# ON EIGENVALUES OF REPRESENTATIONS OF REFLECTION GROUPS AND WREATH PRODUCTS

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(This is a two-part lecture series on John Stembridge's paper of the same name. It was given in the student-run CAT seminar at the University of Washington on February 11th and 18th, 2016.)

## Lecture 1

**Outline.**

(1) Cyclic exponents, branching rule application
(2) (Co)invariants, degrees, exponents
(3) Fake degree polynomials

**Definition 1.** Let $G$ be a finite group, $V$ a finite dimensional complex vector space, and $G \to \mathrm{GL}(V)$ a representation. The minimal polynomial of (the image of) $g \in G$ divides $x^{|g|} - 1$ which splits with distinct roots, so $g$ is diagonalizable.

Say $m := |g|$, set $\omega_m := \exp(2\pi i/m)$, and suppose $g \sim \mathrm{diag}(\omega_m^{e_1}, \ldots, \omega_m^{e_{\dim V}})$. Define the **cyclic exponents of** $g$ as the multiset

$$E_{g,V} := \{e_1, \ldots, e_{\dim V}\}.$$

We will "normalize" so that $0 \leq e_i < m$.

(Note: $E_{g,V}$ only depends on the conjugacy class of $g$ and the representation or its character, so sometimes we'll use slightly different notation. Hopefully this will not cause confusion.)

**Example 2.** Let $G = S_n$ act on $V = \mathbb{C}^n$ as permutation matrices. The matrix of $(12 \cdots n)$ is a companion matrix with minimal polynomial $x^n - 1 = \prod_{i=0}^{n-1}(x - \omega_n^i)$, so

$$E_{(12\cdots n),\mathbb{C}^n} = \{0, 1, 2, \ldots, n-1\}.$$

**Question 3.** What are the cyclic exponents...

(1) ...when $G$ is a reflection group acting naturally on $\mathbb{C}^n$?
(2) ...when $V$ is an irrep of such $G$?

**Motivation 4.** Pick $g \in G$ and consider $\langle g \rangle \subset G$. Let $\rho \colon G \to \mathrm{GL}(V)$. There are $m = |g|$ irreps of $\langle g \rangle$, which are of the form $\phi^r \colon \langle g \rangle \to \mathrm{GL}(\mathbb{C})(= \mathbb{C}^\times)$ where $\phi^r(g) := \omega_m^r$. The irreducible decomposition of the restricted representation is

$$\rho\downarrow^G_{\langle g \rangle} \cong \sum_{e \in E_{g,V}} \phi^e.$$

Hence the cyclic exponents are recording the branching rules from $G$ down to its cyclic subgroups. By Frobenius Reciprocity, we get rules to induce from $\langle g \rangle$ up to $G$, which can be summarized neatly with the formal generating function

$$\sum_{r=0}^{m-1} q^r (\phi^r\uparrow^G_{\langle g \rangle}) = \sum_{\text{irreps } \psi \text{ of } G} E_{g,\psi}(q)\ \psi$$

where $E_{g,\psi}(q) := \sum_{e \in E_{g,\psi}} q^e$.

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**Example 5.** Restricting the natural representation of $S_n$ on $\mathbb{C}^n$ to $\langle(12\cdots n)\rangle$ gives the regular representation on $\langle(12\cdots n)\rangle$. Inducing the regular representation on $\langle(12\cdots n)\rangle$ up to $S_n$ gives $\mathbb{C}S_n$, but now "graded" by the degree of $q$.

In order to state the second main application of cyclic exponents, we first give some background on complex reflection groups and coinvariant algebras.

**Definition 6.** Let $G$ be a finite group, $V$ a finite dimensional $\mathbb{C}$-vector space, and $G \to \mathrm{GL}(V)$ be a representation. Let $S(V)$ denote the *symmetric algebra* of $V$ (which is isomorphic to $\mathbb{C}[x_1, \ldots, x_{\dim V}]$). $S(V)$ is naturally a graded $G$-module, via

$$g \cdot v_1 \cdots v_k := g(v_1) \cdots g(v_k).$$

Let $S(V)^G := \{p \in S(V) : g \cdot p = p, \forall g \in G\}$ be the $G$-invariants of $S(V)$. A *reflection* is an element $T \in \mathrm{GL}(V)$ of finite order whose fixed point set is a hyperplane.

