N! CONJECTURE SEMINAR

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Abstract. These are lecture notes for a seminar organized around Mark Haiman's 2001 proof [Hai01] of the Macdonald Positivity Conjecture. It was held at the University of Washington during Spring and Summer, 2016.

Contents

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1. Milieu: So many symmetric functions, q, t-Kostka polynomials, and graded representations.

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Our first objective is to summarize some of the story behind Haiman's proof [Hai01] of the Macdonald Positivity Conjecture [Mac95, "(8.18?)"]. Our goals include defining $K_{\lambda\mu}$, $K_{\lambda\mu}(t)$, $K_{\lambda\mu}(q,t)$, and associated symmetric functions. Along the way we give representation-theoretic interpretations in the 0- and 1-parameter cases and discuss a "proof prototype" due to Garsia-Procesi [GP92] which in many ways mirrors Haiman's proof. We conclude by stating the 2-parameter interpretation conjectured by Garsia-Haiman [GH96], along with the so-called $n!$ conjecture it was reduced to.

1.1. Symmetric Functions Summary.

We begin with an unavoidable slog through standard notation and concepts. One goal is to illustrate how relevant properties of symmetric functions arise immediately and naturally from representation theory. [Mac95] is a standard reference with an absolute wealth of information, though [Sag91], [Sta99], and [Ful97] are more readable.

We write $\alpha \vDash n$ to mean $\alpha = (\alpha_1, \ldots, \alpha_k)$ is a *weak composition* of n, namely $\alpha_i \geq 0$, $\sum_i \alpha_i = n$. Also write $\mu \vdash n$ to mean $\mu = (\mu_1, \ldots, \mu_k)$ is an *(integer) partition* of n, namely $\mu_1 \geq \mu_2 \geq \cdots \geq 0$, $\sum_i \mu_i = n$. We imagine μ as a subset of lower-left justified boxes in the plane where μ_i denotes the number of boxes in row i. One major source of partitions comes from the conjugacy classes of the symmetric group—the cycle type of such a class is simply a partition which records the cycle lengths.

S_n acts naturally on $R[x_1, \ldots, x_n]$ by $\sigma \cdot x_i = x_{\sigma(i)}$. We're interested in symmetric polynomials, so elements of $R[x_1, \ldots, x_n]^{S_n}$. Some nice ones:

- Set $x^\alpha := x_1^{\alpha_1} \cdots x_k^{\alpha_k}$. Take $m_\alpha := \sum_{\beta \sim \alpha} x^\beta$ to be the *monomial symmetric polynomials.* Obviously $\{m_\lambda\}_{\lambda \vdash m}$ is an R-basis for the degree m component of $R[x_1, \ldots, x_n]^{S_n}$. This is the "most obvious"/"most trivial" R-linear basis, but expanding products $m_\mu m_\nu$ back in this basis is awful/rarely done.
- Let $h_i := \sum_{\alpha \vDash i} x^\alpha$ (where α has at most n parts) and take $h_\mu := \prod_i h_{\mu_i}$ to be the *complete homogeneous symmetric polynomials.* Not obviously an R-basis, but products are trivial.
- $e_\mu := \prod_i e_{\mu_i}$ where

$$\sum_{i=0}^{n} e_i(x_1, \ldots, x_n) t^{n-i} := \prod_{j=1}^{n} (t + x_j)$$

are the *elementary symmetric polynomials.*

- $p_\mu := \prod_i p_{\mu_i}$ where

$$p_i(x_1, \ldots, x_n) = \sum_{j=1}^n x_j^i$$

  are the *power-sum symmetric polynomials.*
- s_μ, the *Schur polynomials*, discussed shortly.

Fact 1.1. Letting μ vary over "certain" partitions, each of these forms an R-basis for $R[x_1, \ldots, x_n]^{S_n}$ (for p_i's, need $\operatorname{char} R = 0$). Particularly, $\{h_1, \ldots, h_n\}$ is thus an algebraic basis for $R[x_1, \ldots, x_n]^{S_n}$. But why? We'll next give a quick representation-theoretic motivation which also constructs the s_μ.

Example 1.2. Let $V := \mathbb{C}^n$ have basis $\{v_1, \ldots, v_n\}$. Let $\mathrm{GL}(V)$ act on $V^{\otimes k}$ by linear substitutions. This descends to an action of $\mathrm{GL}(V)$ on $\mathrm{Sym}^k V$, the kth symmetric power of V, so we have a nice finite-dimensional $\mathrm{GL}(V)$-representation. The *Schur character* of such a representation is the function which records the traces of the actions of diagonal matrices, which we now compute.

A basis for $\mathrm{Sym}^k(V)$ is

$$\{v_{i_1} \cdots v_{i_k} : 1 \le i_1 \le \cdots \le i_k \le n\} =: \{v^\alpha : \alpha \vDash n \text{ has } k \text{ parts}\}$$

(where α_ℓ records the number of times $i_j = \ell$). Now

$$\mathrm{diag}(x_1, \ldots, x_n) v_{i_1} \cdots v_{i_k} = x_1 \cdots x_n (v_{i_1} \cdots v_{i_n}) = x^\alpha v^\alpha,$$

so that

$$\mathrm{Tr}_{\mathrm{Sym}^k(V)}(\mathrm{diag}(x_1, \ldots, x_n)) = \sum_{\alpha \vDash k} x^\alpha = h_k(x_1, \ldots, x_n).$$

We summarize this by saying $\operatorname{ch} \mathrm{Sym}^k(V) = h_k$. Since characters are multiplicative under tensoring, we also have

$$\mathrm{Sym}^\mu(V) := \otimes_k \mathrm{Sym}^{\mu_k}(V)$$
$$\operatorname{ch} \mathrm{Sym}^\mu(V) = h_\mu(x_1, \ldots, x_n).$$

Aside 1.3. Under Schur-Weyl duality, Sym^μ is equivalent to inducing the trivial character of S_μ to $S_{|\mu|}$, which is the same as the natural representation on the parabolic quotient $S_\mu / S_{|\mu|}$. An analogous construction with exterior powers yields e_μ.

Remark 1.4. Why are the h_μ a basis? The reason runs through the irreducible (polynomial, complex) representations of $\mathrm{GL}(V)$. As it turns out, the Sym^μ's are almost never irreducible. One may systematically construct irreducible representations E^λ of $\mathrm{GL}(V)$ naturally indexed by a partition λ (with at most n parts); see [Ful97] for details. The Schur character $\operatorname{ch} E^\lambda$ are certain polynomials $s_\lambda(x_1, \ldots, x_n)$. Indeed, since such $\mathrm{GL}(V)$ representations are completely reducible, we must have

$$h_\mu = \sum_\lambda K_{\lambda\mu} s_\lambda$$

where $K_{\lambda\mu}$ is the multiplicity of E^λ in Sym^μ. These are the classical *Kostka numbers*. As it turns out, $K_{\lambda\mu}$ can be seen as an invertible matrix, and the s_λ must be linearly independent, so the h_μ are linearly independent.

Aside 1.5. A trivial observation: whatever $s_\mu s_\nu$ ends up being, it's certainly a non-negative integer linear combination of s_λ's, by the same argument. In this way representation theory can give positivity results trivially.

Remark 1.6. It gets tedious to carry around a fixed number n of variables (e.g. "certain partitions" above), and more importantly there are natural quotient maps $\mathbb{C}[x_1, \ldots, x_n] \twoheadrightarrow \mathbb{C}[x_1, \ldots, x_{n-1}]$ ($x_n \mapsto 0$), and there are natural inclusions $S_{n-1} \hookrightarrow S_n$ (fix n), which all the above bases respect in natural ways. The literature very often avoids these trivialities by "taking $n \to \infty$" in the following sense.

