SYMMETRIC GROUP CHARACTERS AS SYMMETRIC FUNCTIONS

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Lecture 1

Summary. Main things:

- New non-homogeneous bases $\{\widetilde{s}_\lambda\}$ and $\{\widetilde{h}_\lambda\}$ of SYM.
- "Evaluations" at diagonalized permutation matrices of these give symmetric group characters.
- Structure constants of $\widetilde{s}_\lambda$ are stable Kronecker coefficients
- Structure constants of $\widetilde{h}_\lambda$ are also pretty nice
- They have many transition coefficients, e.g. $h_\lambda \to \widetilde{h}_\mu$.
- They're implemented in Sage.

Outline.

(1) Some Schur-Weyl duality
(2) Evaluations $f[\Xi_\mu]$
(3) Computing $h_\lambda[\Xi_\mu]$
(4) Defining $\widetilde{h}_\lambda$, $\widetilde{s}_\lambda$

Definition 1. Let V be a polynomial $\mathrm{GL}(\mathbb{C}^m)$-module. The *Schur character* of V is the function

$$\mathrm{ch}\, V \colon \left\{ \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_m \end{pmatrix} : x_i \in \mathbb{C}^\times \right\} \to \mathbb{C}$$

given by

$$D \mapsto \mathrm{Tr}(- \mapsto D \cdot -).$$

It turns out that $\mathrm{ch}\, V \in \mathbb{C}[x_1, \ldots, x_m]$. Furthermore, the irreducible polynomial representations of $\mathrm{GL}(\mathbb{C}^m)$ are well-known modules V^λ where λ is a partition with at most m rows (unbounded number of boxes), and

$$\mathrm{ch}\, V^\lambda = s_\lambda(x_1, \ldots, x_m).$$

(See Fulton's Young Tableaux, Chapter 8.) That is, evaluations of Schur functions are characters of $\mathrm{GL}(\mathbb{C}^m)$-modules at diagonal matrices.

Definition 2. Recall that if V_1, V_2 are $\mathrm{GL}(\mathbb{C}^m)$-representations, then $V_1 \otimes_{\mathbb{C}} V_2$ is a $\mathrm{GL}(\mathbb{C}^m)$-representation with action

$$M \cdot (v_1 \otimes v_2) := (M \cdot v_1) \otimes (M \cdot v_2).$$

Indeed,

$$\mathrm{ch}(V_1 \otimes V_2) = (\mathrm{ch}\, V_1)(\mathrm{ch}\, V_2).$$

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Hence the Schur polynomial structure constants $c^\lambda_{\mu\nu}$ defined by

$$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda$$

can be interpreted as saying

$$c^\lambda_{\mu\nu} = \{\text{multiplicity of } V^\lambda \text{ in } V^\mu \otimes V^\nu\}.$$

Question 3. Our motivation is the following: what if we evaluate $f \in \mathbb{C}[x_1, \ldots, x_m]$ at just diagonalizations of permutation matrices? Can we do this for $f \in \mathrm{SYM}$, say for our usual bases?

Notation 4. Let Ξ_k denote the (multi)set of eigenvalues of the permutation matrix of a k-cycle, written

$$\Xi_k := 1, e^{2\pi i/k}, e^{2\cdot 2\pi i/k}, \ldots, e^{(k-1)2\pi i/k}.$$

Let Ξ_μ denote the multiset of eigenvalues of the matrix of a permutation of cycle type μ,

$$\Xi_\mu := \Xi_{\mu_1}, \Xi_{\mu_2}, \ldots.$$

Definition 5. Fix a partition μ. Define a map $\mathrm{SYM} \to \mathbb{C}$ by

$$p_k \mapsto (x_1^k + \cdots + x_{|\mu|}^k)(\Xi_\mu),$$

extended algebraically. Denote this as

$$f \mapsto f[\Xi_\mu].$$

Intuitively, we imagine plugging in $\Xi_\mu, 0, 0, \ldots$ to a symmetric function.

Question 6. What is $h_\lambda[\Xi_\mu]$?

To answer the question, we first introduce a slew of notation.

Notation 7.