**Theorem 7** (Chevalley-Shephard-Todd)**.** *Suppose $G \to \mathrm{GL}(V)$ is faithful. The following are equivalent:*
*(a) $S(V)^G$ is a (free) polynomial algebra.*
*(b) $G$ is generated by reflections. (We call such $G$ a complex reflection group.)*

**Example 8.** Let $S_n$ act on $\mathbb{C}^n$ naturally. Then $S(V) = \mathbb{C}[x_1, \ldots, x_n]$ and $S(V)^G$ is the symmetric polynomials on $x_1, \ldots, x_n$. This is freely generated by, say, $h_1 \ldots, h_n$, so $S_n$ must be generated by reflections.

**Definition 9.** Let $G$ be a complex reflection group in $V = \mathbb{C}^n$. As it turns out, the multiset of degrees $\{d_1, \ldots, d_n\}$ of any homogeneous algebraically independent generating set of $S(V)^G$ is uniquely determined. They are called the *degrees of $G$*.

If $G$ is a finite Coxeter group, a *Coxeter element $c$* is a product of all simple reflections taken in any order. The *exponents* of $G$ are $E_{c,\mathbb{C}^n}$. All Coxeter elements are conjugate, so this is well-defined.

**Remark 10.** In fact, $|G| = d_1 \cdots d_n$. More generally, the Poincare polynomial of the coinvariant algebra of $G$ is the product of the $q$-analogues of the degrees.

**Theorem 11** (Coxeter)**.** *Let $G$ be a finite Coxeter group. Indeed, $G$ is a complex reflection group acting on some $\mathbb{C}^n$. Let $c \in G$ be a Coxeter element (i.e. a product of all simple reflections taken in any order). Then*

$$\textit{the degrees of } G = 1 + \textit{the exponents of } G$$

**Remark 12.** Springer generalized the preceding theorem to *regular elements* of $G$. Precisely, if $g \in G$ contains an eigenvector which is not contained in any hyperplane fixed by the reflections of $G$ with eigenvalue $\omega$, it is called $\omega$-regular. He showed that if $g \in G$ is $\omega_n^r$-regular, then

$$r \cdot (1 - \textit{the degrees of } G) = E_{g,\mathbb{C}^n} \ (\textit{mod } n).$$

As it happens, Coxeter elements are $\omega_n^{-1}$-regular.

**Example 13.** For $G = S_n$ acting on $\mathbb{C}^n$, the degrees are $\{1, 2, \ldots, n\}$, and we computed $E_{(12\cdots n),\mathbb{C}^n} = \{0, 1, \ldots, n-1\}$. Consequently, $|S_n| = n!$.

**Definition 14.** With $G$ as above, the *coinvariant algebra* of $G$ is

$$S(V)_G := S(V)/S(V)^G_+$$

where $S(V)^G_+$ denotes ideal generated by invariants without constant term. It is naturally a graded $G$-module.

**Theorem 15** (Chevalley)**.** *If $G$ is a complex reflection group, $S(V)_G$ is isomorphic as a $G$-module to the regular representation of $G$.*

For instance, $\dim S(V)_G = |G|$ is finite. We think of $S(V)_G$ as a "graded regular representation."

**Notation 16.** Given a (finite) set $X$ and a function $\mathrm{stat}\colon X \to \mathbb{Z}$, write

$$X^{\mathrm{stat}}(q) := \sum_{x \in X} q^{\mathrm{stat}\, x}.$$

Extend this notation to $\mathrm{stat}\colon X \to \mathbb{Z}^n$ in the obvious way, or to multiple statistics. If $\mathrm{stat}\colon X \hookrightarrow \mathbb{Z}$ is the inclusion of a subset, we omit it from the notation. We sometimes allow the codomain to be a set other than $\mathbb{Z}$.