Definition 1.7. Let Λ denote the ring of formal power series in $x_1, x_2, \ldots$ with coefficients in R of bounded total degree which are fixed under the natural $S_\infty := \varinjlim S_n$-action. Λ is the ring of *symmetric functions*. All of our previous bases lift naturally to Λ. For instance,

$$p_3 = x_1^3 + x_2^3 + \cdots, \qquad e_2 = x_1x_2 + x_1x_3 + \cdots + x_2x_2 + \cdots.$$

("Bounded total degree" can safely be ignored virtually always.)

Fact 1.8. Some facts to keep things in perspective:

- $s_{(i)} = h_i$, $s_{(1^i)} = e_i$
- $h_\mu = s_\mu + \sum_{\lambda > \mu} K_{\lambda\mu} s_\lambda$
- $s_\lambda = m_\lambda + \sum_{\mu < \lambda} K_{\lambda\mu} m_\mu$
- s_λ is the "content generating function for semi-standard Young tableaux of shape λ"
- $K_{\lambda\mu}$ counts the number of "semi-standard Young tableaux of shape λ and weight μ"

1.2. **Hall-Littlewood Functions, $K_{\lambda\mu}(t)$.** Symmetric functions and the bases just described are shockingly ubiquitous. We next describe Hall-Littlewood polynomials which generalize Schur polynomials and which come with polynomials $K_{\lambda\mu}(t)$. Along the way we introduce an algebra, the Hall algebra, which is graded by the Grothendieck group. [Mac95] is again the standard reference, though see also [Sch12] for a more modern, general, and readable account.

Definition 1.9. Fix p prime. An abelian p-group A has $A \cong \oplus_i \mathbb{Z}/p^{\lambda_i}$ for some $\lambda \vdash n$ with $|A| = p^n$ called the *type* of A. Write the isomorphism class of A as $[\lambda]$. Let H_p be the free $\mathbb{Z}$-module with basis $\{[\lambda]\}$. Define a product on H_p by

$$[\mu][\nu] := \sum_{\lambda} G^{\lambda}_{\mu\nu}(p)[\lambda]$$

where having fixed $A \in [\lambda]$ we set

$$G^{\lambda}_{\mu\nu}(p) := \#\{N \subset A : N \in [\nu], A/N \in [\mu]\}.$$

(Roughly, $[\mu][\nu]$'s $[\lambda]$ coefficient is the number of distinct extensions $0 \to [\nu] \to [\lambda] \to [\mu] \to 0$, where twisting $[\nu]$ or $[\mu]$ by automorphisms does not count as distinct.)

Theorem 1.10 (Hall [Hal59]; see also Steinitz [Ste01])**.** *H_p is a unital, associative, commutative algebra. Furthermore, $G^{\lambda}_{\mu\nu}(t) \in \mathbb{Z}[t]$, so there is a "universal" (classical) Hall algebra H, which is a $\mathbb{Z}[t]$-algebra with structure constants $G^{\lambda}_{\mu\nu}(t)$. In fact, H has algebraic basis $\{[(1^i)]\}_{i=1}^{\infty}$.*

In light of Hall's theorem, there is a relatively natural algebra isomorphism

$$H \xrightarrow{\pi} \Lambda[t, t^{-1}]$$
$$[(1^r)] \mapsto t^{-r(r-1)/2} e_r.$$

(where on the left we view H as a $\mathbb{Z}[t, t^{-1}]$-algebra).

Aside 1.11. Why that particular t factor? Some t-dependence is good, and as it turns out there is a natural coalgebra structure on H and $\Lambda[t, t^{-1}]$ for which this map yields a bialgebra morphism.

Definition 1.12 (See [Mac95, III, (3.4), p. 217])**.** The *Hall-Littlewood symmetric functions* are functions $P_\lambda \in \Lambda[t]$ defined by

$$t^{-n(\lambda)} P_\lambda(x_1, x_2, \ldots; t^{-1}) := \pi([\lambda]) \in \Lambda[t]$$

where $n(\lambda) := \sum_i (i-1)\lambda_i$.

Example 1.13. At $\lambda = (1^r)$, we have $n(\lambda) = 0+1+\cdots+(r-1) = r(r-1)/2$. It follows that $P_{(1^r)}(x; t) = e_r$.

Aside 1.14. There is a direct formula for (the polynomial version of) P_λ due to Littlewood [Lit61], which is discussed in [Mac95, III, §1]. Explicitly, if $\lambda = (\lambda_1, \ldots, \lambda_n) = 1^{m_1} 2^{m_2} \cdots$, then

$$P_\lambda(x_1, \ldots, x_n, 0, 0, \ldots; t) = \left(\prod_{i \geq 1} [m_i]_t! \right)^{-1} \sum_{w \in S_n} w \left(x^\lambda \prod_{i<j} \frac{x_i - t x_j}{x_i - x_j} \right) \in \mathbb{Z}[x_1, \ldots, x_n; t]$$

where $[c]_t! := [c]_t [c-1]_t \cdots [1]_t$ and $[c]_t := 1 + t + \cdots + t^{c-1}$ as usual. From either our definition or (the $n \to \infty$ limit of) Littlewood's formula, we only have $P_\lambda \in \Lambda[t, t^{-1}]$ or $\Lambda(t)$.

There is a very similar formula which is a sum over a quotient of S_n by a Young subgroup [Mac95, III, (2.2)] which manifestly exhibits the stability $P_\lambda(x_1, \ldots, x_n, 0; t) = P_\lambda(x_1, \ldots, x_n; t)$. Without dividing off the extra t term, the result is not stable.

The following suggests this set of symmetric functions is interesting:

Fact 1.15. (See [Mac95, III, §2])
- $\{P_\lambda\}$ is a $\mathbb{Z}[t]$-basis for $\Lambda[t]$
- $P_\lambda(t = 1) = m_\lambda$
- $P_\lambda(t = 0) = s_\lambda$
- $P_\lambda(t = -1) =$"Schur P-function of shape λ
- $P_{(1^r)}(t) = e_r$

Aside 1.16. See Stembridge [Ste89] for more information about Schur P-functions. Very roughly, they play a role analogous to the Schur functions but in the *projective* representation theory of symmetric groups. For instance, they have a nice combinatorial generating function formula in terms of "shifted marked tableaux."

There is a vast generalization of the above Hall algebra construction that works for abelian categories with appropriate finiteness constraints that often yields algebras with more structure, "Hall Hopf algebras." In general the structure constants involve a certain scalar product involving Ext of the subgroup and quotient, though this part is trivial in the classical case above. Some naturality arises already from observing that the Hall algebra is graded by the Grothendieck group. A particular case of the construction is for the category of coherent sheaves on a smooth projective curve over $\mathbb{F}_q$, which has been somewhat famously considered by Kapranov [Kap97] for $\mathbb{P}^1$. For more information, see Schiffmann [Sch12].

Definition 1.17. Since $\{P_\lambda\}$ is a $\mathbb{Z}[t]$-basis for $\Lambda[t]$, we have transition coefficients

$$s_\lambda =: \sum_\mu K_{\lambda\mu}(t) P_\mu(t)$$

for some $K_{\lambda,\mu}(t) \in \mathbb{Z}[t]$. We call these the *t-Kostka polynomials*. For instance,

$$K_{\lambda\mu}(1) = K_{\lambda\mu}, \qquad K_{\lambda\mu}(0) = \delta_{\lambda\mu}.$$

The change-of-basis matrix from $\{s_\lambda\}$ to $\{P_\mu\}$ is again upper-triangular. In fact, we have [Mac95, III, (6.5)(ii)] $K_{\lambda\mu}(t)$ is monic of degree $n(\mu) - n(\lambda)$ when $\lambda \geq \mu$ and is zero otherwise.

Remark 1.18. Macdonald [Mac95, III, §6] lists all $K_{\lambda\mu}(t)$ for $n \leq 6$, and they have non-negative coefficients. We have both combinatorial and representation theoretic interpretations of $K_{\lambda\mu}(1) = K_{\lambda\mu}$ and $K_{\lambda\mu}(0) = \delta_{\lambda\mu}$, so we could hope the same is true of $K_{\lambda\mu}(t)$ in general. Indeed, Lascoux-Schützenberger [LS78] gave a proof (outline) of a combinatorial interpretation, namely $K_{\lambda\mu}(t)$ is the *charge* generating function on $\mathrm{SSYT}(\lambda, \mu)$, which we will not define.

