Group Cohomology Lecture Notes

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Summary The Serre spectral sequence effectively computes cohomology rings for numerous classical spaces. As a sample application, we prove the Gysin sequence and use it to compute $H^*(\mathbb{C}P^n; R) \cong R[x]/(x^{n+1})$ ($\deg x = 2$) explicitly. We also summarize numerous related computations, taken from McCleary's "A User's Guide to Spectral Sequences." We discuss the cohomology of homogeneous spaces with the flag manifold as a running example, and end with some remarks on the cohomology of Grassmannians.

1 Notation

R will refer to a commutative unital ring.

If X is a topological space, $H^*(X; R)$ denotes the singular cohomology of X with coefficients in R, which is a graded, graded-commutative R-algebra using the cup product for multiplication.

Definition 2. A map $\pi \colon E \to B$ of topological spaces has the $\boxed{\text{homotopy lifting property}}$ with respect to a space Y if, given any homotopy $G \colon Y \times I \to B$ and an "initial lift" $\ell \colon Y \to E$ (meaning $\pi \ell \colon Y \to B$ is $G(-, 0) \colon Y \to B$), there is a "full lift" $\tilde{G} \colon Y \times I \to E$ (meaning $\pi \tilde{G} = G$) starting at ℓ (meaning $\tilde{G}(-, 0) = \ell$).

Definition 3. A map $\pi \colon E \to B$ with the homotopy lifting property with respect to all spaces is a $\boxed{\text{Hurewicz fibration}}$ or just a $\boxed{\text{fibration}}$. If it only has the property with respect to closed unit spheres in $\mathbb{R}^n$ (equivalently, finite CW complexes) it is a $\boxed{\text{Serre fibration}}$.

E is called the $\boxed{\text{total space}}$ and B is called the $\boxed{\text{base space}}$.

4 Remark

Suppose $\pi \colon E \to B$ is a fibration. Let $F_b := \pi^{-1}(b)$ for $b \in B$. If B is path-connected, each F_b has the same homotopy type (eg. $H^*(F_b, R)$ is constant up to isomorphism). In this case, we write $\boxed{F \hookrightarrow E \xrightarrow{p} B}$ and call F the $\boxed{\text{fiber}}$, without having any particular F_b in mind.

5 Theorem (Cohomological Serre Spectral Sequence)

Suppose we have a fibration $F \hookrightarrow E \xrightarrow{\pi} B$ where B is path-connected and F is connected. Further suppose B is simply-connected. Then there exists a first quadrant spectral sequence of algebras with

$$E_2^{p,q} \cong H^p(B; H^q(F; R)) \Rightarrow H^*(E; R).$$

Indeed, we have a multiplicative structure on $E_2^{*,*}$:

$$E_2^{p,q} \otimes E_2^{p',q'} \to E_2^{p+p',q+q'}$$
$$u \otimes v \mapsto u \cdot_2 v = (-1)^{pq'} u \smile v$$

where $\smile$ is given as before by

$$H^p(B; H^q(F; R)) \otimes H^{p'}(B; H^{q'}(F; R))$$

$$\downarrow {\scriptstyle \smile_B}$$

$$H^{p+p'}(B; H^q(F; R) \otimes H^{q'}(F; R))$$

$$\downarrow {\scriptstyle \smile_F}$$

$$H^{p+p'}(B; H^{q+q'}(F; R)).$$

6 Remark

The convergence is as algebras, which roughly means that each page of the spectral sequence has a differential bigraded algebra structure which induces the next pages' structure, and the E_∞ page is isomorphic to the induced algebra of the associated graded object of $H^*(E; R)$. See McCleary for details.

7 Proposition

$E_2^{*,0} \cong H^*(B; R)$ and $E_2^{0,*} \cong H^*(F; R)$ as algebras, using the product structure on $E_2^{*,*}$ on the left and the cup product structures on the right.