**Definition 17.** Let $G$ be a complex reflection group acting naturally on $\mathbb{C}^n$. Let $\psi$ be an irrep of $G$. The corresponding *fake degree polynomial* is

$$\begin{aligned} G_\psi(q) &:= \{\text{irreps in } S(V)_G \text{ iso. to } \psi\}^{\deg}(q) \\ (&:= \sum_{i=0}^{N} (\text{multiplicity of } \psi \text{ in } i\text{th homog. comp. of } S(V)_G) q^i) \end{aligned}$$

**Example 18.** $G_\psi(1) = \dim \psi$.

**Motivation 19.** The fake degree polynomials are in fact generating functions for cyclic exponents. Precisely:

**Theorem 20** (Springer)**.** *Let $G$ be a finite reflection group and let $g \in G$ have order $m$. If $g$ is $\omega_m^s$-regular, then for any irrep $\psi$ of $G$,*

$$G_\psi(q^s) = E_{g,\psi}(q) \ (\mathit{mod}\ q^m - 1).$$

**Example 21.** At $q = 1$, this says $\dim \psi = \dim \psi$.

Let $G := S_n, g := (12 \cdots n)$, and let $\psi := \mathrm{sgn}\colon g \mapsto \mathrm{sgn}(g) \in \mathbb{C}^\times$ be the sign representation. Here $c \sim \mathrm{diag}(\mathrm{sgn}(s_1 \cdots s_{n-1}) = (-1)^{n-1})$. Hence

$$E_{(12\cdots n),\mathrm{sgn}}(q) = \begin{cases} q^0 & n \text{ odd} \\ q^{n/2} & n \text{ even.} \end{cases}$$

On the other hand, one-dimensional representations occur uniquely in $\mathbb{C}G$, and $S_n$ has just two since its abelianization is $C_2$. In $S(\mathbb{C}^n)_{S_n}$, the degree 0 component carries the trivial representation, and the top-degree component is spanned by the "staircase monomial" $x_1^{n-1} \cdots x_n^{n-n}$ which must then carry the sign representation. Hence

$$G_{\mathrm{sgn}}(q^{-1}) = q^{-\binom{n}{2}} \equiv \begin{cases} q^0 & n \text{ odd} \\ q^{n/2} & n \text{ even} \end{cases} \ (\mathrm{mod}\ q^n - 1).$$

*Note to self*: Ended here, about an hour in, going at a reasonable pace. Depending on audience, may need to add a review of the representation theory of $S_n$ or remove the background material on coinvariant algebras (though note the use of $V$ instead of $V^*$ for the symmetric algebra).

## Lecture 2

**Outline.**
(1) Summary of last time
(2) Fake degree polynomials and maj
(3) Cyclic exponents for $S_n$
(4) Wreath products, $G \wr S_n$-rep theory
(5) Cyclic exponents for $C_a \wr S_b$

**Summary.** Let $G$ be a finite group, $V$ a finite dimensional complex vector space, $G \to \mathrm{GL}(V)$ a representation. The *cyclic exponents* of $(g, V)$ are the multiset $E_{g,V}$ of exponents of eigenvalues of $g$ relative to, say, $\omega_{|g|} := \exp(2\pi i/|g|)$. We saw them appearing in several contexts:

(i) Branching rules inducing between $G$ and $\langle g \rangle$ (completely general)

(ii) For regular elements in reflection groups (e.g. Coxeter elements) *acting naturally* on $\mathbb{C}^n$, they're the *degrees* (up to an affine transformation, mod $n$).

(iii) For regular elements in reflection groups *acting via an irrep*, they track which irreps occur in which homogeneous components of the coinvariant algebra:

$$G_\psi(q^s) = E_{g,\psi}(q) \pmod{q^m - 1}$$

where $G_\psi(q) := \{\text{irreps in } S(V)_G \text{ iso. to } \psi\}^{\deg}(q)$, $E_{g,\psi}(q) := \sum_{e \in E_{g,\psi}} q^e$, $S(V)_G := S(V)/S(V)^G_+$.

(iv) To illustrate these points, we considered $S_n$ acting naturally on $\mathbb{C}^n$ with regular (Coxeter) element $(12\cdots n)$. For (ii), we computed the degrees of this action as $\{1, 2, \ldots, n\}$ (since $\mathbb{C}[x_1, \ldots, x_n]^{S_n} = \mathbb{C}[h_1, \ldots, h_n]$), and we computed $E_{(12\cdots n),\mathbb{C}^n} = \{0, 1, \ldots, n-1\}$ directly from the permutation matrix. For (iii), the sgn representation appears uniquely in the $\binom{n}{2}$-degree component of $S(\mathbb{C}^n)_{S_n}$, so

$$G_{\text{sgn}}(q^{-1}) = q^{-\binom{n}{2}} \equiv \begin{cases} q^0 & n \text{ odd} \\ q^{n/2} & n \text{ even} \end{cases} \pmod{q^n - 1}.$$