Aside 1.19. There is at present no combinatorial analogue of the charge statistic for the q, t-Kostkas defined below. Charge itself is quite complicated to describe, and Lascoux-Schützenberger's identification of charge is celebrated as a particularly deep combinatorial result.

1.3. **Springer Fibers: $K_{\lambda\mu}(t) \in \mathbb{N}[t]$; Garsia-Procesi.** A "baby" version of Haiman's proof is Garsia-Procesi's argument [GP92] for the positivity of $K_{\lambda\mu}(t)$. It built on a lengthy and intricate series of topological and algebro-geometric arguments due to Kostant [Kos63], Steinberg [Ste76], Hotta-Springer [HS77], Kraft [Kra81], de Concini-Procesi [DCP81], and others. Garsia-Procesi decided the barrier to entry to understand those arguments was too high and replaced as much as possible with explicit presentations of rings and the construction of a basis. We outline both approaches—the geometric approach serves as motivation, and the explicit approach was more or less extended by Haiman.

The original geometric argument is described briefly in [Mac95, III, §7, Ex. 8] which we now summarize.

Definition 1.20. Set $V := \mathbb{C}^n$ and pick $\mu \vdash n$. Pick a matrix u in Jordan Normal Form with 1's along the diagonal whose block sizes are recorded by μ. (That is, u is a unipotent endomorphism of V of type μ.) Let X_μ be the set of complete flags on V fixed by u, which is called a *Springer fiber.* (These are geometrically fibers of a resolution of singularities introduced by Springer [Spr69].)

Fact 1.21. X_μ is a closed subvariety of the complete flag manifold of complex dimension $n(\mu)$. There is a graded S_n-action on the rational cohomology $H^*(X_\mu)$; see [HS77]. In fact, X_μ has one connected component for each standard Young tableau of shape μ. One might then guess that $H^{2n(\mu)}(X_\mu)$ might have the same dimension as S^μ, so perhaps the action is precisely S^μ. (Warning: the X_μ are not in general smooth.)

Example 1.22. At $\mu = (1^n)$, we have $u = 1$, so $X_{(1^n)}$ is the complete flag manifold, which is in particular smooth and connected of complex dimension $n^2 - (1+2+\cdots+n) = n(n-1)/2$. A famous presentation for this $H^*(X_{(1^n)})$ due to Borel [Bor53] is $\mathbb{Q}[x_1, \ldots, x_n]/(e_1, \ldots, e_n)$, which comes with a natural grading and S_n-action. It is well-known that the top-dimensional component carries the sign representation, i.e. is isomorphic to $S^{(1^n)}$.

Jusifying the 2 parameter version of the following 1 parameter theorem is essentially the point of this seminar:

Theorem 1.23 ([GP92])**.** *The coefficient of t^k in $K_{\lambda\mu}(t)$ is the multiplicity of S^λ in $H^{2i}(X_\mu)$ where $i := n(\mu) - k$.*

Example 1.24. Set $t = 0$ to get $K_{\lambda\mu}(0) = \delta_{\lambda\mu}$, so $H^{2n(\mu)}(X_\mu)$ is S^μ as an S_n-module. Our earlier guess was hence correct!

We will now summarize [GP92]. Many of the large-scale ideas recur in Haiman's proof [Hai01], though the specifics are significantly different. Garsia-Procesi begin with a short, explicit presentation of $H^*(X_\mu)$ (due to de Concini-Procesi [DCP81]) generalizing the above presentation for the complete flag manifold.

Definition 1.25. Fix a partition $\mu \vdash n$. Let

$$R_\mu := \mathbb{Q}[x_1, \ldots, x_n]/I_\mu$$

where I_μ is a homogeneous ideal generated by certain "$e_r(S)$'s"; we will not be concerned with the precise definition, which is in [GP92, (I.5)]. This comes with the natural induced graded S_n-action, and $H^*(X_\mu) \cong R_\mu$ as graded S_n-modules.

Example 1.26. At $\mu = (1^n)$, one has $I_{(1^n)} = (e_1, \ldots, e_n)$, so $R_{(1^n)}$ is the coinvariant algebra of S_n. By Chevalley [Che55], $R_{(1^n)}$ is isomorphic to the regular representation $\mathbb{C}S_n$ as S_n-modules, which is the $\mu = (1^n)$ case of the next theorem.

Theorem 1.27 (de Concini-Procesi [DCP81])**.** $R_\mu \cong 1{\uparrow}_{S_\mu}^{S_n}$ *(which is the Schur-Weyl dual of* Sym^μ *above).*

Aside 1.28. This result serves as motivation for the [GP92] result above, since it is the $t = 1$ ("ungraded") case.

Again using the coinvariant algebra as a model, the Poincare polynomial of the coinvariant algebra is well-known to be $[n]_q! = \binom{n}{1,\ldots,1}_q$. In particular it has dimension $n!$, and interestingly it comes with a basis

$$\{x_1^{a_1} \cdots x_n^{a_n} : a_i \leq n - i\}.$$

Garsia-Procesi showed the Poincare polynomial of R_μ is another (generally different) q-analogue of $\binom{n}{\mu} := \binom{n}{\mu_1,\ldots,\mu_k}$, so in particular $\dim R_\mu = \binom{n}{\mu}$. Garsia-Procesi give an explicit algorithm for constructing a proposed monomial basis for R_μ generalizing the $\mu = (1^n)$ case above, which has the desired number of elements and is a spanning set. (We have no need to describe the specifics, though see [GP92, §1].)

To show linear independence, they use an elementary version of an argument due to Kraft [Kra81]. He gave a construction analogous to Springer's, but this time considering the coordinate ring of the scheme-theoretic intersection of a conjugacy class closure and a Cartan subalgebra. Garsia-Procesi write this ring as a certain very explicit quotient $\mathcal{A}_\mu$, which is an ungraded S_n-module. They use the natural filtration on $\mathcal{A}_\mu$ and essentially show R_μ is the corresponding associated graded algebra, while simultaneously deducing a graded branching rule for restricting R_μ to S_{n-1} which becomes the key ingredient for proving the above theorem.

Aside 1.29. Discussing the geometric details behind the constructions of Springer [Spr69] and Kraft [Kra81] and their relation to Weyl group actions on cohomology could be a lovely later lecture.

Corollary 1.30 ([GP92])**.** *If $\nu \geq \mu$ in dominance order, then $I_\nu \supset I_\mu$, so $R_\nu \cong R_\mu/I_\nu$, and $K_{\lambda\nu}(q) \leq K_{\lambda\mu}(q)$ (coordinate-wise).*

1.4. **Macdonald Symmetric Functions; $K_{\lambda\mu}(q,t)$.** Haiman [Hai01] in some sense pushed through the Garsia-Procesi approach to a 2-parameter generalization of $K_{\lambda\mu}(t)$, which we now describe. Macdonald [Mac95, VI] was combinatorially motivated to define a two-parameter basis of symmetric functions including most of the ones we've already encountered as specializations. Here we will take $R = \mathbb{Q}$ so that we may freely use the p_μ basis. Our exposition follows Macdonald's.

Definition 1.31. There is a "natural" scalar product on Λ given by

$$\langle s_\mu, s_\nu \rangle := \delta_{\mu\nu},$$

or alternatively $\langle p_\lambda, p_\mu \rangle := \delta_{\lambda\mu} z_\lambda$ where $z_\lambda :=$ the order of the centralizer of any permutation of cycle type λ.

