SCHUBERT MULTIPLICATION RULES AND BRUHAT CHAINS

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Abstract. Seminar and pre-seminar presentation notes.

1. Pre-seminar

Definition 1. *Young's lattice* is the lattice consisting of integer partitions. Meet is intersection of Young diagrams, or component-wise min; join is union, or component-wise max. Covers add a single box. We tend to think in French notation. Examples: a row; a column; a hook. Content is constant along diagonals, increasing to the southeast, starting at, say, 0.

Definition 2. A *standard Young tableau* of skew shape λ/μ (SYT) is a saturated chain through Young's lattice starting at μ and ending at λ. Nice enumerative formula:

$$n! = \sum_{\lambda \vdash n} \#\,\mathrm{SYT}(\lambda)^2.$$

Also, the hook length formula: $\#\,\mathrm{SYT}(\lambda) = n!/\prod_c h_c$ where c ranges over the cells of λ and h_c is the length of the "hook" through c.

Definition 3. A *semi-standard Young tableaux* of skew shape λ/μ and weight ν (SSYT) is a standard Young tableaux with the additional constraint that the first ν_1 boxes are added left-to-right, the next ν_2 are added left-to-right, etc. Note that ν is a composition in general.

Definition 4. The *skew Schur functions* $s_{\lambda/\mu}$ are the weight generating functions for the semi-standard tableaux of skew shape λ/μ. Precisely,

$$s_{\lambda/\mu} = \sum_{T \in \mathrm{SSYT}(\lambda/\mu, \nu)} x^\nu$$

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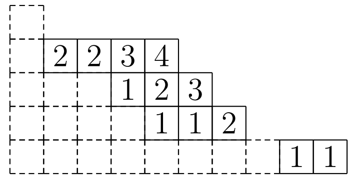

FIGURE 1. A tableau T of skew shape λ/μ in French notation with $\mu = (8,4,3,1,1) \subset (10,7,6,5,1) = \lambda$. Dashed boxes belong to μ, non-dashed boxes belong to $\lambda - \mu$. This tableau is semi-standard of weight $\alpha(T) = \nu = (5,4,2,1)$, so $K_{\lambda/\mu,\nu} \geq 1$. It has reverse reading word 112113214322, which is Yamanouchi, so $c^{\lambda}_{\mu,\nu} \geq 1$. We also have $\mathrm{ht}(\mu) = 5 = \mathrm{ht}(\lambda)$ and $|\lambda/\mu| = 29 - 17 = 12$.

where $x^\nu := \prod_i x_i^{\nu_i}$.

Example 5. $s_{(1)} = x_1 + x_2 + \cdots$, $s_{(2)} = \sum_{i \leq j} x_i x_j$. In general, $s_{(m)} = h_m$, $s_{(1^m)} = e_m$, $s_{(p,1^{q-1})} =?$. The last one is alright as a sum, though in terms of other symmetric functions, the best relation is probably the Murnaghan-Nakayama base case, $p_m = s_{(m)} - s_{(m-1,1)} + s_{(m-2,1^2)} - \cdots + (-1)^{m-1} s_{(1^m)}$.

These functions are symmetric, i.e. $s_{\lambda/\mu}(x_1, x_2, \ldots) = s_{\lambda/\mu}(w(x_1), w(x_2), \ldots)$ for any $w \colon \mathbb{P} \xrightarrow{\sim} \mathbb{P}$ (or equivalently for any $w \in S_\infty$). These functions are elements of a power series ring and actually form a $\mathbb{Z}$-basis for the ring of symmetric power series in $x_1, x_2, \ldots$ of bounded degree. *Schur polynomials* are given by restricting to just variables $x_1, \ldots, x_k$, or equivalently by setting $0 = x_{k+1} = x_{k+2} = \cdots$.

Definition 6. The Littlewood–Richardson coefficients $c^{\lambda}_{\mu,\nu}$ are defined via $s_\mu s_\nu =: \sum_\lambda c^{\lambda}_{\mu,\nu} s_\lambda$. For instance, $c^{\lambda}_{\mu,\nu} = c^{\lambda}_{\nu,\mu}$.