8 Theorem (The Gysin Sequence)

Suppose $F \hookrightarrow E \xrightarrow{\pi} B$ is a fibration, where B is path-connected and simply connected. If F is a $\boxed{\text{homology } n\text{-sphere}}$ for $n \geq 1$, then there is an exact sequence
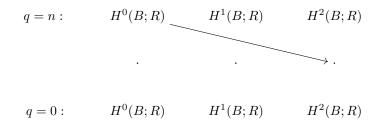
$$\cdots \to H^k(B; R) \xrightarrow{\gamma} H^{n+1+k}(B; R) \xrightarrow{\pi^*} H^{n+1+k}(E; R) \to H^{k+1}(B; R) \to \cdots,$$

where indeed $\gamma(-) = z \smile -$ for some $z \in H^{n+1}(B; R)$. Moreover, if n is even, then in fact $2z = 0$.

9 Remark

z in the theorem is called an $\boxed{\text{Euler class}}$. If we have a sphere bundle $S^n \hookrightarrow E \xrightarrow{\pi} B$ and happen to know $H^{n+1}(B; R)$ has trivial 2-torsion (for $n \geq 1$ even), then $z = 0$ so $\gamma = 0$. This severely restricts the possible sphere bundles of spheres over spheres; indeed, the four Hopf fibrations corresponding to the division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ give the only possible dimensions of the various spheres involved.

PROOF By definition, a homology n-sphere is a space X where $H_k(X; \mathbb{Z})$ is 0, except we get one copy of $\mathbb{Z}$ for $k = n$ and one more for $k = 0$ (hence $H_*(X; \mathbb{Z}) = H_*(S^n; \mathbb{Z})$). Likewise $H^k(X; R)$ is 0, except we get one copy of R for $k = n$ and for $k = 0$. The Serre spectral sequence associated to this fibration is thus mostly zero, since $H^k(F; R) = 0$ unless $k = 0, n$. Indeed, the E_2 page is in part

$$q = n: \qquad H^0(B; R) \qquad H^1(B; R) \qquad H^2(B; R)$$

$$. \qquad\qquad . \qquad\qquad \to .$$

$$q = 0: \qquad H^0(B; R) \qquad H^1(B; R) \qquad H^2(B; R)$$

It follows that $E_2 \cong \cdots \cong E_{n+1}$ and $H(E_{n+1}, d_{n+1}) \cong E_{n+2} \cong \cdots \cong E_\infty$. That is, the E_∞ page is

$$q = n: \qquad \begin{array}{ccc} \ker d_{n+1} & \ker d_{n+1} & \ker d_{n+1} \\ \cong H^n/F^1 H^n & \cong F^1 H^{n+1}/F^2 H^{n+1} & \cong F^2 H^{n+2}/F^3 H^{n+2} \end{array}$$

$$. \qquad\qquad . \qquad\qquad .$$

$$q = 0: \qquad \begin{array}{ccc} H^0(B;R)/\operatorname{im} d_{n+1} & H^0(B;R)/\operatorname{im} d_{n+1} & H^0(B;R)/\operatorname{im} d_{n+1} \\ \cong H^0 & \cong F^1 H^1 & \cong F^2 H^2 \end{array}$$

Note that for $k \geq 0$

$$0 \to E_\infty^{k,n} \to E_2^{k,n} \stackrel{d_{n+1}^{k,n}}{\to} E_2^{n+1+k,0} \to E_\infty^{n+1+k,0} \to 0$$

is an exact sequence, since

$$0 \to \ker d_{n+1}^{k,n} \hookrightarrow E_2^{k,n} \to E_2^{n+1+k,0} \twoheadrightarrow E_2^{n+1+k,0}/\operatorname{im} d_{n+1} \to 0.$$
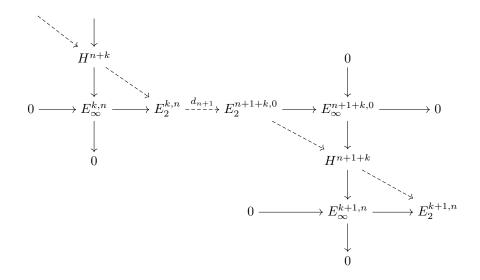
Moreover, looking at a fixed antidiagonal of degree $n+k$ for $k \geq 0$, there are two places where the containments of the filtration $F^* H^{n+k}$ might be proper, namely at $F^k H^{k+n} \supset F^{k+1} H^{k+n}$ and at $F^{k+n} H^{k+n} \supset 0$. Thus