Moreover, $\{s_\lambda\}$ is the unique basis of Λ such that

(1) the $\{s_\lambda\}$ are pairwise orthogonal under $\langle -, - \rangle$
(2) there is a strictly upper unitriangular change of basis

$$s_\lambda = m_\lambda + \sum_{\mu < \lambda} c_{\lambda\mu} m_\mu$$

for some $c_{\lambda\mu}$ $(= K_{\lambda\mu})$.

Indeed, existence would follow from the Gram-Schmidt procedure if we were using a *total* order, though uniqueness follows from unitriangularity in general. Hence these conditions *overdetermine* $\{s_\lambda\}$, and in that sense their existence is remarkable.

The Hall-Littlewood polynomials satisfy the same uniqueness statement for $\Lambda(t)$ (replacing s_λ with P_λ but keeping the m_λ) using

$$\langle p_\lambda, p_\mu \rangle_t := \delta_{\lambda\mu} z_\lambda(t) := \delta_{\lambda\mu} z_\lambda \prod_{i=1}^{\ell(\lambda)} (1 - t^{\lambda_i})^{-1}.$$

The "zonal polynomials" and more generally "Jack's symmetric functions" have very similar possible definitions. A relatively natural, common generalization of all of these is to use $\Lambda(q,t)$ with scalar product

$$\langle p_\lambda, p_\mu \rangle_{q,t} := \delta_{\lambda\mu} z_\lambda(q,t) := \delta_{\lambda\mu} z_\lambda \prod_{i=1}^{\ell(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}.$$

Theorem 1.32 (Macdonald, [Mac95, IV, (4.7)])**.** *There is a unique* $\mathbb{Q}(q,t)$*-basis* $\{H_\lambda(x_1, x_2, \ldots; q, t)\}$ *for* $\Lambda(q,t)$ *which is pairwise orthogonal under* $\langle -, - \rangle_{q,t}$ *and where*

$$H_\lambda(x;q,t) = m_\lambda(x) + \sum_{\mu<\lambda} c_{\lambda\mu}(q,t) m_\mu(x).$$

These are called the Macdonald symmetric functions.

Example 1.33. Limiting cases:

- $q = t \Rightarrow H_\lambda(q,q) = s_\lambda$
- $q = 0 \Rightarrow H_\lambda(0,t) = P_\lambda(t)$
- $q = t^\alpha, t \to 1$ gives Jack's symmetric functions

Macdonald defined the *q,t-Kostka polynomials* [Mac95, VI, (8.11)], written $K_{\lambda\mu}(q,t)$, as the coefficients of a certain transition matrix from the *integral form* Macdonald polynomials J_μ, which are a slight rescaling of the H_μ, to (an easy plethysm of) the Schur functions. Garsia-Procesi's result was naturally phrased in terms of a mild variant on $K_{\lambda\mu}(t)$, namely $t^{n(\mu)} K_{\lambda\mu}(t^{-1})$, and the same is true in Haiman's work, where he used the *modified Macdonald polynomials* $\widetilde{H}_\lambda$. These are a simple plethystic substitution away from the Macdonald polynomials; we will define these notions precisely as needed, but those details are not presently important.

Aside 1.34. Of the three variants $H_\mu, J_\mu, \widetilde{H}_\mu$, the Macdonald polynomials H_μ seem to mainly arise naturally from certain considerations from theoretical physics. Indeed, Macdonald constructed them as the eigenvectors of a certain self-adjoint operator [Mac95, VI, (4.7)]. The modified Macdonald polynomials seem to be more useful for representation theory and combinatorics, as we shall see. In particular, we will see a combinatorial interpretation of $\widetilde{H}_\mu(x;q,t)$ due to Haglund-Haiman-Loehr [HHL05a]. The integral forms rarely seem to appear except as a stepping stone between other points of interest.

Warning: Beware of conflicting terminology in the literature for the Macdonald polynomials, their integral forms, and the modified Macdonald polynomials. In particular some authors write H_μ for what we have called $\widetilde{H}_\mu$, and [Mac95] writes P_μ for what we have called H_μ.

Definition 1.35. The modified Macdonald polynomials come with *modified q,t-Kostka polynomials*

$$\widetilde{K}_{\lambda\mu}(q,t) := t^{n(\mu)} K_{\lambda\mu}(q,t^{-1})$$

which in fact satisfy

$$\widetilde{H}_\mu(x;q,t) = \sum_\lambda \widetilde{K}_{\lambda\mu}(q,t) s_\lambda(x).$$

Some facts:

- $K_{\lambda\mu}(0,t) = K_{\lambda\mu}(t)$, so $K_{\lambda\mu}(0,1) = K_{\lambda\mu}$.
- $K_{\lambda\mu}(q,t) = K_{\lambda'\mu'}(t,q)$, where λ' denotes the conjugate partition of λ.
- $K_{\lambda\mu}(1,1) = \dim S^\lambda$ $(= K_{\lambda,(1^n)})$
- *Warning:* $K_{\lambda\mu}(0,t)$ is upper triangular while $K_{\lambda\mu}(q,0)$ is lower triangular, which is a consequence of $\lambda \geq \mu \Leftrightarrow \lambda' \leq \mu'$ and the preceding bullet point.

1.5. **Garsia-Haiman's attack on $K_{\lambda\mu}(q,t) \in \mathbb{N}[q,t]$.** Macdonald conjectured $K_{\lambda\mu}(q,t) \in \mathbb{N}[q,t]$. From the definition, we only have $K_{\lambda\mu}(q,t) \in \mathbb{Q}(q,t)$. Garsia-Haiman essentially conjectured that a variation on Garsia-Procesi's argument should exist to handle this conjecture, which they were quickly able to make precise and which we next describe.

Aside 1.36. Five separate papers appeared in 1996-1998 showing $K_{\lambda\mu}(q,t) \in \mathbb{Z}[q,t]$, but positivity remained open until Haiman's 2001 proof [Hai01]. See the introduction to [Hai01] for further references.

Definition 1.37. Fix n and let $\mathbb{C}[\boldsymbol{x},\boldsymbol{y}] := \mathbb{C}[x_1,\ldots,x_n,y_1,\ldots,y_n]$. Now S_n acts "diagonally" on $\mathbb{C}[\boldsymbol{x},\boldsymbol{y}]$ via

$$\sigma \cdot x_i := x_{\sigma(i)}, \sigma \cdot y_i := y_{\sigma(i)}.$$

A detailed study of the coinvariant algebra proved very fruitful above, so perhaps we should consider the ring of *diagonal coinvariants*

$$\mathbb{C}[\boldsymbol{x},\boldsymbol{y}]/\mathbb{C}[\boldsymbol{x},\boldsymbol{y}]_+^{S_n}$$

where $\mathbb{C}[\boldsymbol{x},\boldsymbol{y}]_+^{S_n}$ denotes the ideal generated by elements of $\mathbb{C}[\boldsymbol{x},\boldsymbol{y}]^{S_n}$ without constant term. Indeed, this is a bigraded S_n-module.

Aside 1.38. *Warning:* In contrast to the coinvariant algebra, $\mathbb{C}[\boldsymbol{x},\boldsymbol{y}]^{S_n}$ is not a polynomial ring. This is an instance of the Chevalley-Shephard-Todd theorem; see [Hum90, Thm. 3.11] for a precise statement. The trouble is essentially that S_n acting diagonally on $\mathrm{Span}_{\mathbb{C}}\{e_1,\ldots,e_n,f_1,\ldots,f_n\}$ is *not* generated by pseudoreflections. The denominator at least has an "obvious" generating set (attributed to Weyl) along the lines of the p_i, namely

$$\mathbb{C}[\boldsymbol{x}, \boldsymbol{y}]_+^{S_n} = \left(\sum_{i=1}^n x_i^h y_i^k : h, k \geq 0, h+k > 0 \right).$$

Definition 1.39. Pick a set of n points in $\mathbb{N} \times \mathbb{N}$,

$$D := \{(p_1, q_1), \ldots, (p_n, q_n)\}.$$

Define

$$\Delta_D := \det(x_i^{p_j} y_i^{q_j})_{1 \leq i,j \leq n}.$$

(This is well-defined up to ± 1 since we didn't order our set.) Given a partition μ, take $\mu \subset \mathbb{N} \times \mathbb{N}$ by indexing *from zero*.

