CYCLIC SIEVING AND SPRINGER'S REGULAR ELEMENTS

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Outline.

(1) Coxeter group CSP
(2) Complex reflection group background
(3) RSW proof
(4) Springer's regular elements

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1. Primary goal

Our basic definition is the following:

Definition 1 (Reiner-Stanton-White)**.** Given $(W, C_n, f(q))$ where W is a finite C_n-set, and $f(q) \in \mathbb{N}[q]$, this triple exhibits the *cyclic sieving phenomenon* if

$$f(\omega_n^k) = \#\{w \in W : \sigma_n^k \cdot w = w\}$$

where $C_n = \langle \sigma_n \rangle$ and $\omega_n \in \mathbb{C}$ is a primitive nth root of unity.

Our primary goal is to prove the following theorem:

Theorem 2 ([RSW04, Thm. 1.6])**.** *Let (W, S) be a finite Coxeter system and $J \subset S$. Let C be a cyclic subgroup generated by a regular element of W. Let W^J be the set of minimal length coset representatives for W/W_J. Define*

$$W^J(q) := \sum_{w \in W^J} q^{\ell(w)}.$$

Then the triple

$$(W, C, W^J(q))$$

exhibits the cyclic sieving phenomenon (CSP).

(Why? In tomorrow's talk, we'll generalize the type A case when $C = \langle (1\ 2\ \cdots\ n) \rangle$.)

A down-to-earth special case:

Corollary 3. *Let $\mathbb{Z}/n$ act on $\binom{\mathbb{Z}/n}{k}$ naturally. Then*

$$\left(\binom{\mathbb{Z}/n}{k}, \mathbb{Z}/n, \binom{n}{k}_q \right)$$

exhibits the CSP.

Remark 4. The following are regular for any (W, S):

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- the longest element w_0;
- any Coxeter element, i.e. the product $\prod_{s\in S} s$ taken in any order, all of which are conjugate.

In type A_{n-1}, $w_0 = n(n-1)\cdots 21 \in S_n$, and the Coxeter elements are the n-cycles. Indeed, the regular elements in S_n are precisely the elements with cycle types of the form (a^b) or $(a^b, 1)$.

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2. Complex reflection groups

We first deduce the preceding theorem from a more general result. We give some background before stating it.

Definition 5. $w \in \mathrm{GL}(\mathbb{C}^n)$ is a *(pseudo-)reflection* if $|w| < \infty$ and the stabilizer of w has codimension 1; this stabilizer is called a *reflecting hyperplane.*

A *(complex) reflection group* is a finite subgroup W of $\mathrm{GL}(\mathbb{C}^n)$ generated by reflections. The *coinvariant algebra* of W is

$$A := \mathbb{C}[x_1, \ldots, x_n]/I_+^W$$

where

$$I_+^W := (p : W \cdot p = \{p\}, p(0, \ldots, 0) = 0)$$

and $w \cdot p(x_1, \ldots, x_n) = p(w \cdot x_1, \ldots, w \cdot x_n)$ using $\mathbb{C}^n = \langle e_1, \ldots, e_n\rangle \cong \mathbb{C}x_1 \oplus \cdots \oplus \mathbb{C}x_n$. This is a graded $\mathbb{C}$-algebra and a graded W-module.

Note that each $w \in W$ is diagonalizable. An element $w \in W$ is *regular* if some eigenvector of w does not lie in any of the reflecting hyperplanes of the reflections of W.

Example 6. Let $W = S_n \subset \mathrm{GL}(\mathbb{C}^n)$ be given by permutation matrices. W is generated by $s_i = (i\ i+1)$ for $1 \le i < n$ which has reflecting hyperplane $x_i = x_{i+1}$. The reflections of W are the transpositions $(i\ j)$ for $1 \le i < j \le n$ with reflecting hyperplane $x_i = x_j$. The eigenspaces of $h = (1\ 2\ \cdots\ n)$ are spanned by $(1, \omega_n^k, \ldots, (\omega_n^k)^{n-1})$, so h is regular.

I_+^W is generated by the homogeneous symmetric polynomials of positive degree, so $I_+^W = (e_1, e_2, \ldots, e_n)$. It is not immediately obvious, but a basis for the coinvariant algebtra A is given by the “staircase monomials” $\{x_1^{a_1} \cdots x_n^{a_n} : a_i \le n-i\}$, and in particular $\dim_{\mathbb{C}} A = n!$.