$$0 \to E_\infty^{n+k,0} \to H^{n+k} \to E_\infty^{k,n} \to 0$$

is an exact sequence, since

$$0 \to F^{k+n} H^{k+n} \hookrightarrow F^k H^{k+n} \twoheadrightarrow F^k H^{k+n}/F^{k+1} H^{k+n} \to 0.$$

Splice these two sequences together via

to get the long exact sequence

$$\cdots \to H^{n+k} \to E_2^{k,n} \stackrel{d_{n+1}}{\to} E_2^{n+1+k,0} \to H^{n+1+k} \to \cdots$$

which in our case is

$$\cdots \to H^{n+k}(E;R) \to H^k(B;R) \stackrel{d_{n+1}}{\to} H^{n+1+k}(B;R) \to H^{n+1+k}(E;R) \to \cdots$$

To describe d_{n+1}, let $h \in H^n(F; R) \cong R$ be a generator. Identify $E_2^{*,n} = H^*(B; R) \otimes h$ and $E_2^{*,0} = H^*(B; R) \otimes 1$, so that (for instance)

$$\smile: E_2^{0,n} \otimes E_2^{k,0} \to E_2^{k,n}$$
$$(1 \otimes h) \smile (x \otimes 1) = (-1)^{n \deg x}(x \otimes h).$$

d_{n+1} is a differential in the multiplicative sense, i.e. it satisfies a Leibniz rule (up to a sign) with the multiplication $\cdot$ on $E_2^{*,*}$. Since $\smile$ is just $\cdot$ up to a sign, d_{n+1} satisfies a Leibniz rule with respect to $\smile$. Letting $d_{n+1}(1 \otimes h) = z \otimes 1$ for some $z \in H^{n+1}(B; R)$ and putting it all together, we now compute

$$
\begin{aligned}
(-1)^{n \deg x} d_{n+1}(x \otimes h) &= d_{n+1}((1 \otimes h) \smile (x \otimes 1)) \\
&= [d_{n+1}(1 \otimes h)] \smile (x \otimes 1) + (-1)^n (1 \otimes h) \smile [d_{n+1}(x \otimes 1)] \\
&= (z \otimes 1) \smile (x \otimes 1) + 0 \\
&= (z \smile x) \otimes 1.
\end{aligned}
$$

(Here $d_{n+1}(x \otimes 1) = 0$ since it lands below the x-axis.) Let $\gamma(x) = (-1)^{n \deg x} d_{n+1}(x \otimes h)$ to get the map in the theorem statement.

Finally, if n is even, since $h \smile h \in H^{2n}(F; R) = 0$, we have

$$
\begin{aligned}
0 &= d_{n+1}(1 \otimes (h \smile h)) \\
&= d_{n+1}((1 \otimes h) \smile (1 \otimes h)) \\
&= (z \otimes 1) \smile (1 \otimes h) + (-1)^n (1 \otimes h) \smile (z \otimes 1) \\
&= (2z) \otimes h.
\end{aligned}
$$

Hence $2z = 0$. (If $2 = 0$ in R, this is trivial.)

10 Example ($H^*(\mathbb{C}P^n; R)$)

Claim: $H^*(\mathbb{C}P^n; R) \cong R[x]/(x^{n+1})$ with $\deg x = 2$, for $n \geq 0$.

PROOF The $n = 0$ case is trivial. Since $\mathbb{C}P^1 \cong S^2$ via stereographic projection, the graded structure of $H^*(S^2; R)$ forces the ring structure to be trivial. So, take $n > 1$.