Example 1.40. If $D = \{(0,0), (0,1), \ldots, (0, n-1)\}$, then $\Delta_D = \det(y_i^j) = \Delta_{(1^n)}$ is the Vandermonde determinant. In general, $\Delta_\mu(x,y) = \Delta_{\mu'}(y,x)$.

Definition 1.41. Now let

$$D_\mu := \mathbb{C}[\partial \boldsymbol{x}, \partial \boldsymbol{y}] \Delta_\mu$$

be the space spanned by all iterated partial derivatives of Δ_μ. D_μ has a doubly-graded S_n-action inherited from $\mathbb{C}[\boldsymbol{x}, \boldsymbol{y}]$.

Example 1.42. For $\mu = (2,1)$, we find

$$\Delta_{(2,1)} = \begin{bmatrix} 1 & y_1 & x_1 \\ 1 & y_2 & x_2 \\ 1 & y_3 & x^3 \end{bmatrix} = y_2 x_3 - x_2 y_3 - y_1 x_3 + y_1 x_2 + x_1 y_3 - x_1 y_2.$$

Taking partial derivatives yields

$$\{\Delta_{(2,1)}, y_3 - y_2, y_1 - y_3, y_2 - y_1, x_3 - x_2, x_1 - x_3, x_2 - x_1, 1\}.$$

Note that $y_2 - y_1$ and $x_2 - x_1$ (say) are redundant, so D_μ has a basis of precisely 6 elements.

Aside 1.43. The diagonal coinvariants are a quotient of $\mathbb{C}[\boldsymbol{x}, \boldsymbol{y}]$, and one may express the D_μ in the same way. Namely, [Hai99, Prop. 3.4] D_μ is isomorphic as a doubly graded S_n-representation to the quotient of $\mathbb{C}[\boldsymbol{x}, \boldsymbol{y}]$ by the ideal of polynomials $p(x,y)$ for which $p(\partial \boldsymbol{x}, \partial \boldsymbol{y})$ annihilates Δ_μ, which we will use later. This can be run in reverse, and one often hears about the *diagonal harmonics*, which are roughly the subset of $\mathbb{C}[\boldsymbol{x}, \boldsymbol{y}]$ consisting of polynomials f such that $\mathbb{C}[\partial \boldsymbol{x}, \partial \boldsymbol{y}]_+^{S_n} \cdot f = 0$. See [Hai99, §7] for a more complete discussion. The name arises from the fact that the diagonal harmonics are in particular harmonic polynomials in the classical sense.

Indeed, the Δ_μ form a basis for the space of alternating polynomials in $\mathbb{C}[\boldsymbol{x}, \boldsymbol{y}]$, i.e. the subspace of $f \in \mathbb{C}[\boldsymbol{x}, \boldsymbol{y}]$ where $w \cdot f = \operatorname{sgn}(w) f$ for all $w \in S_n$. We will encounter this subspace again in the next section.

Describing Haiman's proof of the following conjecture is the main goal of this seminar:

Conjecture 1.44 (Garsia-Haiman [GH96])**.**

$$\widetilde{K}_{\lambda\mu}(q,t) = \sum_{r,s} t^r q^s \textit{(multiplicity of } S^\lambda \textit{ in } (D_\mu)_{r,s}\textit{)}$$

Corollary 1.45. *The Macdonald positivity conjecture is true.*

Describing Haiman's reduction of the preceding conjecture to the following conjecture and certain geometric equivalents will be one of our main topics:

Conjecture 1.46 (Garsia-Haiman [GH96]; "n! conjecture")**.** $\dim D_\mu = n!$.

Remark 1.47. The first conjecture implies the n! conjecture as follows. Setting $t = q = 1$, $\widetilde{K}_{\lambda\mu}(1,1) = \dim S^\lambda$, so D_μ has the same number of copies of S^λ as $\mathbb{C}S_n$. Hence $D_\mu \cong \mathbb{C}S_n$ as S_n-modules, so in particular their dimensions agree. Remarkably, [GH96] proved the reverse implication.

Haiman [Hai99], following up on a suggestion of Procesi, conjectured a connection between the n! conjecture, and the "isospectral Hilbert scheme" X_n arising from the Hilbert scheme of points in the plane $\mathbb{C}^2$. In particular, Haiman showed that the n! conjecture is equivalent to the statement that "X_n is Cohen-Macaulay," which implies the first conjecture and hence the Macdonald positivity conjecture. The details of this connection will be the focus of later lectures.

2. Aftermath: Diagonal coinvariants, k-Schur functions, and equivalences of derived categories

Lecturer: Josh Swanson.

Having motivated Haiman's proof [Hai01], we pause to describe three very different directions follow-up work has taken. Indeed, to avoid getting too side-tracked, we only discuss k-Schur functions and diagonal coinvariants in any detail here. Dan will give a later lecture on derived category equivalences.

2.1. Equivalences of derived categories.

A sample paper in this direction is [GS04], and the abstract gives a flavor for these results: "We give an equivalence of triangulated categories between the derived category of finitely generated representations of symplectic reflection algebras associated with wreath products (with parameter $t = 0$) and the derived category of coherent sheaves on a crepant resolution of the spectrum of the centre of these algebras." See Dan's lecture for more.

2.2. ***k*-Schur functions.** The k-Schur functions $A_\mu^{(k)}(X;t)$ were introduced in [LLM03] as roughly

$$A_\mu^{(k)}(x;t) := \sum_{T \in \mathbb{A}_\mu^{(k)}} t^{\mathrm{charge}(T)} s_{\mathrm{shape}(T)}(x)$$

In fact, one has $A_\mu^{(k)}(x;t) = s_\mu(x)$ when $k \geq |\mu|$; the only term in the sum is the "trivial" semistandard tableau of shape μ whose ith row consists of i's (which has charge 0). See [LLM03, Property 6] for more details.

Recall that $K_{\lambda\mu}(t)$ is a charge generating function; indeed $A_\mu^{(k)}(x;t)$ were defined to be "pieces of Macdonald polynomials" in a sense made precise by [LLM03, (1.6), (1.9)]. They form a basis for the subspace of symmetric functions spanned by $\{s_\lambda[X/(1-t)]\}_{\lambda_1 \leq k}$. They are related to yet another slight modification of the Macdonald polynomials via

$$H'_\mu[X;q,t] = \sum_\lambda K_{\lambda\mu}^{(k)}(q,t) A_\lambda^{(k)}[X;t]$$

with (conjecturally)

$$0 \leq K_{\lambda\mu}^{(k)}(q,t) \leq K_{\lambda\mu}(q,t)$$

pointwise. Indeed, the conjecture is implied by the at first glance weaker conjecture $K_{\lambda\mu}^{(k)}(q,t) \in \mathbb{N}[q,t]$, which in the "$k \to \infty$ limit" gives Macdonald's positivity conjecture.

The k-Schur functions share an enormous number of similarities with the Schur functions. They have more than half a dozen conjecturally equivalent definitions; the standard reference is [LLM$^+$14].

We mention one further topological connection. Bott [Bot58] famously showed the homology and cohomology of the "affine Grassmannian" can be realized naturally as a subring and quotient of the ring of symmetric functions, respectively. The affine Grassmannian comes with Schubert classes indexed by Grassmannian elements of the underlying affine Weyl group. Lam [Lam08] showed that the homology classes of these Schubert classes under Bott's isomorphism map precisely to k-Schur functions.

2.3. **Diagonal coinvariants.** We encountered diagonal coinvariants above. For brevity, now write

$$R_n := \mathbb{C}[\boldsymbol{x}, \boldsymbol{y}]/\mathbb{C}[\boldsymbol{x}, \boldsymbol{y}]_+^{S_n}$$

using the previous notation, where for instance $\boldsymbol{x}$ stands for $x_1, x_2, \ldots, x_n$. A number of questions of varying difficulty quickly come to mind:

Question 2.1. What are...

