27 Diagonal Coinvariants and Tesler Matrices

The following is a brief account of the relationship between Tesler matrices and diagonal coinvariants. There are no proofs.

Definition *A Tesler matrix is an upper triangular matrix with non-negative integer entries whose "hook sums" are all* 1*. The* i*th hook sum is the difference between the sum of the* i*th row and the non-diagonal elements of the* i*th column.*

Example *There are seven* 3×3 *Tesler matrices. This list includes the identity matrix and the matrix whose third column is* $1, 1, 3$ *with* 0*'s everywhere else.*

Tesler counted these matrices and entered them into OEIS in the late 90's. His motivating work was unpublished and nothing more happened until Haglund (2011) found Tesler's entry while working on expressions for the diagonal coinvariants. Tesler's OEIS entry motivated Haglund to express the Hilbert series of the diagonal coinvariants as a generating function for Tesler matrices. This note is aimed at explaining the previous sentence and highlighting a few related results. (Tesler himself had completely forgotten his unpublished work by the time Haglund—who was working down the hall—asked him about it.)

Let L be a semisimple Lie algebra over $\mathbb{C}$ with CSA H, base $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$, and Weyl group $\mathcal{W}$. Take $\mathbb{C}[H^*]$ to be the algebra of polynomials on H, namely $\mathbb{C}[\alpha_1, \ldots, \alpha_\ell]$. Since $\mathcal{W}$ acts on H^*, it acts on $\mathbb{C}[H^*]$. In fact, it acts by $(\sigma \cdot p)(h) = p(\sigma^{-1}(h))$—see the lemma in the above exposition of §23.3.

Definition

- *Call* $\mathbb{C}[H^*]^{\mathcal{W}} := \{p \in \mathbb{C}[H^*] : \sigma \cdot p = p \text{ for all } \sigma \in \mathcal{W}\}$ *the algebra of* $\mathcal{W}$*-invariants.*
- *Write* $\mathbb{C}[H^*]^{\mathcal{W}}_+$ *for the ideal generated by homogeneous* $\mathcal{W}$*-invariants of positive degree.*
- *Call* $\mathbb{C}[H^*]/\mathbb{C}[H^*]^{\mathcal{W}}_+$ *the algebra of coinvariants.*

Given a graded R-algebra $A = \oplus_{i=0}^{\infty} A_i$, the Hilbert series of A is the dimension generating function of the homogeneous components:

$$\mathrm{Hilb}(A; q) := \oplus_{i=0}^{\infty} (\dim_R A_i) q^i.$$

For instance, $\mathrm{Hilb}(A; 1) = \dim_R A$. (For concreteness, let's say that R is a domain, and $\dim_R A$ denotes the dimension of A over the field of fractions of R.)

The algebra of coinvariants is graded, so we may ask about its Hilbert series.

Theorem 127 (Chevalley (1955))

(a) $\mathbb{C}[H^*]^{\mathcal{W}}$ *is generated (as an algebra) by* ℓ *algebraically independent, homogeneous elements of (unique) positive degrees* $d_1, \ldots, d_\ell$.

(b) $\mathbb{C}[H^*]/\mathbb{C}[H^*]^{\mathcal{W}}_+$ *is equivalent as an* S_n*-module to the regular representation.*

(c) $\mathrm{Hilb}(\mathbb{C}[H^*]/\mathbb{C}[H^*]^{\mathcal{W}}_+; q) = \prod_{i=1}^{\ell}[d_i]_q$ *where* $[d]_q := 1 + q + \cdots + q^{d-1} = (q^d - 1)/(q-1)$.

Uniqueness in (a) means the degrees are unique as a multiset regardless of the choice of algebraic generators. They are called the degrees of $\mathcal{W}$. The proof of (b) uses Galois theory.

In light of (b), the algebra of coinvariants is thought of as a graded analogue of the group algebra $\mathbb{C}[\mathcal{W}]$. For the rest of the talk, we'll restrict to type A_{n-1}, so $\ell = n-1$ and $\mathcal{W} = S_n$. Write R_n for the corresponding algebra of coinvariants.