- We write multisets as $\{\{b_1, b_2, \ldots, b_r\}\}$, or as $\{\{1^{a_1}, 2^{a_2}, \ldots, \ell^{a_\ell}\}\}$.
- A set partition of a set S is a set of subsets $\{S_1, \ldots, S_\ell\}$ such that $\varnothing \neq S_i \subset S$, $S_i \cap S_j$ for all $i \neq j$, and $S_1 \cup \cdots \cup S_\ell = S$.
- By contrast, a multiset partition of a multiset S is a multiset $\pi = \{\{S_1, \ldots, S_\ell\}\}$ such that $\varnothing \neq S_i \subset S$ and $S_1 \cup \cdots \cup S_\ell = S$, both in the multiset sense. We write $\pi \Vdash S$.
- Write $\widetilde{m}(\pi)$ to denote the partition obtained from the list of multiplicities of the multiset π. For example,

$$\widetilde{m}(\{\{\{\{1,1,2\}\}, \{\{1,1,2\}\}, \{\{1,3\}\}\}\}) = (2,1).$$

We also need some symmetric function identities. Recall that

$$p_\mu = \sum_\lambda \chi^\lambda(\mu) s_\lambda$$

where $\chi^\lambda(\mu)$ is the character of the Specht module indexed by λ at a permutation of cycle type μ. Consequently,

$$\langle s_\lambda, p_\mu \rangle = \chi^\lambda(\mu)$$

(where we've used the Hall Inner Product, which can be defined by $\langle s_\lambda, s_\mu \rangle := \delta_{\lambda=\mu}$). Similarly,

$$\langle h_\lambda, p_\mu \rangle = (\mathrm{ch}\, 1\!\uparrow_{S_\lambda}^{S_n})(\mu)$$

where $\lambda, \mu \vdash n$, $S_\lambda := S_{\lambda_1} \times S_{\lambda_2} \times \cdots$ is a Young subgroup, and ch denotes the usual character of a representation.

With all this notation, we can write out a nice answer to the above question:

Theorem 8 (Theorem 2 in O-Z). *For partitions λ, μ (not necessarily of the same size), write*

$$H_{\lambda\mu} := \langle h_{|\mu|-|\lambda|} h_\lambda, p_\mu \rangle.$$

(This is 0 if $|\lambda| > |\mu|$.) Then

$$h_\lambda[\Xi_\mu] = \sum_{\pi \Vdash \{\{1^{\lambda_1}, \ldots, \ell^{\lambda_\ell}\}\}} H_{\widetilde{m}(\pi),\mu}.$$

The theorem motivates the following recursive definition:

Definition 9. Let $\widetilde{h}_\mu$ be the family of symmetric functions defined by

$$h_\lambda = \sum_{\pi \Vdash \{\{1^{\lambda_1}, \ldots, \ell^{\lambda_\ell}\}\}} \widetilde{h}_{\widetilde{m}(\pi)}.$$

Example 10. At $\lambda = (2)$, we consider multiset partitions $\pi \Vdash \{\{1^2\}\}$, of which there are

$$\pi = \{\{\{\{1\}\}, \{\{1\}\}\}\}, \qquad \{\{\{\{1,1\}\}\}\}$$

and which have $\widetilde{m}(\pi) = (2), (1)$, respectively. Hence

$$h_{(2)} = \widetilde{h}_{(2)} + \widetilde{h}_{(1)}.$$

Similarly we find $h_{(1)} = \widetilde{h}_{(1)}$, so that we are forced to use

$$\widetilde{h}_{(2)} = h_{(2)} - h_{(1)}.$$

Corollary 11. *$\{\widetilde{h}_\mu\}$ is a well-defined basis of* SYM*, where in fact*

$$\widetilde{h}_\lambda[\Xi_\mu] = H_{\lambda\mu} = (\mathrm{ch}\, 1\uparrow_{S_{(|\mu|-|\lambda|,\lambda)}}^{S_{|\mu|}})(\mu)$$

(where the last equality requires $|\mu| \geq |\lambda|$).

The basis $\{\widetilde{h}_\mu\}$ is thus called the **induced trivial character basis**. Note that infinitely many of the evaluations of $\widetilde{h}_\lambda$ are all specified simultaneously as μ varies over all partitions.

We can do the same thing but with S^λ instead of $1\uparrow_{S_\lambda}^{S_n}$. Formally,

Definition 12. Choose $n \geq 2|\mu|$ and require

$$\widetilde{h}_\mu = \sum_{\lambda : |\lambda| \leq |\mu|} K_{(n-|\lambda|,\lambda),(n-|\mu|,\mu)} \widetilde{s}_\lambda.$$

One may check the coefficients are independent of n and that this linear system can be inverted.

Corollary 13. *$\{\widetilde{s}_\mu\}$ is a well-defined basis of* SYM*, where in fact*

$$\widetilde{s}_\lambda[\Xi_\nu] = \langle s_{(n-|\lambda|,\lambda)}, p_\nu \rangle = \chi^{(n-|\lambda|,\lambda)}(\nu)$$

where $\nu \vdash n$, and the last equality requires $n \geq |\lambda| + \lambda_1$.

The basis $\{\widetilde{s}_\mu\}$ is thus called the **irreducible character basis**.

Lecture 2

Outline.