Theorem 7 (Pieri's rule)**.** *$s_\mu s_{(m)} = \sum s_\lambda$ where the sum is over λ obtained from μ by adding m boxes, no two in the same column. Equivalently, $s_\mu s_{(m)} = \sum_\lambda s_\lambda \# \mathrm{SSYT}(\lambda/\mu, (m))$.*

Theorem 8 (Littlewood–Richardson rule)**.** *$s_\mu s_\nu = \sum s_\lambda$ where the sum is over semi-standard tableaux T of skew shape λ/μ with weight ν which are Yamanouchi, meaning as we read the reverse reading word of T, the number of i's we've encountered is always at least as large as the number of $i-1$'s. (Ex: the first letter we read must be 1.) Equivalently, $s_\mu s_\nu = \sum_\lambda s_\lambda \# \mathrm{SSYT}(\lambda/\mu, \nu; Yam)$.*

Definition 9. The symmetric group S_n on n letters has a standard presentation as follows. Its generators are $s_1, \dots, s_{n-1}$ subject to the relations $s_i^2 = 1$, $s_i s_j = s_j s_i$ if $|i-j| > 1$, and $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ (the braid relations). Every $w \in S_n$ has a minimum length as a word in these generators called its *length*, denoted $\ell(w)$.

Example 10. $\ell(uv) = \ell(u) + \ell(v)$ fails miserably in general, ex. $\ell(s_1 s_1) = 0 \neq 1+1$. Equality does hold mod two—why? Basically, because each relation preserves parity.

The *right descent set* of $w \in S_n$ is the set of s_i such that $\ell(ws_i) < \ell(w)$. Can read them off.

Definition 11. The *Bruhat order* on S_n is given by the *subword criterion*: we declare $u \leq v$ whenever there is some minimal length word for u which is a subword for a minimal length word of v. Draw S_3's Bruhat order. The covering relations in Bruhat order are of the form $u \to ut_{ab}$ where t_{ab} interchanges a, b, $a < b$, and the permutation matrix for u is as drawn—or when $\ell(ut_{ab}) = \ell(u) + 1$. For us, the *content* of a covering relation $u \to ut_{ab}$ is $u(b)$.

Definition 12. *Schubert polynomials* are elements of $\mathbb{Z}[x_1, x_2, \dots]$ indexed by $w \in S_\infty$. They form a $\mathbb{Z}$-basis for this ring in much the same way Schur functions form a $\mathbb{Z}$-basis for their power series ring. We'll give the divided difference definition at the start of the real seminar. You can also recall Sara's definition from last week in terms of "reduced pipe dreams". In particular, the BJS formula says $\mathfrak{S}_w$ is the weight generating function for reduced pipe dreams for w, where the weight is composition formed by the number of plusses in each row of the pipe dream.

Example 13. List the rc-graphs for S_3 and the corresponding Schubert polynomials.

Definition 14. The *k-Bruhat order* on S_n for $1 \leq k < n$ is given by the following covering relations: $u \leq_k v$ whenever $v = ut_{ab}$ is a Bruhat covering relation and $a \leq k < b$. Draw 2-Bruhat order on S_3.

Proposition 15. *For $1 \leq k < n$, there is a natural inclusion of labeled directed graphs $v(-;k)\colon Y_{n,k} \hookrightarrow B_{n,k}$. It is given by $v(\lambda;k)(i) - i = \lambda_{k+1-i}$ for $1 \leq i \leq k$ with the remaining* *entries arranged in increasing order. A covering relation $\mu \to \lambda$ with content c is sent to a covering relation $\nu(\mu) \to \nu(\lambda)$ with content $c+1$. This map is rank-preserving, has image consisting of all elements comparable to the identity in k-Bruhat order, and is an isomorphism onto its image. The maps $v(-;k)$ are compatible as $n > k$ varies, so we frequently refer to the k-Grassmannian of shape λ (where $\mathrm{ht}(\lambda) \leq k$) as $v(\lambda;k) \in S_\infty$.*

In fact, if $\mu \subset \lambda$, a reduced word for $v(\lambda;k)v(\mu;k)^{-1}$ is given by forming a tableau of skew shape λ/μ, filling each box with s_{c+k} where c is the content of that box, and reading off the entries of the tableau from the topmost row to the bottommost, going right to left along each row.