(i) the dimension of R_n, $\dim_{\mathbb{C}} R_n$?

(ii) the doubly-graded Hilbert series (polynomial) for R_n,

$$H_n(t,q) := \sum_{i,j} t^i q^i (\dim_{\mathbb{C}}(R_n)_{i,j}) \in \mathbb{N}[t,q]?$$

(iii) the doubly-graded Frobenius series for R_n,

$$F_n(x;t,q) := \sum_{i,j} t^i q^i (\mathrm{ch}(R_n)_{i,j}) \in \Lambda[t,q]?$$

(Here ch denotes the Frobenius characteristic map, which sends the Specht module S^λ to the Schur function s_λ and is extended additively.)

(iv) combinatorial generating functions for $H_n(t,q)$ and $F_n(x;t,q)$?

Example 2.2. First consider the analogous questions for the coinvariant algebra of S_n.

- (i) is just $n!$, and we exhibited an explicit basis above.
- (ii) is $[n]_q!$, the usual q-analogue of $n!$, which is also the major index or inversion number generating function for S_n. Indeed, it is also the dimension generating function for Schubert varieties in the complete flag manifold.
- (iii) is given essentially by the shape, maj joint generating function on standard Young tableaux of size n, which is due independently to Lusztig (unpublished) and Stanley [Sta79, Prop. 4.11].

Everything about R_n has proven much harder.

We sample some of the early conjectured answers to these questions:

Conjecture 2.3 (See [Hai94] for precise statements)**.** *We have...*

(a) $\dim_{\mathbb{C}} R_n = (n+1)^{n-1}$.

(b) $H_n(1,q)$ *is the inversion generating function on spanning trees on the set* $\{0,1,\ldots,n\}$ *rooted outward from* 0.

(c) ("Master formula.") Define an operator ∇ *on symmetric functions over* $\mathbb{Q}(t,q)$ *by*

$$\nabla \widetilde{H}_\mu := t^{n(\mu)} q^{n(\mu')} \widetilde{H}_\mu.$$

Then $F_n(x;t,q) = \nabla e_n(x)$.

Aside 2.4. The expression $(n+1)^{n-1}$ appears more often than you'd except. In particular it counts trees as above, or equivalently certain forests; parking functions of length n; maximal chains of non-crossing partitions of $[n+1]$; regions in the Shi hyperplane arrangement; sand pile partitions; etc. In fact, Haiman showed that the S_n action on R_n is just the S_n-action on parking functions of length n tensored with the sign representation. Note that this generalizes (a). See [Sta] for a parking function survey.

Indeed, (c) was proved by Haiman in follow-up work [Hai02] to [Hai01], which was known to imply (a) and (b). This provided at least an implicit answer to (i)-(iii). A more satisfying answer for (ii) was relatively recently provided by Haglund, which we now summarize.

Definition 2.5. A *Tesler matrix* is an upper triangular matrix with non-negative integer entries whose "hook sums" are all 1. The ith hook sum is the difference between the sum of the ith row and the non-diagonal elements of the ith column. Write $\mathrm{Tes}(n)$ for the set of $n \times n$ Tesler matrices.

Example 2.6. There are seven 3×3 Tesler matrices. This list includes the identity matrix and the matrix whose third column is 1, 1, 3 with 0's everywhere else.

Theorem 2.7 ([Hag11]). *The doubly-graded Hilbert series for the diagonal coinvariants R_n is*

$$H_n(t, q) = \sum_{T \in \mathrm{Tes}(n)} \mathrm{weight}(T)$$

where

$$\mathrm{weight}(T) := (-(1-q)(1-t))^{\#\,\mathrm{Pos}(T)-n} \prod_{(i,j) \in \mathrm{Pos}(T)} [T_{i,j}]_{q,t}$$

where $\mathrm{Pos}(T)$ denotes the set of indexes of positive entries of $T \in \mathrm{Tes}(n)$ and where

$$[k]_{q,t} := q^{k-1} + q^{k-2}t + \cdots + qt^{k-2} + t^{k-1} = \frac{q^k - t^k}{q-t}.$$

Note, however, that Haglund's result still involves subtractions. It takes some effort to recover (a) from the $t = q = 0$ case of Haglund's formula. A vast literature has sprung up around (iv), with the following likely being the most famous part:

Conjecture 2.8 (Shuffle Conjecture [HHL$^+$05b]). *The bigraded Frobenius series for diagonal coinvariants is given by*

$$F_n(x; q, t) = \sum_{\lambda \subset \delta_n} \sum_{T \in \mathrm{SSYT}(\lambda + (1^n)/\lambda)} t^{|\delta_n/\lambda|} q^{\mathrm{dinv}(T)} z^T.$$

where $\mathrm{dinv}(T)$ counts the number of "d-inversions" of T and δ_n is the staircase shape with longest part $n-1$.

Wonderfully, we have the following very recent result due to Carlsson and Mellit:

Theorem 2.9 ([CM15]). *The shuffle conjecture is true.*

Aside 2.10. They actually proved a generalization known as the "compositional shuffle conjecture." We will likely have a seminar over the summer dedicated to [CM15] and related theory.

There is another thread giving rise to a q,t-analogue of the Catalan numbers which we next summarize.

Definition 2.11. Let Γ be the "anti-invariant subspace" of R_n,

$$\Gamma := \{f \in R_n : \sigma \cdot f = \operatorname{sgn}(\sigma) f \quad \forall \sigma \in S_n\}.$$

The determinants Δ_μ above form a basis for the corresponding subspace of $\mathbb{C}[\boldsymbol{x}, \boldsymbol{y}]$. We have

Theorem 2.12 ([GH02], [Hai02]). *The bigraded Hilbert series of* Γ *is the generating function*

$$\sum_{\pi} t^{\operatorname{area}(\pi)} q^{\beta(\pi)}$$

where π *ranges over Dyck paths in the* $n \times n$ *square,* $\operatorname{area}(\pi)$ *is the number of boxes strictly above the diagonal and at or below* π*, and* $\beta(\pi)$ *is a statistic involving a sum over locations of "diagonal corners" of a "billiard ball" launched through* π*.*

In particular, $\dim \Gamma = \frac{1}{n+1}\binom{2n}{n}$ is a Catalan number, though this was known earlier.

We cannot help concluding with another combinatorial result not directly related to diagonal coinvariants. The modified Macdonald polynomials have a beautiful combinatorial generating function due to Haglund-Haiman-Loehr [HHL05a]. We will not take the time to define the statistics precisely, but [HHL05a] is quite lucid.

Theorem 2.13 ([HHL05a]). *We have*

$$\widetilde{H}_\mu(x; q, t) = \sum_{\sigma \colon \mu \to \mathbb{Z}_+} q^{\operatorname{inv}(\sigma)} t^{\operatorname{maj}(\sigma)} x^\sigma$$

where μ *is viewed as a set of cells,* $\operatorname{inv}(\sigma)$ *is a certain sum of "arm lengths" over "attacking pairs" which give inversions of the reading word, and* $\operatorname{maj}(\sigma)$ *is a sum over "leg lengths" for descents in the usual sense.*

Consequently, $\widetilde{H}_\mu(x; q, t) \in \mathbb{N}[x; q, t]$.

Since $\widetilde{H}_\mu(x; q, t) = \sum_\lambda \widetilde{K}_{\lambda\mu}(q, t) s_\lambda(x)$, a combinatorial interpretation of the Schur decomposition of the preceding combinatorial sum is equivalent to a combinatorial interpretation of the q,t-Kostka polynomials, which is a big open problem in algebraic combinatorics.

(Note to self: read Haglund's "Genesis" survey article.)

3. Overview of Haiman's Proof

Lecturer: Josh Swanson.