Definition 7. Let $R = \oplus_{i\ge 0} R_d$ be a graded $\mathbb{C}$-algebra. Its *Hilbert series* is

$$\mathrm{Hilb}(R; q) := \sum_{i\ge 0} (\dim_{\mathbb{C}} R_i) q^i.$$

Theorem 8 ([RSW04, Thm. 8.2])**.** *Let W be a complex reflection group, $\sigma \in W$ regular, $W' \le W$ any subgroup, $A^{W'}$ the W'-invariant (graded) subalgebra of the coinvariant algebra of W. Then*

$$(W/W', \langle c\rangle, \mathrm{Hilb}(A^{W'}, q))$$

exhibits the CSP.

We can deduce the first theorem from the second using the following two observations. First, finite Coxeter groups are precisely those complex reflection groups which arise from extending scalars for real reflection groups. Second:

Fact 9 ([Hil82, §IV.4])**.** Let (W, S) be a finite Coxeter system, $J \subset S$. Then

$$\mathrm{Hilb}(A^{W_J}, q) = \sum_{w\in W^J} q^{\ell(w)} = W^J(q).$$

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3. RSW proof

For the second theorem, the key is a generalization of the following well-known result:

Theorem 10 ([Che55]). *Let W be a complex reflection group. Then $A \cong \mathbb{C}W$ as W-modules.*

Theorem 11 ([Spr74, Prop. 4.5], cf. [KW01]). *Let W be a complex reflection group, $c \in W$ regular of order n, and A the coinvariant algebra of W. Let A be a $W \times C$-module via*

$$c \cdot x_i := \omega_n x_i$$

for a fixed primitive nth root of unity ω_n. Let $\mathbb{C}W$ be a $W \times C$-module via

$$(w, c) \cdot u = wuc.$$

Then $A \cong \mathbb{C}W$ as $W \times C$-modules.

Given this, we'll now sketch the proof of the second theorem:

Proof. Consider $(W, C_n, f(q))$ with $C_n = \langle \sigma_n \rangle$. Let $\chi^i \colon C_n \to \mathbb{C}^\times$ via $\chi^i(\omega_n) := \omega_n^i$.

It's straightforward to check that $(W, C_n, f(q))$ exhibits the CSP if and only if

$$f(q) \equiv \sum_{i=0}^{n-1} a_i q^i \pmod{q^n - 1}$$

where a_i is the multiplicity of χ^i in the C_n-module $\mathbb{C}[W]$. (Note: $\mathbb{C}[W]$ is the $\mathbb{C}$-vector space with basis W and the induced $\mathbb{C}C_n$-module action; $\mathbb{C}W$ is the group algebra of W.)

A restatement of this observation is the following. Let $X = \oplus_{i=0}^{n-1} X_i$ be the $\mathbb{C}$-vector space where $\dim X_i = a_i$ and σ_n acts on X_i as multiplication by ω_n^i. The triple $(W, C_n, f(q))$ exhibits the CSP if and only if $W \cong X$ as C_n-modules.

Now consider $(W/W', \langle c \rangle, \mathrm{Hilb}(A^{W'}, q))$. By Springer's theorem, $A \cong \mathbb{C}W$ as $W \times C$-modules, so $A^{W'} \cong (\mathbb{C}W)^{W'}$ as C-modules. Notice that ω_n acts on the ith component of $A^{W'}$ by multiplication by ω_n^i. It follows that $A^{W'} \cong X$ as C-modules. So, we must only show $(\mathbb{C}W)^{W'} \cong \mathbb{C}[W/W']$ as C-modules. Indeed, it's straightforward to check that

$$\Phi \colon \mathbb{C}[W/W'] \to (\mathbb{C}W)^{W'}$$

given by

$$\Phi(wW') := \sum_{u \in wW'} u$$

gives an isomorphism of C-modules. (Technically, $(\mathbb{C}W)^{W'}$ are the right W'-invariants here, but $\mathbb{C}W$ is abelian, so it's not an issue.) ∎

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4. Springer's Result

It takes some effort to translate Springer's actual result into the theorem above. This step doesn't seem to have been written down (e.g. it's not in [RSW04]), but here's my write up. We first state Springer's result in his language and then describe the translation.

Definition 12. Let G be a complex reflection group. Suppose ρ is an irreducible complex representation of G with character χ. Let $A = \oplus_{i \geq 0} A_i$ be the coinvariant algebra of G graded by degree. The *χ-exponents* of G form the multiset $\{p_j(\chi)\}_j$ of degrees of the copies of χ in A.