Example 16. $v((1);1) \in S_3$ is just s_1. Indeed, $v(-;1)$ and $v(-;2)$ together cover all of S_3. Won't really explain, but great picture to have for how these things fit together: compositions in staircase shape yield all of S_n; the "partitions" are the image of the v's; the lowest occupied row is the k.

Correspondingly, $\mathfrak{S}_{v(\lambda;k)}(x_1, \ldots, x_n) = s_\lambda(x_1, \ldots, x_k)$. Hence in S_3 all Schubert polynomials are just Schur polynomials, which we've already computed. This begins to fail in S_4: example?

Remark 17. I very much doubt there will be extra time, but if there happens to be, jdt via growth diagrams would be a nice addition.

2. Seminar

Remark 18. Outline:

1. k-Bruhat order and Young's lattice; Schubert and Schur polynomials
2. Schubert varieties, intersections, cohomology of Grassmannians and flag manifolds
3. Monk's rule, Sottile's Pieri rule, and a conjectured generalization
4. Pattern algebras

Rationale: first five minutes understandable, so start explicit. We'll explain the geometric underpinnings of (1) in (2) which also motivates the study of multiplication rules. In (3) we give some Schubert multiplication rules and in (4) we'll introduce pattern algebras as a tool for studying such rules.

Definition 19. (Pictures: Bruhat order on S_3; descents in S_3; Schubert poly's in S_3; 1- and 2-Bruhat order on S_3 using $v(-;k)$'s.)

- Length: $\ell(w) = \#$ inv.
- Longest element: $w_0 \in S_n$ means $[n, n-1, \ldots, 1]$.
- Bruhat order: covering relations: $u \to v$ iff $v = ut_{ab}$ and $\ell(v) = \ell(u) + 1$.
- (Right) descents: for $w \in S_n$, $\mathrm{Des}(w) = \{i : w(i) > w(i+1)\} \subset [n-1]$.
- Divided difference operators: for $f \in \mathbb{Z}[x_1, x_2, \ldots]$, $\partial_i := (f - s_i \cdot f)/(x_i - x_{i+1})$. Verbal: $\partial_i^2 = 0$, etc.
- Schubert polynomials: $\mathfrak{S}_{w_0} := x_1^{n-1} x_2^{n-2} \cdots x_{n-1}^1$. Ex: (see picture)

$$\mathfrak{S}_{[231]} = \partial_2 \mathfrak{S}_{[321]} = (x^2 y - x^2 z)/(y - z) = x^2.$$

- k-Grassmannians: $w \in S_n$ such that $\mathrm{Des}(w) \subset \{k\}$.
- Bijection: Y = Young's lattice, $Y_{n,k} = \{\lambda \subset ((n-k)^k)\}$. Have bijection

$$v(-;k)\colon Y_{n,k} \to \{k\text{-Grassmannians in } S_n\}.$$

- k-Bruhat order: $u \to_k v$ iff $v = ut_{ab}$, $\ell(v) = \ell(u) + 1$, and $a \le k < b$.
- Littlewood–Richardson coefficients: $\sum_{\lambda \in Y} c^{\lambda}_{\mu,\nu} s_\lambda := s_\mu s_\nu$ and $\sum_{w \in S_\infty} c^w_{u,v} \mathfrak{S}_w := \mathfrak{S}_u \mathfrak{S}_v$.