The quotient $\mathbb{C}^{n+1} - \{0\} \twoheadrightarrow (\mathbb{C}^{n+1} - \{0\})/\sim := \mathbb{C}P^n$ can be interpreted as

$$S^1 \hookrightarrow S^{2n+1} \to \mathbb{C}P^n,$$

a fibration (indeed, a fiber bundle). $\mathbb{C}P^n$ is in general simply-connected (and path-connected), so the Gysin sequence applies. It starts with

$$0 \to H^0(\mathbb{C}P^n; R) \xrightarrow{\gamma} H^2(\mathbb{C}P^n; R) \to H^2(S^{2n+1}; R)$$
$$\to H^1(\mathbb{C}P^n; R) \xrightarrow{\gamma} H^3(\mathbb{C}P^n; R) \to H^3(S^{2n+1}; R) \to \cdots$$

Since $2n + 1 \geq 5$, $H^2(S^{2n+1}; R) = H^3(S^{2n+1}; R) = 0$. Hence $x := \gamma(1) \in H^2(\mathbb{C}P^n; R)$ generates $H^2(\mathbb{C}P^n; R) \cong R$, and $\gamma(-) = z \smile -$ says $z = x$ in the notation of the Gysin sequence. Moreover, $H^1(\mathbb{C}P^n; R) = 0$, which can be seen in a few ways; for instance, $\mathbb{C}P^n$ has a CW complex decomposition with no one-dimensional cells. Hence $0 = H^1(\mathbb{C}P^n; R) \cong H^3(\mathbb{C}P^n; R)$. Now consider

$$\cdots \to H^{2k+1}(S^{2n+1}; R) \to H^{2k}(\mathbb{C}P^n; R) \xrightarrow{\gamma} H^{2k+2}(\mathbb{C}P^n; R) \to H^{2k+2}(S^{2n+1}; R) \to \cdots$$

If $n < k$, the first term is 0, and the last term is 0 generally, so γ is an isomorphism. Suppose inductively $H^{2k}(\mathbb{C}P^n; R) \cong R$ is generated by x^k. γ sends a generator to a generator, so $\gamma(x) = x \smile x^k = x^{k+1}$ generates $H^{2k+2}(\mathbb{C}P^n; R)$. Similarly in odd dimensions the cohomology groups are trivial.

4

On the other hand, if $n = k$, we have

$$\cdots \to H^{2n-1}(\mathbb{C}P^n; R) \to H^{2n+1}(\mathbb{C}P^n; R) \to H^{2n+1}(S^{2n+1}; R)$$
$$\to H^{2n}(\mathbb{C}P^n; R) \xrightarrow{\gamma} H^{2n+2}(\mathbb{C}P^n; R) \to H^{2n+2}(S^{2n+1}; R) \to \cdots$$

which is

$$0 \to 0 \to H^{2n}(\mathbb{C}P^n; R) \to R \xrightarrow{\gamma} 0 \to 0$$

where we've used the fact that $H^i(M; R) = 0$ for a manifold M if $i > \dim M$. It follows that $\gamma(x^k) = x \smile x^k = x^{k+1} = 0$, completing the result.

11 Theorem

Here is a summary of cohomology computations which can also be carried out with the Leray-Serre spectral sequence, taken from McCleary:

(i) Let $\boxed{SU(n)} \subset M_n(\mathbb{C})$ be the Lie group of unitary matrices of determinant 1, called the $\boxed{\text{special unitary group}}$. Then

$$H^*(SU(n); R) \cong \Lambda(x_3, x_5, \ldots, x_{2n-1}),$$

where $\deg x_i = i$ (throughout) and Λ refers to the exterior algebra (over R). (Recall this is given by formal linear combinations of k-fold tensors of the generators, subject to the relation $x \otimes x = 0$.)