Alternatively, we noted that under the sign representation, $(12\cdots n) \sim \operatorname{diag}((-1)^{n-1})$, so

$$E_{(12\cdots n),\text{sgn}}(q) = \begin{cases} q^0 & n \text{ odd} \\ q^{n/2} & n \text{ even.} \end{cases}$$

**Remark 22.** Today, we'll fully describe $G_\psi(q)$ and $E_{g,\psi}(q)$ for $G = S_n$ in terms of maj on SYT ($g$ need not be regular). We'll motivate this with a result of Lusztig, Stanley and deduce a result of Kraskiewicz-Weyman as a sample corollary. We'll then give an analogous result for "most" complex reflection groups, which will require summarizing the irreps of certain wreath products.

**Definition 23.** Let $\lambda \vdash n$ denote a partition of $n$. We will use (SSYT) SYT to denote (semi)standard Young tableaux of a given shape or size. The *descent set* of $T \in \text{SYT}(n)$ is the subset of $[n-1]$ consisting of all $i$ for which the box labeled $i+1$ appears strictly below the box labeled $i$ (in English notation). The *major index* $\operatorname{maj} T$ is the sum of the descents of $T$.

**Theorem 24** (Lusztig, Stanley)**.** *Let $S_n$ act naturally on $\mathbb{C}^n$. Then*

$$\{\textit{irreps } \psi \textit{ in } S(\mathbb{C}^n)_{S_n}\}^{\deg,\text{type}}(q,t) = \text{SYT}(n)^{\text{maj},\text{shape}}(q,t),$$

*where* deg *refers to the degree of the homogeneous component containing $\psi$ and* $\text{type}(S^\lambda) := \lambda$.

**Remark 25.** The LHS is by definition the formal sum $\sum_{\lambda \vdash n} G_{S^\lambda}(q) t^\lambda$. Evaluating the parameters at 1, the LHS gives $\#\{\text{irreps in } \mathbb{C}S_n\} = \sum_{\lambda \vdash n} f^\lambda$ and the RHS gives $\#\,\text{SYT}(n)$, which agree.

**Example 26.** We have $\text{type}(\text{sgn}) = (1^n)$. Now $\text{SYT}((1^n))$ has one element with descent set $\{1, 2, \ldots, n-1\}$ and major index $\binom{n}{2}$. Correspondingly, sgn appears precisely once in the coinvariant algebra, namely in degree $\binom{n}{2}$, as we computed last week.

As a sample corollary, we have the following pretty branching rule:

**Theorem 27** (Kraskiewicz-Weyman)**.** *Let $\chi_r \colon C_n \to \text{GL}(\mathbb{C}) = \mathbb{C}^\times$ by $\chi_r(1) := \omega_n^r$. Then*

$$\chi_r\!\uparrow_{C_n}^{S_n} = \sum_{\lambda \in \text{SYT}(n)} \#\{T \in \text{SYT}(\lambda) : \operatorname{maj}(T) \equiv_n r\} S^\lambda$$

*where $C_n := \langle (12\cdots n) \rangle \subset S_n$. Moreover, the multiplicities depend only on* $\gcd(n,r)$.

*Proof.* The formula is equivalent to

$$\sum_{r=0}^{n-1} q^r (\chi_r\!\uparrow_{C_n}^{S_n})^{\text{type}}(t) \equiv \text{SYT}(n)^{\text{maj},\text{shape}}(q,t) \pmod{q^n - 1}.$$

From last time, the left-hand side is

$$\sum_{\lambda \in \text{SYT}(n)} E_{(12\cdots n),\psi}(q) t^\lambda.$$

From Springer's theorem, $E_{(12\cdots n),\psi}(q) \equiv G_{S^\lambda}(q^{-1}) \pmod{q^n - 1}$, so the left-hand side is

$$\{\text{irreps } \psi \text{ in } S(\mathbb{C}^n)_{S_n}\}^{\text{deg,type}}(q^{-1}, t).$$

Now apply the Lusztig, Stanley result, which gives the required formula up to replacing $q$ with $q^{-1}$ on the right.