Proposition 20. *Facts:*

- $(Y_{n,k}, \subset)$ *embeds in* $(S_n, \le_k)$ *via* $v(-;k)$.
- $\mathfrak{S}_{v(\lambda;k)} = s_\lambda(x_1, \ldots, x_k)$. *Verbal: there are* $2^n - n$ *Grassmannians in* S_n, *a worse-than-exponentially-decaying fraction of* $n!$.
- $\mathfrak{S}_w$ *well-defined under* $S_n \hookrightarrow S_{n+1}$.
- $\{\mathfrak{S}_w\}_{w \in S_\infty}$ *is* $\mathbb{Z}$*-basis for* $\mathbb{Z}[x_1, x_2, \ldots]$. *Compare with* $\{s_\lambda\}_{\lambda \in Y}$ *as* $\mathbb{Z}$*-basis for* SYM.

Remark 21. Verbal: Schubs generalize Schurs, Schurs are interesting, ergo Schubs are hopefully interesting. $c^\lambda_{\mu,\nu}$ show up in the representation theory of S_n or $\mathrm{GL}_n(\mathbb{C})$, Horn's conjecture, abelian p-group extensions, and Schubert calculus.

Schubert polynomials come from generalizing Schubert calculus. The underlying geometry is intrinsically interesting to many and at any rate gives a geometric guarantee that the combinatorial study of Schubert polynomials will prove interesting and non-trivial. It also suggests an enormous number of generalizations and variations, which we'll very briefly mention later.

Remark 22. Schubert calculus motivation. Classical case: Grassmannians and Schur functions; modern case: flag manifolds and Schubert polynomials.

Question: # lines passing through four generic lines in $\mathbb{R}^3$?

$$X = \text{affine lines through a fixed line in } \mathbb{R}^3.$$

Projectivize; identify affine lines in $\mathbb{R}^3$ with non-affine 2-planes in $\mathbb{R}^4$, so $X \subset \mathrm{Gr}_2(\mathbb{R}^4)$. Fact: X is an irreducible subvariety. WLOG,

$$X = \{V \in \mathrm{Gr}_2(\mathbb{R}^4) : \dim V \cap \mathrm{Span}\{e_1, e_2\} \geq 1\}.$$

Need to compute # points in intersection of four generic translates of X. This is a job for the Chow ring.

Remark 23. Classical Schubert calculus summary:

- Schubert varieties in $\mathrm{Gr}_k(K^n)$: closed subvarieties $\{\Omega_\lambda\}_{\lambda \in Y_{n,k}}$; $X = \Omega_{(1)}$ for $K = \mathbb{R}$.
- Chow ring: formal sums of subvarieties up to rational equivalence, graded by codimension, with multiplication (in the transverse case) being literal intersection.
- For $K = \mathbb{C}$, have $A^*(\mathrm{Gr}_k(\mathbb{C}^n), \mathbb{Z}) \cong H^*(\mathrm{Gr}_k(\mathbb{C}^n), \mathbb{Z})$.
- Schubert classes: σ_λ is the cycle of Ω_λ in A^*. Verbal: $\Omega_\lambda \hookrightarrow \mathrm{Gr}_k(K^n)$ induces injection on homology; realizes $[\Omega_\lambda] \in H_{2(k(n-k)-|\lambda|}(\mathrm{Gr}_k(\mathbb{C}^n), \mathbb{Z})$; get $\sigma_\lambda \in H^{2|\lambda|}$ by Poincare duality. $\sigma_\lambda \mapsto \sigma_\lambda$ under the above isomorphism.