For instance, if $\langle \chi, A_1 \rangle = 2$, $\langle \chi, A_2 \rangle = 3$, and the rest are 0, then $\{p_j(\chi)\} = \{1, 1, 2, 2, 2\}$.

Proposition 13 ([Spr74, Prop. 4.5])**.** *Let G be a complex reflection group. Suppose $g \in G$ has an eigenvector with eigenvalue ζ contained in no reflecting hyperplane of G. Let ρ be an irreducible complex representation of G, with character χ. Then the eigenvalues of g in the representation ρ are the $\zeta^{-p_j(\chi)}$, where the $p_j(\chi)$ are the χ-exponents of G.*

We now show the two results' relatively straightforward equivalence.

Remark 14. Continue the notation of Springer's result. A consequence of [Spr74, Thm. 4.2] is that ζ above has the same multiplicative order as g, call it d.

Let $C := \langle g \rangle$ and set

$$\begin{aligned} \chi^i \colon C &\to \mathbb{C}^\times \\ g &\mapsto \zeta^{-i}. \end{aligned}$$

for $0 \leq i < d$. Note that the irreducible representations of $G \times C$ are precisely of the form $\rho \boxtimes \chi^i$ for some unique irreducible ρ of G and i, where

$$(\rho \boxtimes \chi^i)(w, c) := \chi^i(c)\rho(w).$$

The eigendecomposition of $\rho(g)$ gives the irreducible decomposition of $\rho\downarrow_C^G$. A restatement of Springer's result is then:

$$\langle \chi^i, \rho\downarrow_C^G \rangle = \text{multiplicity of } \zeta^{-i} \text{ in } \{\zeta^{-p_j(\chi)}\} = \sum_{j : j \equiv_d i} \langle \rho, A_j \rangle.$$

Giving A_j the above $G \times C$-action,

$$(w, g) \cdot p(x_1, \ldots, x_n) := p(w \cdot \zeta x_1, \ldots, w \cdot \zeta x_n),$$

we again find that g acts on A_j as multiplication by ζ^j. Hence any G-submodule of A_j is indeed a $G \times C$-submodule. It follows that

$$\langle \rho \boxtimes \chi^i, A_j \rangle = \begin{cases} \langle \rho, A_j \rangle & \text{if } i \equiv_d j \\ 0 & \text{otherwise.} \end{cases}$$

Hence another restatement of Springer's result is

$$\langle \chi^i, \rho\downarrow_C^G \rangle = \langle \rho \boxtimes \chi^i, A \rangle.$$

On the other hand, write $\underline{\mathbb{C}G}$ to denote $\mathbb{C}G$ with the above $G \times C$-action,

$$(w, g) \cdot u := wug.$$

Hence $\underline{\mathbb{C}C} \cong A$ if and only if

$$\langle \rho \boxtimes \chi^i, A \rangle = \langle \rho \boxtimes \chi^i, \underline{\mathbb{C}G} \rangle.$$

We'll directly show

$$\langle \chi^i\uparrow_C^G, \rho \rangle = \langle \rho \boxtimes \chi^i, \underline{\mathbb{C}G} \rangle$$

which, using Frobenius reciprocity, will establish the equivalence of Springer's result and the RSW restatement.

First, note that as left C-modules

$$\mathbb{C}C \cong \oplus_{j=0}^{d-1} \chi^j$$

so that as left G-modules

$$\mathbb{C}G \cong \mathbb{C}C\uparrow_C^G = \mathbb{C}G \otimes_{\mathbb{C}C} \mathbb{C}C \cong \oplus_{j=0}^{d-1} (\mathbb{C}G \otimes_{\mathbb{C}C} \chi^j).$$

Write $\underline{\chi^j\!\uparrow_C^G}$ to denote $\chi^j\!\uparrow_C^G$ with the $G \times C$-action

$$(w, g) \cdot u \otimes v := (wug) \otimes v = (wu) \otimes (g \cdot v) = \zeta^j (wu \otimes v)$$

The preceding G-module isomorphism then yields a $G \times C$-module isomorphism

$$\underline{\mathbb{C}G} \cong \oplus_{j=0}^{d-1} \underline{\chi^j\!\uparrow_C^G}.$$

Hence it suffices to show

$$\langle \rho \boxtimes \chi^i, \underline{\chi^j\!\uparrow_C^G} \rangle = \delta_{ij} \langle \rho, \chi^i\!\uparrow_C^G \rangle.$$

Since g acts on $\underline{\chi^j\!\uparrow_C^G}$ as multiplication by ζ^j, this follows as before for $\langle \rho \boxtimes \chi^i, A_j \rangle$.

References

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