Algebraic Combinatorics Lecture Notes

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Summary Presenting on Malvenuto and Reutenauer's "Duality between Quasi-Symmetric Functions and the Solomon Descent Algebra".

There is a bialgebra structure $(\mathrm{QSYM}_{\mathbb{Q}}, m, u, \gamma, \epsilon)$ which dualizes to give a bialgebra

$$(\mathrm{QSYM}_{\mathbb{Q}}^*, \gamma^*, \epsilon^*, m^*, u^*).$$

This is isomorphic as a bialgebra to the bialgebra $\mathbb{Q}\langle t_1, t_2, \ldots\rangle$ where $\deg t_i = i$ and each t_i is primitive. They are both Hopf algebras; the latter is called the concatenation Hopf algebra. Note that the main difference between

$$\mathrm{NSYM}_{\mathbb{Q}} := \mathbb{Q}\langle H_1, H_2, \ldots\rangle$$

and the concatenation Hopf algebra is that the coproducts differ (rock-breaking vs. primitive).

There is a second coproduct γ' on QSYM with counit ϵ'. Dualizing gives a ring structure $(\mathrm{QSYM}^*, (\gamma')^*, (\epsilon')^*)$. Gessel showed the Solomon Descent Algebras Σ_n have a ring structure isomorphic to $(\mathrm{QSYM}_n^*, (\gamma')^*, (\epsilon')^*)$. (Indeed, this second coproduct and counit give a second bialgebra structure $(\mathrm{QSYM}, m, u, \gamma', \epsilon')$, which gives a second bialgebra structure $(\mathrm{QSYM}^*, (\gamma')^*, (\epsilon')^*, m^*, u^*)$.)

There is a different multiplication $*$ and a coproduct Δ on Σ (with unit and counit) such that $(\Sigma, *, \Delta)$ forms a Hopf algebra isomorphic to $(\mathrm{QSYM}^*, \gamma^*, \epsilon^*, m^*, u^*)$.

1 Remark
Outline:

1. Define QSYM^* with bialgebra structure.
2. Show (1) is naturally isomorphic to the concatenation Hopf algebra $\mathbb{Q}\langle T\rangle$.
3. Corollaries: $\mathrm{QSYM}_{\mathbb{Q}}$ is a free algebra and a free $\mathrm{SYM}_{\mathbb{Q}}$-module; antipode formula for QSYM.
4. Define QSYM^* with second algebra structure.
5. Define Solomon Descent Algebra Σ, which is naturally isomorphic to (4).
6. Define a Hopf algebra structure on Σ agreeing with (1).

Definition 2. Let T be a countable totally ordered set. $\boxed{\mathrm{QSYM}(T)}$ is the subring of formal power series $F(T)$ over $\mathbb{Z}$ in commuting variables T which are of finite degree and which have the property that, if $t_1^{c_1} \cdots t_k^{c_k}$ is a monomial in $F(T)$ (here $c_i \geq 1$), then $u_1^{c_1} \cdots u_k^{c_k}$ is a monomial in $F(T)$ with the same coefficient, for any $u_1 < \cdots < u_k$ in T.

To each (strong) composition $C = (c_1, \ldots, c_k)$, we associate a quasisymmetric function $\boxed{M_C^T}$ in $\mathrm{QSYM}(T)$ given by $\sum_{t_1 < \cdots < t_k} t_1^{c_1} \cdots t_k^{c_k}$. These are the monomial quasisymmetric functions, and they form a $\mathbb{Z}$-basis for $\mathrm{QSYM}(T)$.

Note that $\mathrm{QSYM}(X)$ and $\mathrm{QSYM}(Y)$ are canonically isomorphic, for any totally ordered sets X and Y (even if there is no order-preserving bijection between them). In particular, send M_C^X to M_C^Y.

3 Remark

Consider the tensor product $\mathrm{QSYM}(X) \otimes \mathrm{QSYM}(Y)$ (tensored over $\mathbb{Z}$). An element $x_1^{a_1} \cdots \otimes y_1^{b_1} \cdots$ with $x_1 < x_2 < \cdots$, $y_1 < y_2 < \cdots$ can naturally be identified with the element $x_1^{a_1} \cdots y_1^{b_1} \cdots$ of $\mathrm{QSYM}(X \cup Y)$ where $X \cup Y$ is totally ordered by declaring $x_i < y_j$ for all i, j. This gives a ring isomorphism $\mathrm{QSYM}(X) \otimes \mathrm{QSYM}(Y) \cong \mathrm{QSYM}(X \cup Y)$. It operates via

$$M_C^{X \cup Y} \mapsto \sum_{C=AB} M_A^X \otimes M_B^Y$$

where AB denotes the concatenation of compositions A and B.

Definition 4. Define $\boxed{\gamma}$: $\mathrm{QSYM}(T) \to \mathrm{QSYM}(X) \otimes \mathrm{QSYM}(Y)$ to be the composite

$$\mathrm{QSYM}(T) \xrightarrow{\sim} \mathrm{QSYM}(X \cup Y) \to \mathrm{QSYM}(X) \otimes \mathrm{QSYM}(Y) \xrightarrow{\sim} \mathrm{QSYM}(T) \otimes \mathrm{QSYM}(T).$$

We call γ the $\boxed{\text{outer coproduct}}$. One can check

$$\gamma(M_C^T) = \sum_{C=AB} M_A^T \otimes M_B^T.$$

From now on, we drop the alphabet from the notation when disambiguation is not needed.

5 Remark

The counit ϵ of γ is evaluation at 0, i.e. it gives the constant coefficient. Indeed, $(\mathrm{QSYM}, m, u, \gamma, \epsilon)$ is a bialgebra, where m denotes the usual multiplication and u the natural inclusion $\mathbb{Z} \to \mathrm{QSYM}$.