Summary. Today we give a "panoramic overview" of Haiman's proof of the Macdonald positivity conjecture [Hai01], [Hai99]. We will encounter many "black boxes," and in the next few lectures we will open some of them, as interest and time permits. See also Procesi's overview [Pro03], which has a different emphasis.

Remark 3.1. The overarching strategy is roughly as follows:

(1) Identify properties of the bigraded Frobenius series of D_μ which make it yield q,t-Kostka's.
(2) Reduce (1) to the study of schemes X_n, H_n, in particular that X_n is Cohen-Macaulay, or equivalently the $n!$ conjecture.
(3) Reduce (2) to the observation that a certain sequence of global sections on X_n yield regular sequences locally.
(4) Reduce (3) to showing a certain ideal J in $\mathbb{C}[\boldsymbol{x},\boldsymbol{y}]$ has J^d free over $\mathbb{C}[\boldsymbol{y}]$ for all d. (In fact, $J^d = \bigcap_{1\le i<j\le n}(x_i - x_j, y_i - y_j)^d$.)
(5) Reduce (4) to a similar statement for global sections of "polygraphs."
(6) Very carefully inductively construct bases to verify (5).

(1)-(3) were in [Hai99]. (5) has an important connection to the *generalized McKay correspondence*, which we will leave for later lectures. (6) is highly technical and takes 30 pages in [Hai01]. It is essentially the key contribution of [Hai01], though it is also perhaps the least appealing part of the overall argument, and it is logically distinct from the preceding parts. We will hopefully discuss it further towards the end of the seminar.

We will discuss (1) most extensively today, since the following lectures will cover (2)-(5), and since we may forget about Macdonald polynomials after (1) is complete. Nonetheless, we will also introduce some of the key players in parts (2)-(4) in discussing (1).

3.1. Frobenius Series of D_μ.

For convenience, we briefly recall our earlier notation. Fix n and let $\mathbb{C}[\boldsymbol{x},\boldsymbol{y}] := \mathbb{C}[x_1,\ldots,x_n,y_1,\ldots,y_n]$. Let S_n act diagonally on $\mathbb{C}[\boldsymbol{x},\boldsymbol{y}]$ via $\sigma\cdot x_i := x_{\sigma(i)}$, $\sigma\cdot y_i := y_{\sigma(i)}$. If $D := \{(p_1,q_1),\ldots,(p_n,q_n)\}\subset\mathbb{N}\times\mathbb{N}$, we set $\Delta_D := \det(x_i^{p_j}y_i^{q_j})_{1\le i,j\le n}$. We further set $D_\mu := \mathbb{C}[\partial\boldsymbol{x},\partial\boldsymbol{y}]\Delta_\mu$, which is a doubly-graded S_n-module. The Frobenius characteristic is the map which sends the irreducible S_n-representation (Specht module) S^λ to the Schur function $s_\lambda(x) = s_\lambda(x_1,x_2,\ldots)\in\Lambda$.

Definition 3.2. Let D be a finite dimensional doubly-graded S_n-module. The *Frobenius series* of D is

$$\mathcal{F}_D(x;q,t) := \sum_{r,s} t^r q^s \operatorname{ch}(D_{r,s}) \in \Lambda[q,t].$$

Recall that

$$\widetilde{H}_\mu(x;q,t) = \sum_\lambda \widetilde{K}_{\lambda\mu}(q,t) s_\lambda(x) \in \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}(q,t).$$

Garsia-Haiman's conjecture [GH96] was that not only is $\widetilde{K}_{\lambda\mu} \in \mathbb{N}[q,t]$, but the coefficient on $t^r q^s$ in $\widetilde{K}_{\lambda\mu}$ is the multiplicity of S^λ in $(D_\mu)_{r,s}$. Equivalently, one must show

$$\widetilde{H}_\mu(x;q,t) = \mathcal{F}_{D_\mu}(x;q,t). \tag{1}$$

The modified Macdonald polynomials $\widetilde{H}_\mu$ have the following characterization, which is quite reminiscent of the original characterization above of the Macdonald polynomials H_μ:

Proposition 3.3 ([Hai01, Prop. 2.1.1])**.** *The $\widetilde{H}_\mu(x;q,t)$ satisfy*

(a) $\widetilde{H}_\mu(x;q,t) \in \mathbb{Q}(q,t)\{s_\lambda[X/(1-q)] : \lambda \geq \mu\}$,
(b) $\widetilde{H}_\mu(x;q,t) \in \mathbb{Q}(q,t)\{s_\lambda[X/(1-t)] : \lambda \geq \mu'\}$, *and*
(c) $\widetilde{H}_\mu[1;q,t] = 1 \quad (= \widetilde{H}(1,0,0,\ldots;q,t))$.

These conditions characterize $\widetilde{H}_\mu$ uniquely.

Aside 3.4. It is perhaps time to discuss plethystic notation. The idea is that given a symmetric function $f(x_1, x_2, \ldots)$ and another function $g(x_1, x_2, \ldots) = x^\alpha + x^\beta + \cdots$ (where $x^\alpha := \prod_i x_i^{\alpha_i}$), we may consider

$$f[g] := f(x^\alpha, x^\beta, \ldots),$$

which is well-defined precisely because f is symmetric. This is the *plethysm* of f and g. One makes this precise by defining $p_k[g]$ for formal Laurent series g by replacing the indeterminates in g with their kth powers and extending to all f using the fact that the p_k form an algebraic basis for Λ (over $\mathbb{Q}$, say).

In any case, one often writes $X := x_1 + x_2 + \cdots$, so that $f[X] = f(x_1, x_2, \ldots)$. More pertinently,

$$\begin{aligned} X/(1-q) &= (x_1 + x_2 + \cdots)/(1-q) \\ &= (x_1 + x_1 q + x_1 q^2 + \cdots) + (x_2 + x_2 q + \cdots) + \cdots \end{aligned}$$

and

$$s_\lambda[X/(1-q)] = s_\lambda(x_1, x_1 q, x_1 q^2, \ldots, x_2, x_2 q, \ldots).$$

One great convenience of this notation is that plethystically substituting $X/(1-q)$ is invertible by plethystically substituting $X(1-q)$. Indeed, $f[-X] = (-1)^d \omega f$ for degree d elements $f \in \Lambda$, where $\omega\colon s_\lambda \mapsto s_{\lambda'}$. See [Hai99, §2] for more information. Plethysms arise naturally in the representation theory of S_n and $\mathrm{GL}_m(\mathbb{C})$ quite frequently, though they are also most often extremely difficult to decompose explicitly.

We first note the $q = 0$ specialization of (1),

$$\mathcal{F}_{D_\mu}(x; 0, t) = t^{n(\mu)} P_\mu[X/(1 - t^{-1}); t^{-1}], \tag{2}$$

where P_μ is a Hall-Littlewood function. Assuming (1), from the triangularity requirements above (after "inverting" the formula $s_\lambda(x) = \sum_\mu K_{\lambda\mu}(q) P_\mu(x; q)$, which accounts for the plethysm), it must then be that the only irreducible representations S^λ appearing in the y-degree zero part of D_μ have $\lambda \geq \mu$. This is similar in flavor to condition (a) above.

We first replace D_μ with a quotient:

Definition 3.5. Define an ideal $J_\mu := \{p \in \mathbb{C}[\boldsymbol{x}, \boldsymbol{y}] : p(\partial \boldsymbol{x}, \partial \boldsymbol{y}) \Delta_\mu = 0\}$. Let

$$R_\mu := \mathbb{C}[\boldsymbol{x}, \boldsymbol{y}]/J_\mu,$$

which is a doubly-graded S_n-module. (Indeed, R_μ is a further quotient of the diagonal coinvariants R_n.) Note that R_μ is a doubly-graded S_n-module.