For the "moreover," consider the permutation matrix of $(12\cdots n)$. It has an eigenvector $(1, \omega_n^s, \omega_n^{2s}, \ldots)$ for each $1 \leq s \leq n$ with eigenvalue $\omega_n$. It's easy to see that this eigenvector is fixed by no transposition $x_i \leftrightarrow x_j$ if and only if $\gcd(s, n) = 1$. Hence $(12\cdots n)$ is $\omega_n^s$-regular precisely when $\gcd(s, n) = 1$. Now we may apply Springer's theorem with $s$ in place of $-1$ above, which replaces $-r$ with $rs$. The result now follows from elementary number theory. □

In light of these results, we may expect the cyclic exponents for $g \in S_n$ acting on $S^\lambda$ to be a generating function on tableau related to the major index in general. In fact:

**Definition 28.** Fix a partition $\mu \vdash n$. Set $m := \operatorname{lcm}(\mu_1, \mu_2, \ldots)$. Define $\operatorname{maj}_\mu \colon \operatorname{SYT}(n) \to \mathbb{Z}/m$ as follows.

Given $T \in \operatorname{SYT}(n)$, let $D_j \in \operatorname{SYT}(\mu_1 + \cdots + \mu_j)$ consist of those entries of $T$ from 1 to $\mu_1 + \cdots + \mu_j$. Let $T_j$ be the standard skew tableaux corresponding to $D_j/D_{j-1}$ where the entries have been renumbered from 1 to $\mu_j$. Define

$$\operatorname{maj}_\mu(T) := \sum_j \frac{m}{\mu_j} \operatorname{maj} T_j \pmod{m}.$$

Note that whether or not there is a descent in $T$ at $\mu_1, \mu_1 + \mu_2, \ldots$ does not matter mod $m$.

**Theorem 29** (Stembridge; conjectured by Stanley)**.** *Let $E_{\mu,S^\lambda}$ be the multiset of cyclic exponents of any permutation in $S^n$ of cycle type $\mu$ and order $m$ acting on the irrep $S^\lambda$. Then*

$$E_{\mu,S^\lambda}(q) = \operatorname{SYT}(\lambda)^{\operatorname{maj}_\mu}(q) \pmod{q^m - 1}$$

**Example 30.** Let $\mu = (n), \lambda = (1^n)$. Then $\operatorname{maj}_\mu \equiv_n \operatorname{maj}$, so we find $E_{(12\cdots n),\text{sgn}} = \{\binom{n}{2} \pmod{n}\}$ as before.

Evaluating this expression at $q = \omega_m$ gives the character $\chi^\lambda_\mu$ of $S^\lambda$ at $\mu$, an integer. Hence the right-hand side as an element of $\mathbb{Q}(\omega_m)$ is fixed by the Galois action. This says... something, though not that $\#\{T \in \operatorname{SYT}(\lambda) : \operatorname{maj}_\mu T \equiv_m r\}$ depends only on $\gcd(r, m)$. (Hard?) Exercise: determine for which $\mu$ this is true.

We now switch gears and describe the complex reflection groups and their irreps.

**Definition 31.** Let $N, H$ be groups and let $X$ be a set with an $H$-action. Consequently, $H$ acts on $\prod_{x \in X} N$ by permuting terms $(h \cdot (n_x)_{x\in X} := (n_{h^{-1}\cdot x})_{x \in X})$. A group acting on another group is precisely a semi-direct product (decomposition), so we may define the *wreath product* of $N$ and $H$ as

$$N \wr H := \left(\prod_{x \in X} N\right) \rtimes H.$$

(The $X$-action is often left implicit.)

**Example 32.** For our purposes, we'll use $H \leq S_n$, $X = [n]$ with the natural $H$-action. For instance, $N \wr S_n$ can be thought of as the "pseudo-permutation matrices" whose non-zero entries are taken from $N$. Even more concretely, we may realize $C_a \wr S_b$ as a subgroup of $S_{ab}$: these are the permutations in $S_{ab}$ which first cyclically permute the size-$a$ blocks (independently) and then permute the $b$ size-$a$ blocks amongst themselves.

For instance, $C_2 \wr S_2 \cong \langle (12), (34), (13)(24) \rangle \cong D_8$ is a Sylow-2 subgroup of $S_4$. Indeed, the Sylow subgroups of symmetric groups are generally direct products of iterated wreath products of cyclic groups.

