k_i - 1]}{\delta_{i+1}}, \qquad R \in \left(\!\!\binom{[0, k_{i+1} - 1]}{\alpha_{i+1} - \delta_{i+1}}\!\!\right)$$

and leaves in

$$\{w \in W_{\alpha,\delta} : w \text{ ends in } 1\}.$$

For this full $T_{\alpha,\delta}$, the root has $\binom{4}{2} = 6$ children since 1111 has 4 falls. Each child of the root itself has $\binom{4}{1}\left(\!\binom{2}{2}\!\right) = 12$ children. Hence, $T_{\alpha,\delta}$ has 72 leaves.

Remark 18. The product formula follows from the tree decomposition relatively easily:

- compute changes in maj upon traversing edges;
- recognize $\left(\!\binom{n}{k}\!\right)_q = \left(\!\binom{[0,n-1]}{k}\!\right)^{\mathrm{sum}}(q)$, etc.

Product formula suggests comparing C_n action on $W_{\alpha,\delta}$ to some cyclic action on

$$\prod_{i=2}^m \binom{[0, n_{i-1} - k_{i-1} - 1]}{\delta_i} \times \left(\!\!\binom{[0, k_i - 1]}{\alpha_i - \delta_i}\!\!\right).$$

Let

$$g := \gcd(\alpha_1, \ldots, \alpha_m, \delta_1, \ldots, \delta_m).$$

- C_g acts on Cartesian product diagonally;
- Tree essentially encodes a bijection from $W_{\alpha,\delta}$ to this product; can check that stabilizer sizes are preserved under this bijection.
- $(W_{\alpha,\delta}, C_g, W^{\mathrm{maj}}_{\alpha,\delta}(q))$ CSP.
- Need to extend CSP from C_g to C_n.

Definition 19 (*Modular Periodicity*)**.** We say $\mathrm{stat}\colon W \to \mathbb{Z}$ has *period a modulo b* if for all $i \in \mathbb{Z}$,

$$\#\{w \in W : \mathrm{stat}(w) \equiv_b i\} = \#\{w \in W : \mathrm{stat}(w) \equiv_b i + a\}.$$

Similarly, $f(q) \in \mathbb{C}[q]$ has *period a modulo b* if

$$q^a f(q) \equiv f(q) \qquad (\mathrm{mod}\ q^b - 1).$$

Lemma 20. *Period a modulo b and period b modulo c implies period a modulo c.*

Corollary 21. *$W^{\mathrm{maj}}_{\alpha,\delta}(q)$ has period g modulo n.*

Lemma 22. *Suppose C_n acts on W, $g \mid n$, $C_g \subset C_n$. If*

(i) $(W, C_g, f(q))$ exhibits the CSP;
(ii) $f(q)$ has period g modulo n;
(iii) and for all orbits $\mathcal{O} \subset W$, $\frac{n}{|\mathcal{O}|} \mid g$,

then $(W, C_n, f(q))$ exhibits the CSP.

Remark 23. Main result follows from lemma. Condition (iii) is that $\mathrm{freq}(N)$ divides each α_i and δ_i, hence also g.

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4. NSP

A key building block of the preceding argument is:

Theorem 24 (RSW, Thm. 1.1)**.** *Let C_n act on $\binom{[n]}{k}$ and $\left(\!\binom{[n]}{k}\!\right)$ by rotation. Then*

$$\left(\binom{[n]}{k}, C_n, \binom{n}{k}_q\right)$$

and

$$\left(\left(\!\!\binom{[n]}{k}\!\!\right), C_n, \left(\!\!\binom{n}{k}\!\!\right)_q\right)$$

exhibit the CSP.

Remark 25. The first is the $\alpha = (k, n-k)$ case of RSW's result. Can give straightforward derivations of both using explicit evaluations at roots of unity, or using some simple representation theory involving kth symmetric or exterior powers of $\mathbb{C}^n$. (See Sagan's CSP survey for a very nice, gentle exposition.) We give a purely combinatorial proof.

Definition 26. Let $(W, \mathcal{S}_n, f(q))$ where W is a finite set, $f(q) \in \mathbb{N}[q]$, and $\mathcal{S}\colon W \to \{r \in [n] : r \mid n\}$ is some function. This triple exhibits the *numerical sieving phenomenon* (NSP) if:

$$f(q) \equiv \sum_{r \mid n} \#\mathcal{S}^{-1}(r) \cdot \frac{r}{n} \frac{q^n - 1}{q^r - 1} \qquad (\mathrm{mod}\ q^n - 1).$$

Remark 27.

- A straightforward computation shows this formula is equivalent to the orbit-sum CSP formula when $\mathcal{S}(w)$ is the order of the stabilizer of $w \in W$.
- However, the NSP has no obvious representation theoretic interpretation in general–there need not be a cyclic action.