Example *In type* A_{n-1}*, we may identify* $\mathbb{C}[H^*]$ *with* $\mathbb{C}[x_1, \ldots, x_{n-1}]$*. We may add a variable* $x_n := -x_1 - \cdots - x_{n-1}$ *so that* $\mathbb{C}[H^*] = \mathbb{C}[x_1, \ldots, x_n]/(p_1)$ *has its natural* S_n *action, where* $p_k := \sum_{i=1}^n x_i^k$*. Here the* $\mathcal{W}$*-invariants are generated by* $p_2, \ldots, p_n$ *(or by* $h_2, \ldots, h_n$*), so* $d_i = i+1$*. Hence*

$$R_n \cong \frac{\mathbb{C}[x_1, \ldots, x_n]}{(p_1, \ldots, p_n)}.$$

By (b), $R_n \cong \mathbb{C}[S_n]$ *as* $\mathbb{C}$*-modules, so*

$$\dim_{\mathbb{C}} R_n = |S_n| = n!.$$

More generally, (c) says

$$\mathrm{Hilb}(R_n; q) = [n]_q!.$$

(The "missing" $[1]_q$ *causes no harm.)*

Comment *Interestingly, the length generating function of* S_n *is*

$$\sum_{\sigma \in S_n} q^{\ell(\sigma)} = [n]_q! = \mathrm{Hilb}(R_n; q).$$

There is a geometric interpretation of the equality of the left and right sides. The short version:

$$\mathrm{Hilb}(R_n; q) = P(H^*(\mathrm{Flag}(\mathbb{C}^n); \mathbb{C}); q^{1/2}) = \sum_{\sigma \in S_n} q^{|[X_\sigma]|/2}$$

where H^* *denotes the singular cohomology ring,* $\mathrm{Flag}(\mathbb{C}^n)$ *is the complete flag variety,* P *is the Poincaré polynomial, and* $|[X_\sigma]|$ *is the degree of the cohomology class of the Schubert variety* X_σ*, namely the codimension of* X_σ*, which is* $2\ell(w)$*.*

Indeed, a nice basis for R_n *is given by monomials whose exponents are a composition fitting in a staircase, i.e.* $\{x_1^{a_1} \cdots x_n^{a_n} : 0 \leq a_i \leq n-i\}$*, of which there are* $n!$*. This connects to Schubert polynomials and rc-graphs, namely the BJS formula expressing Schubert polynomials as a weight generating function for rc-graphs uses precisely these monomials.*

Twice the variables means twice the fun. So, let S_n act diagonally on

$$\mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n] \qquad \text{by} \qquad \sigma \cdot x_i := x_{\sigma(i)}, \qquad \sigma \cdot y_i := y_{\sigma(i)}.$$

(The "diagonal" here is reflecting the fact that $S_n \times S_n$ acts naturally, while we're restrincting to the action induced by the diagonal inclusion $S_n \hookrightarrow S_n \times S_n$ given by $\sigma \mapsto (\sigma, \sigma)$.)

The S_n-invariants here are generated algebraically by

$$\{p_{k,\ell} := \sum_{i=1}^{n} x_i^k y_i^\ell \mid k, \ell \geq 0, k + \ell > 0\},$$

though these are not in general algebraically independent (for instance, if $n = 1$, then $p_{1,0}p_{0,1} = xy = p_{1,1}$). (Armstrong attributes this to Weyl, though I believe the proof should be straightforward, along the lines of Humphrey's Exercise 23.5 together with the discussion near the start of §23.1.) The above discussion motivates the following definition:

Definition *The diagonal coinvariants are*

$$\mathrm{DR}_n := \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n]/(p_{k,\ell} : k, \ell \geq 0, k + \ell > 0).$$

Note that DR_n is bigraded, i.e. we can track the x and y degrees separately to give a decomposition into homogeneous subspaces $\mathrm{DR}_n = \oplus_{i,j\geq 0}(\mathrm{DR}_n)_{i,j}$. We may generalize the Hilbert series to this context, namely we say

$$\mathrm{Hilb}(\mathrm{DR}_n; q, t) := \sum_{i,j\geq 0} (\dim_{\mathbb{C}}(\mathrm{DR}_n)_{i,j}) q^i t^j.$$

Example

(i) $\mathrm{Hilb}(\mathrm{DR}_n; t, q) = \mathrm{Hilb}(\mathrm{DR}_n; q, t)$

(ii) $\mathrm{Hilb}(\mathrm{DR}_n; q, 0) = \mathrm{Hilb}(R_n; q) = [n]_q!$

(iii) $\mathrm{Hilb}(\mathrm{DR}_n; q, q^{-1}) = ([n + 1]_q)^{n-1}/q^{\binom{n}{2}}$

(iv) $\mathrm{Hilb}(\mathrm{DR}_n; 1, 1) = \dim_{\mathbb{C}} \mathrm{DR}_n = (n + 1)^{n-1}$.