The following theorem motivates our interest in such wreath products:

**Theorem 33** (Shephard-Todd)**.** *The complete list of irreducible finite complex reflection groups is as follows:*

*(i) Groups of the form* $C_a \wr S_b$*;*
*(ii) Groups of finite index* $d \mid a$ *in* $C_a \wr S_b$*, namely where* $C_a^b$ *is replaced by the subgroup of elements which sum to* 0 *mod* $d$*.*
*(iii) 34 exceptions, the largest of which is* $E_8$*, of order* $696729600 = 2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$.

**Example 34.** Type $A_{n-1}$ is $C_1 \wr S_n$; type $B_n$ or $C_n$ is $C_2 \wr S_n$; type $D_n$ is the index $d = 2$ subgroup of $C_2 \wr S_n$. Where are the dihedral groups?

**Question 35.** Let $C_G$ denote the set of conjugacy classes of $G$, so $C_{S_n} \cong \{\lambda \vdash n\}$, which in general are equinumerous with the inequivalent irreps of $G$. What are the conjugacy classes of $G \wr S_n$?

**Proposition 36.** *The conjugacy classes of* $G \wr S_n$ *are naturally indexed by partition-valued functions on the conjugacy classes of* $G$*, namely*

$$C_{G \wr S_n} \cong \{\underline{\mu} \colon C_G \to \{\textit{partitions}\} : \sum_{c \in C_G} |\underline{\mu}(c)| = n\}.$$

*Proof.* We define the map; bijectivity is left as an exercise. View $G \wr S_n$ in terms of "pseudo" permutation matrices. Pick $g \in G \wr S_n$ and let $(i_1 \cdots i_k)$ be a cycle in the underlying permutation with corresponding entries $g_1, \ldots, g_k \in G$. Call the conjugacy class of $g_k \cdots g_1$ the $G$-class of the cycle $(i_1 \cdots i_k)$ in $G$. This is well-defined under cyclic rotations. Now define $\underline{\mu}(c)$ to be the partition formed from the lengths of the cycles of $g$ whose $G$-class is $c$.

("Naturally" here just means we made no choices aside from viewing $G \wr S_n$ concretely as pseudo-permutation matrices.) □

**Example 37.** For $C_a \wr S_b$, we may take the conjugacy classes of $C_a$ to be $[a]$, so the conjugacy classes of $C_a \wr S_b$ can be thought of as partition-valued functions on $[a]$ (or length $a$ sequences of partitions) whose sizes add up to $b$. At $a = 1$, this reduces to tracking cycle types.

**Question 38.** Given a complete list of irreps for $G$ and $S_n$, can we construct such a list for $G \wr S_n$? We have one more or less obvious construction.

(In the following, we mean $\mathbb{C}G$-module when we say $G$-module. The constructions work over arbitrary commutative rings, though the representation theory can differ drastically.)

**Definition 39.** Let $U$ be a $G$-module, $V$ an $S_n$-module. Define the *wreath product* $U \wr V$ as the following $G \wr S_n$-module. As a $\mathbb{C}$-vector space, it is $U^{\otimes n} \otimes V$. We define the $G \wr S_n = G^n \rtimes S_n$-action on each factor as follows. For $(g_1, \ldots, g_n) \in G^n$, define

$$(g_1, \ldots, g_n) \cdot (u_1 \otimes \cdots \otimes u_n \otimes v) := (g_1 \cdot u_1) \otimes \cdots \otimes (g_n \otimes u_n) \otimes v.$$

If $w \in S_n$, define

$$w \cdot (u_1 \otimes \cdots \otimes u_n \otimes v) := u_{w^{-1}(1)} \otimes \cdots \otimes u_{w^{-1}(n)} \otimes (w \cdot v).$$

**Example 40.** It is easy to see that if $U$ and $V$ are irreducible, then $U \wr V$ is irreducible. In general, $\dim U \wr V = (\dim U)^n (\dim V)$.

However, there are typically far more conjugacy classes in $G \wr S_n$ than these pairs can account for. In $C_a \wr S_2$, they give $2a$ representations, all one-dimensional, whereas $|C_a \wr S_2| = 2a^2$.

As another example, the wreath product of the regular representations of $G$ and $S_n$ is the regular representation of $G \wr S_n$.