(ii) Let $\boxed{\text{Sp}(n)} \subset M_n(\mathbb{H})$ be the space of linear transformations which preserve the (quaternionic) inner product, called the $\boxed{\text{symplectic group}}$. Then

$$H^*(\text{Sp}(n); R) \cong \Lambda(x_3, x_7, \ldots, x_{4n-1}).$$

(iii) Let $\boxed{SU}$ denote the $\boxed{\text{infinite special unitary group}}$, which is the direct limit (union) of special unitary groups $SU(2) \subset SU(3) \subset \cdots$ (with the natural inclusions). Then

$$H^*(SU; R) \cong \Lambda(x_3, x_5, x_7, x_9, \ldots).$$

(iv) Let $\boxed{V_k(\mathbb{C}^n)}$ denote the space of orthonormal k-frames (ordered bases) in $\mathbb{C}^n$, called the $\boxed{\text{Stiefel manifold}}$. Then

$$H^*(V_k(\mathbb{C}^n); R) \cong \Lambda(x_{2(n-k)+1}, x_{2(n-k)+3}, \ldots, x_{2n-1}).$$

(v) Let $\boxed{SO(n)} \subset M_n(\mathbb{R})$ denote the space of orthogonal matrices of determinant 1, called the $\boxed{\text{special orthogonal group}}$. Then $H^*(SO(n); \mathbb{F}_2)$ has a "simple system of generators" (see below)

$$\{x_1, x_2, \ldots, x_{n-1}\}, \qquad \deg x_i = i.$$

(vi) Let $V_k(\mathbb{R}^n)$ denote the space of orthonormal k-frames in $\mathbb{R}^n$. Then $H^*(V_k(\mathbb{R}^n); \mathbb{F}_2)$ has a simple system of generators

$$\{x_{n-k}, x_{n-k+1}, \ldots, x_{n-1}\}, \qquad \deg x_i = i.$$

(vii) Let $\boxed{K(\mathbb{Z}, n)}$ denote the $\boxed{\text{Eilenberg-Mac Lane spaces}}$. Then

$$H^*(K(\mathbb{Z}, n); \mathbb{Q}) \cong \begin{cases} \Lambda(x_n) & n \text{ odd} \\ \mathbb{Q}[x_n] & n \text{ even} \end{cases}$$

(viii) Let $B\operatorname{SO}(n)$ denote the classifying space (see below) of the special orthogonal group, and likewise with other groups we've encountered. Similarly, a lack of "(n)" or an (∞) denotes an "infinite" version. Then

$$H^*(B\operatorname{SO}(n); \mathbb{F}_2) \cong \mathbb{F}_2[w_2, \ldots, w_n]$$
$$H^*(B\operatorname{SO}; \mathbb{F}_2) \cong \mathbb{F}_2[w_2, w_3, \ldots]$$
$$H^*(BO(n); \mathbb{F}_2) \cong \mathbb{F}_2[w_1, \ldots, w_n]$$
$$H^*(BO; \mathbb{F}_2) \cong \mathbb{F}_2[w_1, w_2, \ldots]$$
$$H^*(\mathbb{R}P(\infty); \mathbb{F}_2) \cong \mathbb{F}_2[w_1].$$

Definition 12. Let H^* be a graded-commutative algebra, taken over R. A set $\{y_1, y_2, \ldots\}$ is called a $\boxed{\text{simple system of generators}}$ if the elements 1 and $x_{i_1} \cdot x_{i_2} \cdots$ with $i_1 < i_2 < \cdots$ form a basis over R for H^*. (Note: this does not determine the algebra structure fully. For instance, x_i^2 is not determined.)

13 Remark

Next we'll discuss computing the cohomology ring of a flag manifold. This will be a rough overview with many references and little rigor.

Definition 14. The $\boxed{\text{flag manifold}}$ $\boxed{\operatorname{Flag}(n)}$ as a set consists of ordered bases of $\mathbb{C}^n$, or equivalently saturated chains of subspaces in $\mathbb{C}^n$. The (complex) unitary group $U(n)$ acts on $\operatorname{Flag}(n)$ in an obvious way, and has stabilizer $T(n)$, the diagonal (complex) matrices in $U(n)$. Indeed, $\operatorname{Flag}(n) \cong U(n)/T(n)$ is a $\boxed{\text{homogeneous space}}$ and carries a Lie group structure. Note that $U(n)$ is compact, so $\operatorname{Flag}(n)$ is as well. There is an associated fibration

$$T(m) \hookrightarrow U_m(\mathbb{C}) \to U(m)/T(m).$$

15 Proposition

Associated to a Lie group G is a $\boxed{\text{classifying space}}$ BG. Indeed, given a closed subgroup $i \colon H \hookrightarrow G$, there is an associated fibration
$$G/H \hookrightarrow BH \overset{Bi}{\to} BG.$$