In fact, $\mathcal{F}_{D_\mu} = \mathcal{F}_{R_\mu}$, so we may replace D_μ with R_μ in (1) and (2). (D_μ is the first in a long line of friends we will likely never see again.) Moreover, $R_\mu/(\boldsymbol{y})$ where $(\boldsymbol{y}) := (y_1, \ldots, y_n)$ is the ring from Garsia-Procesi [GP92] (unfortunately written R_μ above), so (2) is true. We also find that condition (c) is true for $\mathcal{F}_{R_\mu}$ since condition (c) is equivalent to requiring that the trivial representation appear uniquely in degree 0, and $R_\mu^{S_n}$ consists only of the constants. (b) more or less follows from (a) by symmetry, so we concentrate on amplifying (2) to (a).

3.2. H_n and X_n detour.

To connect (2) to condition (a), we must take a detour through geometry. We will often not give full definitions, there will be no universal properties, etc. Some of this will appear in later lectures in more detail.

Definition 3.6. The *Hilbert scheme* of ("n") points in the plane $H_n := \text{Hilb}(\mathbb{C}^2)$ has closed points given by ideals $\{I \subset \mathbb{C}[x, y] : \dim_{\mathbb{C}} \mathbb{C}[x, y]/I = n\}$. Reduced closed subschemes of $\mathbb{C}^2$ consisting of n distinct points in $\mathbb{C}^2$ form an open, dense subset of H_n; these are intuitively the n-tuples of distinct points in $\mathbb{C}^2$.

A diagram $D(\mu) \subset \mathbb{N} \times \mathbb{N}$ with n boxes determines a *monomial ideal* $I_\mu \in H_n$ given by $I_\mu := (x^p y^q : (p, q) \notin D(\mu))$, and the monomials $x^p y^q$ for $(p, q) \in D(\mu)$ form a $\mathbb{C}$-basis for $\mathbb{C}[x, y]/I_\mu$. Note that I_μ vanishes only at the origin $(0, 0)$, so the corresponding subscheme is "very" non-reduced.

The torus $\mathbb{T}^2 := (\mathbb{C}^*)^2$ acts on $\mathbb{C}[x, y]$ by $(t, q) \cdot (x, y) := (tx, qy)$, and hence on H_n, on closed points by $(t, q) \cdot I = I(x/t, y/q)$. The I_μ are fixed by $\mathbb{T}^2$, and indeed they are the only fixed points, which are hence isolated. The I_μ are fundamental in the sense that every $\mathbb{T}^2$-orbit has some I_μ in its closure. The bigraded Hilbert series of D_μ is precisely its $\mathbb{T}^2$-character (as a function of t, q).

Definition 3.7. Ordered n-tuples of points in $\mathbb{C}^2$ are $(\mathbb{C}^2)^n = \operatorname{Spec}\mathbb{C}[\boldsymbol{x}, \boldsymbol{y}]$, and the obvious S_n-actions coincide. Unordered n-element multisets of points in $\mathbb{C}^2$ are orbits in $(\mathbb{C}^2)^n/S_n =: S^n\mathbb{C}^2$, arising from $\operatorname{Spec}\mathbb{C}[\boldsymbol{x}, \boldsymbol{y}]^{S_n}$. In this way we have a natural map $(\mathbb{C}^2)^n \to S^n\mathbb{C}^2$ both set- and scheme-theoretically.

For $I \in H_n$, each point of the corresponding closed subscheme can be counted with multiplicity equal to the length of the stalk at that point, or of the corresponding artinian local ring factor of $\mathbb{C}[x, y]/I$. We then have a map $\sigma\colon H_n \to S^n\mathbb{C}^2$ which sends an ideal I to the n-element multiset of its points repeated according to their multiplicities. Indeed, σ is a projective morphism, called the *Chow morphism.*

The *isospectral Hilbert scheme* X_n is the *reduced* fiber product

$$\begin{array}{ccc} X_n & \xrightarrow{f} & (\mathbb{C}^2)^n \\ {\scriptstyle\rho}\downarrow & & \downarrow \\ H_n & \xrightarrow{\sigma} & S^n\mathbb{C}^2, \end{array}$$

that is, the reduced closed subscheme of $H_n \times (\mathbb{C}^2)^n$ whose closed points are n-tuples $(I, P_1, \dots, P_n)$ satisfying $\sigma(I) = \{P_1, \dots, P_n\}$ (as a multiset). In this sense f and ρ are projection maps. For instance, we have $\rho^{-1}(I_\mu) = \{(I_\mu, 0, \dots, 0) =: Q_\mu\}$.

A consequence of the $n!$ conjecture is that the coordinate ring of the scheme-theoretic fiber $\rho^{-1}(I_\mu)$ is in fact R_μ. We now return to condition (a) above.

Definition 3.8. Let $S_\mu := \mathcal{O}_{X_n, Q_\mu}$ be the stalk at the fiber of I_μ. There is a notion of a "formal Hilbert (Frobenius) series" for certain modules with equivariant $\mathbb{T}^2$-actions (commuting $\mathbb{T}^2$- and S_n-actions) which coincides with the usual notions in nice situations. See [Hai99, §5] for details.

Indeed, Haiman showed [Hai99] that given the $n!$ conjecture,

$$\mathcal{F}_{S_\mu}(x; q, t) = \mathcal{H}(q, t)\mathcal{F}_{R_\mu}(x; q, t)$$

where $\mathcal{H}(q, t)$ is the formal Hilbert series of $\mathcal{O}_{H_n, I_\mu}$. Now, $x_1, \dots, x_n$ and $y_1, \dots, y_n$ can be considered as coordinate functions in $(\mathbb{C}^2)^n = \operatorname{Spec}\mathbb{C}[\boldsymbol{x}, \boldsymbol{y}]$, so they give global regular functions (denoted by the same symbols) on X_n via $X_n \xrightarrow{f} (\mathbb{C}^2)^n$. Considered in S_μ, they in fact form a regular sequence, and in this situation Haiman showed

$$\mathcal{F}_{S_\mu/(\boldsymbol{y})}(x; q, t) = \mathcal{F}_{S_\mu}[X(1-q); q, t].$$

Moreover, he showed that if S^λ has multiplicity zero in $R_\mu/(\boldsymbol{y})$, then the coefficient of s_λ in $\mathcal{F}_{S_\mu/(\boldsymbol{y})}(x;q,t)$ is zero. Combining these statements and the triangularity result mentioned above yields (a). Since (b) follows essentially by symmetry, this completes the proof.

3.3. **Further highlights.** We briefly mention some of the other key players in the argument.

Remark 3.9. The schemes H_n and X_n can be constructed as quite explicit blowups, as follows.

Let $A := \mathbb{C}[\boldsymbol{x},\boldsymbol{y}]^\epsilon$ be the space of S_n-alternating elements. The Δ_D for $D \subset \mathbb{N}\times\mathbb{N}$ of size n form a $\mathbb{C}$-basis for A. Let A^d be the space spanned by d-fold products of elements of A, with $A^0 := \mathbb{C}[\boldsymbol{x},\boldsymbol{y}]^{S_n}$.

H_n is isomorphic as a scheme projective over $S^n\mathbb{C}^2$ to $\operatorname{Proj} T$ where T is the graded $\mathbb{C}[\boldsymbol{x},\boldsymbol{y}]^{S_n}$-algebra $T := \oplus_{d\geq 0} A^d$.

X_n is isomorphic as a scheme over $(\mathbb{C}^2)^n$ to the blowup of $(\mathbb{C}^2)^n$ at the ideal $J := \mathbb{C}[\boldsymbol{x},\boldsymbol{y}]A$ generated by the alternating polynomials, namely this is $\operatorname{Proj} S[tJ]$ where $S[tJ] \cong \oplus_{d\geq 0} J^d$ is the Rees algebra, $S = \mathbb{C}[\boldsymbol{x},\boldsymbol{y}]$. *Note:* we have $J \subset \cap_{i<j}(x_i - x_j, y_i - y_j)$ trivially. A corollary of Haiman's proof is that equality in fact holds.

There are also "nested" versions of H_n and X_n which are used in essential ways inductively.

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