Definition 28. Let $r \mid n$. The *r-intervals of* $[n]$ are

$$\{1, \ldots, r\}, \{r+1, \ldots, 2r\}, \ldots =: I_r^1, I_r^2, \ldots.$$

For $A \in \binom{[n]}{k}$, let

$$\operatorname{int}(A) := \text{largest } r \mid n \text{ such that } \#(A \cap I_r^j) \in \{0, r\}, \text{ for all } j.$$

Set

$$G_{a,b} := \{A \in \binom{[n]}{k} : \gcd(a, \#(A \cap I_a^1), \cdots, \#(A \cap I_a^{n/a}) = b\}.$$

Theorem 29 (Ahlbach-S. 2016)**.** *Pick $d_{p+1} \mid d_p \mid \cdots \mid d_0 = \gcd(n,k)$. Let*

$$G := G_{n,d_0} \cap G_{d_0,d_1} \cap \cdots \cap G_{d_p,d_{p+1}} \subset \binom{[n]}{k}.$$

Then

$$(G, \operatorname{int}_{d_p}, G^{\mathrm{sum}'}(q))$$

exhibits the NSP. (Here sum' *is subset sum, shifted to have minimum* 0*.)*

Lemma 30.

$$\#\{A \in \binom{[n]}{k} : \mathcal{S}(A) = r\} = \#\{A \in \binom{[n]}{k} : \operatorname{int}(A) = r\}.$$

Corollary 31. *The NSP above is equivalent to the CSP for $G = G_{n,\gcd n,k} = S$. However, this fails in general!*

Remark 32. Similar manipulations can be done for multisubsets to prove the second half of the theorem.

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5. Flex, Bijectivity

Remark 33. Original motivation: find bijective proof of corollary of results due to Kraskiewicz-Weyman, Klyachko (among others):

$$\#\{\text{primitive necklaces of content } \alpha \vDash n\} = \#\{\text{words of content } \alpha \text{ and maj} \equiv_n 1\}.$$

Definition 34. For $w \in W_\alpha$, let

$$\operatorname{flex}(w) := \text{position of } w \text{ in } \textit{multiset } C_n \cdot w.$$

For example, if $w = 551551$, we have

$$\{\!\{551551, 155155, 515515, 551551, 155155, 515515\}\!\}$$

which is ordered in "sports ranking" as

$$\{\!\{4, 0, 2, 4, 0, 2\}\!\}$$

so $\operatorname{flex}(551551) = 4$. ("Frequency times lex.")

Remark 35. flex is a "universal" sieving statistic in the following sense. Let $\mathcal{O}$ be an orbit of length n words under rotation. Then $(\mathcal{O}, C_n, \mathcal{O}^{\mathrm{flex}}(q))$ exhibits the CSP. It is the "most refined" possible CSP statistic.

Remark 36. Generalization of previous result: for all $r \in [0, n-1]$,

$$\#\{\text{words of content } \alpha, \text{ flex} = r\} = \#\{\text{words of content } \alpha \vDash n, \text{ maj} \equiv_n r\}.$$

Corollary 37. flex *is equidistributed with* maj *mod n on $W_{\alpha,\delta}$ if and only if $(W_{\alpha,\delta}, C_n, W_{\alpha,\delta}^{\mathrm{maj}}(q))$ exhibits the CSP.*

Open Problem 38. Find bijections $\Phi\colon W_{\alpha,\delta} \to W_{\alpha,\delta}$ such that

$$\mathrm{maj}(w) \equiv_n \mathrm{flex}(\Phi(w)).$$

Remark 39. Our argument is bijective in many places (e.g. the tree decomposition), but not for instance when applying modular periodicity transitivity. We also use the extended Euclidean algorithm in NSP approach, which is non-canonical.

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6. Future work

This was actually a piece of larger work on the so-called higher Lie multiplicities. An argument due to Klyachko (generalized and rephrased) shows that

$$\sum_{i=0}^{n-1} \text{Schur character of } \chi^i\!\uparrow_{C_n}^{S_n} q^i = \sum_{\alpha \vDash n} \sum_{w \in W_\alpha} q^{\mathrm{flex}\, w} x^\alpha.$$

Flex falls out relatively naturally from this argument. Kraskiewicz-Weyman connected these induced characters to coinvariant algebras and showed

$$\sum_{\lambda \vdash n} \sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}_n T} s_\lambda$$

where $0 \leq \mathrm{maj}_n < n$ is maj modulo n. That flex and maj_n are equidistributed gives a simple equivalence between these formulas using RSK.

Remark 40.

- A solution to the open problem above would give a "bijective" proof of Kraskiewicz-Weyman's result.
- Intuition: flex gives m-expansion, maj_n gives s-expansion, and CSP+RSK allows us to "straighten" from m-basis to s-basis.
- Suggests an attack on higher Lie multiplicities. We've introduced a notion of "group sieving" and identified the analogue of flex for $C_a \wr S_b \hookrightarrow S_{ab}$.
- Also explored this essentially combinatorial approach to higher Lie multiplicities by looking at e.g. 1-dimensional induced $C_a \wr S_b \hookrightarrow S_{ab}$-modules.
- Generalized several results in this setting, e.g. Schocker's formula, graded Frobenius character expansion, some p-expansions.
- Current work is focused on finding an analogue of maj_n in an effort to compute more higher Lie characters. Ongoing!

Thanks for your attention and the invite!