In our case, this looks like
$$\operatorname{Flag}(n) \hookrightarrow BT^n \overset{Bi}{\to} BU(n).$$

16 Theorem (Borel)

Let G be a connected compact Lie group, H a closed connected subgroup of maximal rank, and k a field of characteristic p. Suppose that $p = 0$ or $H^*(G; \mathbb{Z})$ and $H^*(H; \mathbb{Z})$ have no p-torsion. Then

$$H^*(G/H; k) \cong k \otimes_{H^*(BG; k)} H^*(BH; k).$$

PROOF (Statement taken from Frank Neumann's "On the cohomology of homogeneous spaces...", Journal of Pure and Applied Algebra, 1999.) Borel used the Leray-Serre spectral sequence in his Paris thesis in the early 1950's, which is still a classic reference (though it's in French). It can also be proved using the Eilenberg-Moore spectral sequence, which is covered extensively in McCleary.

17 Example

For the flag manifold, this gives

$$H^*(\operatorname{Flag}(n); \mathbb{Q}) \cong \mathbb{Q} \otimes_{H^*(BU(n); \mathbb{Q})} H^*(BT^n; \mathbb{Q}).$$

A general result (Theorem 6.38 in McCleary) gives

$$H^*(U(n); \mathbb{Q}) \cong \mathbb{Q}[y_1, \ldots, y_n],$$

and another general result (Definition 8.4) gives

$$H^*(BU(m); \mathbb{Q}) \cong \mathbb{Q}[y_1, \ldots, y_n]^{S_n},$$

where here S_n is the corresponding $\boxed{\text{Weyl group}}$, with the usual action. Putting it all together,

$$H^*(\text{Flag}(n)) \cong \mathbb{Q} \otimes_{\mathbb{Q}[y_1,\ldots,y_n]^{S_n}} \mathbb{Q}[y_1,\ldots,y_n]$$
$$\cong \frac{\mathbb{Q}[y_1,\ldots,y_n]}{(e_1,\ldots,e_n)}$$

where $y_1,\ldots,y_n$ act on $\mathbb{Q}$ by 0 and e_i is the degree i elementary symmetric polynomial on n variables. That is, we quotient by the ideal of non-constant symmetric polynomials.

18 Example

In algebraic combinatorics, the cohomology ring of both Grassmannians and flag manifolds figure prominently, with bases given by Schur and Schubert polynomials, respectively, both of which have been studied extensively. For instance, for the Grassmannian $\boxed{\text{Gr}(k,n)}$ of k-planes in $\mathbb{C}^n$, we have

$$H^*(\text{Gr}(k,n); \mathbb{Z}) \cong \frac{\mathbb{Z}[x_1,\ldots,x_k]^{S_k}}{\mathcal{I}_{k,n}}$$

where $\mathcal{I}_{k,n}$ is the ideal generated by the Schur polynomials whose diagram does not fit in a box with k rows and $n-k$ columns. The $\boxed{\text{Schur polynomials}}$ s_λ are indexed by integer partitions λ. More precisely,

$$\boxed{s_\lambda} := \sum_{T \in \text{SSYT}(\lambda)} x^T,$$

where $\boxed{\text{SSYT}(\lambda)}$ is the set of all $\boxed{\text{semi-standard Young tableaux}}$ of shape λ. That is, we form a certain diagram out of λ and label boxes with numbers from $1,\ldots,k$ so that rows weakly increase and columns strictly increase. $x^T := x_1^{a_1} \cdots x_k^{a_k}$ where a_i is the number of times the label i appears in T. We frequently forget about the underlying topological interpretation of these polynomials, but it's nice to see where they come from.