The following is a more refined but still generic way to create $G \wr S_n$-modules:

**Definition 41.** Let $\alpha \models n$ denote a (weak) composition of $n$. Write $S_\alpha \leq S_n$ for the Young subgroup of $S_n$ where the first $\alpha_1$ elements of $[n]$ are permuted amongst themselves, the next $\alpha_2$ are permuted amongst themselves, etc. Note that $S_\alpha \cong \prod_i S_{\alpha_i}$.

Let $U_1, \ldots, U_t$ be $G$-modules, let $\alpha \vdash n$ have $t$ parts, and suppose $V_i$ is an $S_{\alpha_i}$-module for each $i \in [t]$. Now $U_i \wr S_{\alpha_i}$ is a $G \wr S_{\alpha_i}$-module. Taking the tensor product of these yields a $\prod_i (G \wr S_{\alpha_i})$-module, which naturally restricts to a $G \wr S_\alpha$-module. We may then induce this to a $G \wr S_n$-module:

$$(U_1 \wr V_1) \otimes \cdots \otimes (U_t \wr V_t)\uparrow_{G \wr S_\alpha}^{G \wr S_n}.$$

**Example 42.** The dimension of the tensor product is $\prod_{i=1}^t (\dim U_i)^{\alpha_i} (\dim V_i)$, and inducing multiplies this further by $|G \wr S_n|/|G \wr S_\alpha| = n!/\prod_i \alpha_i$.

In $C_a \wr S_2$, take $t = a$, let the $U_i$ range over the irreps of $C_a$, and let $V_i$ range over the irreps of $S_{\alpha_i}$. If $\alpha = (0, \ldots, 0, 2, 0, \ldots, 0)$, we get the previous $2a$ one-dimensional irreps. If $\alpha = (1, 1, 0, \ldots)$, we must have $V_1 = V_2 =$ trivial, giving a representation of dimension $1 \cdot (2!)/(1\cdot) = 2$. There are $\binom{a}{2}$ such representations, so we have (probably) accounted for

$$2a + 2^2 \cdot \binom{a}{2} = 2a^2$$

dimensions of $\mathbb{C}(C_a \wr S_2)$.

The content of the next theorem is essentially that this construction works in general.

**Theorem 43** (Specht)**.** *Let $G$ be a finite group, and suppose $U_1, \ldots, U_t$ is a complete list of inequivalent irreps for $G$. A complete list of inequivalent irreps for $G \wr S_n$ arises from considering all $\alpha \models t$, all choices of inequivalent irreps $V_i$ for $S_{\alpha_i}$, and forming all $G \wr S_n$-modules*

$$(U_1 \wr V_1) \otimes \cdots \otimes (U_t \wr V_t)\uparrow_{G \wr S_\alpha}^{G \wr S_n}.$$

**Remark 44.** Having indexed the $U_i$ by $C_G$, Specht's theorem lets us index the irreps of $G \wr S_n$ naturally with the $\underline{\mu}$ above. We are now nearly in a position to generalize Stembridge's theorem to $C_a \wr S_b$.

First, a little notation. Given $\underline{\mu} \in C_{C_a \wr S_b}$, we may think of $\underline{\mu}$ as any skew diagram where the $\underline{\mu}(i)$ have been arranged so that their rows and columns are disjoint. Then $\mathrm{SYT}(\underline{\mu})$ means the standard tableaux on any such skew diagram. Finally, write

$$r(\underline{\mu}) := \sum_{i=0}^{a-1} i|\underline{\mu}(i)|$$

(where $\underline{\mu}(0) = \underline{\mu}(a)$).

**Theorem 45** (Stembridge)**.** *Let $x \in C_a \wr S_b$ be an $n$-cycle of $C_a$-class $1$. Then*

$$E_{x,\underline{\mu}} = \{r(\underline{\mu}) + a \cdot \mathrm{maj}\, T \pmod{ab}\}_{T \in \mathrm{SYT}(\underline{\mu})}.$$

**Example 46.** Show that $\#\,\mathrm{SYT}(\underline{\mu})$ is indeed the dimension of the irrep of type $\underline{\mu}$.

**Remark 47.** What else is in the paper? (We're around halfway through, though we've skipped things.)

- Generalization of the last theorem to arbitrary $x$, $C_a$ replaced by arbitrary $G$, type $D_n$
- Fake degrees for $C_a \wr S_b$, $D_n$
- A Murnaghan-Nakayama rule for $G \wr S_n$
- A description of "difference characters" of $D_n$