Here (i) is obvious and (ii) is easy. (iv) was a conjecture of Haiman and is immediate from (iii), which was proven by Haiman (2002) using AG.

Following up on Haiman's work, Haglund showed:

Theorem 128 (Haglund (2011)) *Let* Tes_n *denote the set of* $n \times n$ *Tesler matrices. Then*

$$\mathrm{Hilb}(\mathrm{DR}_n; q, t) = \sum_{T \in \mathrm{Tes}_n} \mathrm{weight}(T)$$

where

$$\text{weight}(T) := (-(1-q)(1-t))^{\#\,\text{Pos}(T)-n} \prod_{(i,j)\in\text{Pos}(T)} [a_{i,j}]_{q,t}$$

where $\text{Pos}(T)$ *is the set of indexes of positive entries of* $T \in \text{Tes}_n$ *and where*

$$[a]_{q,t} := q^{k-1} + q^{k-2}t + \cdots + qt^{k-2} + t^{k-1} = (q^k - t^k)/(q-t).$$

For instance, at $q = t = 1$, the only contributions come from products of precisely n non-zero entries.

Exercise *Classify such matrices and verify Example (iv) combinatorially.*

Hint/remark: these matrices should be precisely the vertexes of the Tesler polytope, of which there are $n!$ *given by placing precisely one non-zero entry in each row and figuring out what the values must be. There is a recursive way to enumerate Tesler matrices, namely use* $\text{Tes}_n \to \text{Tes}_{n-1}$ *defined by adding* $a_{k,n}$ *to* $a_{k,k}$ *followed by deleting the* n*th row and column. (Note: I have not done the exercise.)*

A final remark: much of the literature talks of "diagonal harmonics" instead of "diagonal coinvariants". See Haiman (1993) for a full discussion, but the summary is that the rings are isomorphic and their Hilbert series are the same. The diagonal harmonics arise from quotienting by polynomials which vanish when acted upon by $\mathbb{C}[H^*]_+^{\mathcal{W}}$, where the action of x_i is partial differentiation by x_i.

Further references:

- Drew Armstrong's "Bruce Saganfest" 2014 slides on Tesler matrices (has some minor typos).
- Jim Haglund's "A polynomial expression for the Hilbert series of the quotient ring of diagonal coinvariants" (2011) for the main theorem.
- Mark Haiman's "Vanishing Theorems and Character Formulas for the Hilbert Scheme of Points in the Plane" (2002) for the hard geometric arguments Haglund's reasoning is founded upon.
- Mark Haiman's "Conjectures on the Quotient Ring by Diagonal Invariants" (1993). Old and of course out of date, but beautifully written with a nice intro to the subject.
- Jim Haglund's research monograph, "The q,t-Catalan Numbers and the Space of Diagonal Harmonics: With an Appendix on the Combinatorics of Macdonald Polynomials" (2008), though it's quite dense.
- Jim Humphreys' "Reflection Groups and Coxeter Groups", Chapter 3, which gives a nice textbook presentation of the theory of invariant and coinvariant algebras for Coxeter groups.

- Claude Chevalley's "Invariants of Finite Groups Generated by Reflections" (1955), a classic five-page paper proving essentially the theorem above; Humphreys includes most of it.
- Karola Mészarós, Alejandro H. Morales, and Brendan Rhoades' "The Polytope of Tesler Matrices" (2014, preprint), for Tesler polytope vertices and more. Interestingly, the h-polynomial of the Tesler polytope is $[n]_q!$, agreeing with the Hilbert series of the coinvariant algebra and of the cohomology of the complete flag manifold, though this is not remarked upon in the paper.