- Overwhelming tradition: use H^* and $K = \mathbb{C}$.
- The cycles of Ω_λ's or σ_λ's form $\mathbb{Z}$-bases for $A^*(\mathrm{Gr}_k(K^n), \mathbb{Z})$ or $H^*(\mathrm{Gr}_k(\mathbb{C}^n), \mathbb{Z})$. Verbal: the open schubert cells yield a CW-decomposition for $\mathrm{Gr}_k(\mathbb{C}^n)$; closures are the Schubert varieties.
- Fact:

$$H^*(\mathrm{Gr}_k(\mathbb{C}^n), \mathbb{Z}) \cong \frac{\mathbb{Z}[x_1, \ldots, x_k]^{S_k}}{\mathrm{Span}_{\mathbb{Z}}\{s_\lambda(x_1, \ldots, x_k) : \lambda \notin Y_{n,k}\}} \cong \frac{\mathrm{SYM}}{\mathrm{Span}_{\mathbb{Z}}\{s_\lambda : \lambda \notin Y_{n,k}\}}.$$

  Here $\sigma_\lambda \mapsto s_\lambda(x_1, \ldots, x_k)$.

Example 24.

- How many lines? Need number of "points" in $\sigma_{(1)}^4 \in H^*(\mathrm{Gr}_2(\mathbb{C}^4), \mathbb{Z})$. A point is $\sigma_{(2,2)}$.
- Expand $s_{(1)}^4$ by Pieri; get 2. Verbal: this is easy to see if the first two lines and the last two lines intersect. The line through these intersections and the line of intersection of the planes these pairs determine are the answers.

Remark 25. Modern Schubert calculus summary:

- Fl_n: saturated chains of subspaces of $\mathbb{C}^n$.
- Schubert varieties in Fl_n: Ω_w for $w \in S_n$.
- Indeed, $\Omega_\lambda \subset \Omega_\mu$ iff $\lambda \supset \mu$, iff $v(\lambda; k) \geq v(\mu; k)$ in Bruhat. Moreover, $\Omega_v \subset \Omega_u$ iff $v \geq u$ in Bruhat.
- σ_w as before in A^* or H^* form a $\mathbb{Z}$-basis.
- Fact: (Borel)

$$H^*(\mathrm{Fl}_n, \mathbb{Z}) \cong \frac{\mathbb{Z}[x_1, \ldots, x_n]}{I_n} \cong \frac{\mathbb{Z}[x_1, x_2, \ldots]}{\mathrm{Span}_{\mathbb{Z}}\{\mathfrak{S}_w : w \notin S_n\}}$$

  where I_n is symmetric polynomials in $x_1, \ldots, x_n$ with no constant term. Here $\sigma_w \mapsto \mathfrak{S}_w$.
- $\mathrm{Fl}_n \twoheadrightarrow \mathrm{Gr}_k(\mathbb{C}^n)$ induces injection in H^* with $s_\lambda(x_1, \ldots, x_k) \mapsto \mathfrak{S}_{v(\lambda;k)}$.
- Verbal: partial flag manifolds and descent sets.

Remark 26. Schubert multiplication rules:

- Classical Schubert calculus: Pieri's rule; Littlewood–Richardson rule solve it.
- Modern Schubert calculus: more or less wide open. I'm interested in improving this situation.
- Verbal segue: have given at least some geometric motivation for everything from first part except perhaps k-Bruhat order.

Theorem 27 (Monk's Rule, 1959)**.** *We have*

$$\mathfrak{S}_u \mathfrak{S}_{s_k} = \mathfrak{S}_u \cdot (x_1 + \cdots + x_k) = \sum_{u \to_k w} \mathfrak{S}_w.$$

Verbal: shockingly powerful; can expand a Schub in the monomial basis (with this, even!) and compute action of each $x_k = \mathfrak{S}_{s_k} - \mathfrak{S}_{s_{k-1}}$ *to compute all* $c_{u,v}^w$*. Many negatives and cancellations—doesn't count.*

Remark 28. Sottile's Pieri rule background:

- Classical Pieri's rule: expand $s_{(2)} s_{(2)}$. Add boxes "no two in the same column", or equivalently add with increasing contents—draw content.
- Label covers in Y with content of added box.
- Label covers in S_n with "content": if $u \to v = ut_{ab}$, the content is $u(b) = v(a)$.
- Verbal: $v(-; k)$ respects this up to an additive constant.