(1) Recap
(2) Stable Kronecker coefficients
(3) $\{\widetilde{s}_\lambda\}$; $\{\widetilde{h}_\lambda\}$ structure constants
(4) A uniqueness theorem
(5) An curious observation

Remark 14. To summarize last lecture, we found:

- Two inhomogeneous bases $\{\widetilde{h}_\lambda\}$, $\{\widetilde{s}_\lambda\}$ of SYM. Top-degree pieces are h_λ, s_λ.
- Evaluations $f[\Xi_\mu]$ defined by
$$p_k[\Xi_\mu] := (x_1^k + \cdots + x_{|\mu|}^k)(\Xi_\mu)$$
where Ξ_μ is the multiset of eigenvalues of a permutation matrix of cycle type μ.
- Nice properties:
$$\widetilde{h}_\lambda[\Xi_\mu] = H_{\lambda\mu} = \langle h_{|\mu|-|\lambda|}h_\lambda, p_\mu\rangle = (\mathrm{ch}\, 1\uparrow_{S_{|\mu|-|\lambda|,\lambda}}^{S_{|\mu|}})$$
and
$$\widetilde{s}_\lambda[\Xi_\mu] = \langle s_{|\mu|-|\lambda|,\lambda}, p_\nu\rangle = \chi^{|\mu|-|\lambda|,\lambda}(\mu).$$
(Technical note: if the Schur function's index is not a partition, we define it formally by the Jacobi-Trudi formula; see equation (9) in Orellana-Zabrocki. Likewise the irreducible character is only valid at partition shapes.)

Question: what are the structure constants for $\{\widetilde{h}_\lambda\}$ and $\{\widetilde{s}_\lambda\}$?

Definition 15. The tensor product of S_n-representations U, V is the S_n-module $U \otimes_{\mathbb{C}} V$ given by
$$\sigma \cdot (u \otimes v) := (\sigma \cdot u) \otimes (\sigma \cdot v).$$
We have some constants $g_{\mu,\nu,\lambda} \in \mathbb{Z}_{\geq 0}$ such that
$$S^\mu \otimes S^\nu = \oplus_\lambda g_{\mu,\nu,\lambda} S^\lambda,$$
which are zero if λ, μ, ν are not all of the same size.

Cute fact: $g_{\mu,\nu,\lambda}$ is independent of the order of the indexes.

Definition 16. The Kronecker product is the bilinear map on SYM given by
$$s_\mu * s_\nu := \sum_\lambda g_{\mu,\nu,\lambda} s_\lambda.$$
Hence the Kronecker product of symmetric functions is the tensor product of the underlying S_n-representations.

Recall that under the Schur character, the tensor product of $\mathrm{GL}(\mathbb{C}^m)$-representations corresponds to the product of symmetric polynomials. Under the Frobenius character, the tensor product of S_n-representations corresponds to the Kronecker product of symmetric polynomials.

Theorem 17 (Murnaghan)**.** *$g_{N\oplus\mu,N\oplus\nu,N\oplus\lambda}$ converges as $N \to \infty$ (indeed, monotonically)!*

Definition 18. The stable or reduced Kronecker coefficients $\widetilde{g}_{\lambda\mu}^\gamma$ are defined by
$$s_{(n-|\lambda|,\lambda)} * s_{(n-|\mu|,\mu)} = \sum_\gamma \widetilde{g}_{\lambda\mu}^\gamma s_{n-|\gamma|,\gamma}$$
and similarly we define $\widetilde{d}_{\lambda\mu}^\gamma$ by
$$h_{(n-|\lambda|,\lambda)} * h_{(n-|\mu|,\mu)} = \sum_\gamma \widetilde{g}_{\lambda\mu}^\gamma h_{n-|\gamma|,\gamma}$$
for $n \gg 0$.

Remark 19. Interestingly, $\widetilde{g}_{\lambda\mu}^\gamma = c_{\lambda\mu}^\gamma$ when $|\lambda| + |\mu| = |\gamma|$. For a nice summary of stable Kronecker coefficients, see Briand-Orellana-Rosas, "The stability of the Kronecker product of Schur functions."