Theorem 29 (Sottile's Pieri rule. Verbal: LS, BB.)**.** *We have*

$$\mathfrak{S}_u s_{(m)}(x_1, \ldots, x_k) = \sum_{*} \mathfrak{S}_w$$

where the sum is over saturated chains starting at u *with* m *covering relations through* k*-Bruhat order on* S_∞ *with increasing sequence of contents.*

Definition 30. Given a saturated chain α through Bruhat order and a partition ν of the same length, call the *filling of* ν *by* α the tableau obtained by making ν's reverse reading word the word of α, that is, its sequence of contents.

Conjecture 31. *We have*

$$c^w_{u,v(\nu;k)} = \#\{\textit{chains } \alpha\colon u \to w \textit{ in } k\textit{-Bruhat order on } S_\infty \textit{ whose filling of } \nu \textit{ by } \alpha \textit{ strictly decreases along rows and columns}\}$$

if either

(1) (Kogan case.) Each descent of u occurs at or before k.

(2) (Two-step case.) $\#(\mathrm{Des}(u) \cup \{k\}) \leq 2$.

Remark 32.

- Generalizes Monk's rule, Sottile's Pieri rule, hook rule, Littlewood–Richardson rule; works in trivial cases.
- Computationally verified through $n = 8$.
- Three-step variant false in $n = 5$ without modification.
- Hope to prove and generalize.
- Example?

Remark 33. Main approaches to Schubert multiplication rules:

- rc-graphs/pipe dreams: Billey–Bergeron, Kogan, Kogan–Kumar.
- Fomin–Kirillov algebra: FK, Postnikov, Meszaros et al.
- Bruhat order and geometry: Sottile, Bergeron–Sottile, Lenart–Sottile. Grassmannian Bruhat order, symmetric function, ABS preprint.
- Pure geometry: Pieri (classical), Vakil, Coskun.
- Knutson–Tao[–Woordward] puzzles and honeycombs; Bernstein–Zelevinski polytopes.
- Generalize and prove analogues of Monk's rule or Pieri's rule: quantum, equivariant, quantum equivariant, K-theoretic, types $B - D$, affine Grassmannians, and a few others.

Remark 34. Pattern algebras:

- Let G be a directed 5-cycle with labeled vertices.

- Let $*^k$ operate on the free $\mathbb{Z}$-module with basis the vertices of G via
$$*^k(u) := \sum_{v \in G} (\#\text{ paths } u \to v) v.$$
Algebra generated by $\{1 = *^0, *^1, *^2, \ldots\}$? Just $\mathbb{Z}[x]/(x^n - 1)$.
- Inspiration from Pieri's rule: don't want to sum over all paths, only some whose labels match certain conditions. In quantum case, don't want to just count paths, but want to weight them.
- Pattern algebra: (1) pattern monoid Γ; (2) weight function W; (3) graph G: $P(G, W)$. Verbal: functorial....
- Verbal: very general, ex. every algebra over a field is a pattern algebra. Can make incidence algebras a special case.
- Up-down algebras: Γ is words in $*, \uparrow, \downarrow$ prefixed by $*$; up-down weight W; $G = Y$ for UD or $Y_{n,k}$ for UD(n, k).
- Thm: UD is abelian; UD $\cong$ SYM; UD$(n, k) \cong H^*(\mathrm{Gr}_k(\mathbb{C}^n), \mathbb{Z})$. ("Natural basis": $\{\widetilde{s}_\lambda\}$; $\widetilde{s}_\lambda \mapsto s_\lambda$.) Verbal: more true: pattern algebras have natural bilinear forms, here gives Hall inner product, etc.
- Glued up-down algebras GUD(n): $G = S_n$, GUD$(n) \cong H^*(\mathrm{Fl}_n, \mathbb{Z})$. Verbal: alternate Sottile Pieri rule route through GUD(n)—commutativity, bijections inspired by pure algebra. Natural basis of Schubs. Hopefully these can generalize to cover conjecture—more work needed!
- GUD(n) suggests $\{h_w\}_{w \in S_n}$ generalizing h_λ's; these satisfy generalization of dominance order.