Theorem 20 (Theorem 4 in Orellana-Zabrocki)**.** *We have*
$$\widetilde{s}_\lambda \widetilde{s}_\mu = \sum_{|\nu| \leq |\lambda|+|\mu|} \widetilde{g}_{\lambda\mu}^\nu \widetilde{s}_\nu$$
and
$$\widetilde{h}_\lambda \widetilde{h}_\mu = \sum_{|\nu| \leq |\lambda|+|\mu|} \widetilde{d}_{\lambda\mu}^\nu \widetilde{h}_\nu.$$

Remark 21. Orellana-Zabrocki also give descriptions for:

- a new $\widetilde{d}^{\nu}_{\lambda\mu}$ interpretation in terms of multiset partition shuffles
- transition coefficients $h_\lambda \to \widetilde{h}_\mu$, $\widetilde{s}_\lambda \to \widetilde{h}_\mu$, $\widetilde{h}_\lambda \to \widetilde{s}_\mu$, $h_\lambda \to \widetilde{s}_\mu$, $e_\lambda \to \widetilde{s}_\mu$, $\widetilde{h}_\lambda \to p_\mu$, $\widetilde{s}_\lambda \to p_\mu$, $p_\lambda \to \widetilde{s}_\mu$.

I'd like to end with an approach to one of Orellana-Zabrocki's results, which potentially suggests consideration of a multivariate generalization of the evaluations $f[\Xi_\mu]$.

Proposition 22 (Proposition 54 in Orellana-Zabrocki)**.** *Let* $f, g \in \mathrm{SYM}$ *with* $\deg f, \deg g \leq n$. *Suppose*

$$f[\Xi_\mu] = g[\Xi_\mu]$$

for all μ *with* $|\mu| \leq n$. *Then* $f = g$.

Proof. (Unfinished; see O-Z for a complete, but different proof.) We need to show that if

$$\sum_{\lambda : |\lambda| \leq n} c_\lambda p_\lambda[\Xi_\mu] = 0$$

for all μ where $|\mu| \leq n$, then $c_\lambda = 0$ for all λ where $|\lambda| \leq n$. That is, we need the coefficient matrix $(p_\lambda[\Xi_\mu])$ to be nonsingular.

One can check that

$$p_\lambda[\Xi_\mu] = \prod_j \sum_{i : \mu_i | \lambda_j} \mu_i.$$

For instance, $p_{(2,1)}[\Xi_2] = (1^2 + (-1)^2)(1 + (-1)) = 0$, and the $j = 2$ factor is an empty sum which is also 0.

After playing around with a few examples, the determinant of this matrix tends to have quite a few factors. Playing around a bit more, let $M_0, M_1, \ldots$ and $X_1, X_2, \ldots$ be formal parameters and consider the power series

$$P_{\lambda\mu} := \prod_j \sum_{i=1}^{\infty} M_{m(\mu,i,\lambda_j)} X_i$$

where

$$\begin{aligned} m(\mu, i, a) &:= \begin{cases} \#\text{ copies of } i \text{ in } \mu & \text{if } i \mid a \\ 0 & \text{otherwise.} \end{cases} \\ &= \#\{j \in \mu : j = i \mid a\}. \end{aligned}$$

(Here partitions have been stripped of zero parts.) For instance,

$$P_{(2,1),(2)} = (M_0 X_1 + M_1 X_2 + M_0 X_3 + M_0 X_4 + M_0 \cdots)(M_0 X_1 + M_0 X_2 + M_0 \cdots).$$

Specializing $M_i \mapsto i, X_i \mapsto i$, $P_{\lambda\mu}$ becomes $p_\lambda[\Xi_\mu]$.

For fixed n, let

$$D_n := \det(P_{\lambda\mu})_{\lambda,\mu : |\lambda|, |\mu| \leq n}.$$

Further, write

$$V(M, n) := \prod_{0 \leq i < j \leq n} (M_j - M_i).$$

By brute force, we have:

$$\begin{aligned}
D_0 &= 1 \\
D_1 &= V(M,1)x_1 \\
D_2 &= V(M,2)V(M,1)x_1^3x_2 \\
D_3 &= V(M,3)V(M,1)^4x_1^7x_2^2x_3 \\
D_4 &= V(M,4)V(M,2)^2V(M,1)^6x_1^{14}x_2^5x_3^2x_4 \\
D_5 &= V(M,5)V(M,3)V(M,2)^3V(M,1)^{12}x_1^{26}x_2^9x_3^4x_4^2x_5.
\end{aligned}$$

Note that X_i for $i > n$ have canceled from D_n.

Hence, we're lead to conjecture that

$$D_n = \prod_{i=1}^{n} V(M,i)^{\alpha_i} x_i^{\beta_i}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a weak composition (probably with $\alpha_n = 1$) and $\beta = (\beta_1, \ldots, \beta_n)$ is a partition (probably with $\beta_n = 1$). The first few α, β:

n	α	β
0	()	()
1	(1)	(1)
2	$(1,1)$	$(3,1)$
3	$(4,0,1)$	$(7,2,1)$
4	$(6,2,0,1)$	$(14,5,2,1)$
5	$(12,3,1,0,1)$	$(26,9,4,2,1)$

Of course, the conjecture explains the factorization of the original coefficient matrix to a large degree and shows it to be nonsingular. ☐