6 Remark

QSYM is a graded $\mathbb{Z}$-algebra, with homogeneous $\mathbb{Z}$-basis consisting of M_C of degree $|C| := \sum c_i$. (We allow $M_\varnothing = 1$.) Moreover each homogeneous component QSYM_n of QSYM is finite dimensional.

Definition 7. The $\boxed{\text{graded dual}}$ of QSYM is

$$\boxed{\mathrm{QSYM}^*} := \bigoplus_{n=0}^{\infty} \mathrm{QSYM}_n^*$$

as a $\mathbb{Z}$-module, where $\mathrm{QSYM}_n^* := \mathrm{Hom}_{\mathbb{Z}}(\mathrm{QSYM}_n, \mathbb{Z})$.

8 Remark

The maps m, u, γ, ϵ above respect the grading, so we may dualize them as well. (Note that $(\mathrm{QSYM} \otimes \mathrm{QSYM})_n := \oplus_{p+q=n} \mathrm{QSYM}_p \otimes \mathrm{QSYM}_q$.) Since each component has finite rank, the natural map $\mathrm{QSYM}_p^* \otimes \mathrm{QSYM}_q^* \to (\mathrm{QSYM}_p \otimes \mathrm{QSYM}_q)^*$ is an isomorphism, allowing us to view m^* as a coproduct with counit u^*. Similarly $(\mathrm{QSYM}^*, \gamma^*, \epsilon^*, m^*, u^*)$ is a bialgebra.

9 Remark

Since QSYM_n is finite dimensional, QSYM_n^* is isomorphic as a $\mathbb{Z}$-module to QSYM_n. Similarly, QSYM^{**} is canonically isomorphic to QSYM as a bialgebra. Graded duals V^* of other graded $\mathbb{Z}$-modules, algebras, or coalgebras, with finite rank in each (free) component, are defined in the same way. Note that $V^* \otimes V^*$ is canonically isomorphic to $(V \otimes V)^*$ for such an object.

10 Remark

We have a dual $\mathbb{Z}$-basis $\{M_C^*\}$ of QSYM^*, which is the $\mathbb{Z}$-linear function $\mathrm{QSYM} \to \mathbb{Z}$ which is 1 on M_C and 0 on M_D for $D \neq C$. Throughout, we use the isomorphism (of $\mathbb{Z}$-modules) $\mathrm{QSYM} \xrightarrow{\sim} \mathrm{QSYM}^*$ given by $M_C \mapsto M_C^*$.

As usual, the dual is non-canonically isomorphic (as a $\mathbb{Z}$-module) to the original object: we seem to like the monomial basis, so we choose it to give a "pseudo-canonical" isomorphism. However, we could theoretically use the fundamental basis, giving a different isomorphism, and neither choice is clearly "correct".

Definition 11. If V is a $\mathbb{Z}$-module, define the $\boxed{\text{pairing}}$ (i.e. $\mathbb{Z}$-bilinear map)

$$\begin{aligned}\langle -, - \rangle \colon V^* \times V &\to \mathbb{Z} \\ \langle \phi, v \rangle &\mapsto \phi(v).\end{aligned}$$

If the graded dual of V exists in the above sense, this naturally induces a pairing

$$\begin{aligned}\langle - \otimes -, - \otimes - \rangle \colon (V^* \otimes V^*) \times (V \otimes V) &\to \mathbb{Z} \\ \langle \phi \otimes \psi, v \otimes w \rangle &\mapsto \langle \phi, v \rangle \langle \psi, w \rangle \\ &= \phi(v)\psi(w).\end{aligned}$$

12 Remark

Suppose V, W have graded duals V^*, W^*. If $\Gamma \colon V \to W$, the dual $\Gamma^* \colon W^* \to V^*$ is defined by

$$(\phi \colon W \to \mathbb{Z}) \mapsto (\phi \circ \Gamma \colon V \to W \to \mathbb{Z}).$$

In particular, $\Gamma^*(\phi)(v) = \phi(\Gamma(v))$. In terms of the pairing,

$$\langle \Gamma^*(\phi), v \rangle = \langle \phi(\Gamma), v \rangle,$$

and this condition characterizes Γ^*.

13 Remark

Denote the product $\gamma^* \colon \mathrm{QSYM}^* \otimes \mathrm{QSYM}^* \to \mathrm{QSYM}^*$ by juxtaposition. It is characterized by

$$\langle \phi\psi, F \rangle = \langle \phi \otimes \psi, \gamma(F) \rangle$$

for all F. Likewise, the coproduct m^* can be defined by requiring

$$\langle \boxed{m^*(\phi)}, F \otimes G \rangle = \langle \phi, FG \rangle$$

for all $F \otimes G$. The unit remains obvious. The counit is given by $u^*(\phi) = \phi(1)$.

Definition 14. Let T be a set of *non*commuting variables and consider the free associative $\mathbb{Z}$-algebra $\mathbb{Z}\langle T \rangle$. Define coproduct $\delta(t_i) = t_i \otimes 1 + 1 \otimes t_i$, counit $t_i \mapsto 0$, $1 \mapsto 1$, and antipode $S(t_1 \cdots t_n) = (-1)^n t_n \cdots t_1$, which gives the $\boxed{\text{concatenation Hopf algebra}}$ over $\mathbb{Z}$. (This is similar to NSYM, except we use a primitive rather than rock-breaking coproduct.)

15 Remark

$\mathbb{Z}$ in the above may be replaced by $\mathbb{Q}$ (or any field) with no change whatsoever. We use $\mathrm{SYM}_{\mathbb{Q}}$, $\mathrm{QSYM}_{\mathbb{Q}}$, $\mathrm{QSYM}^*_{\mathbb{Q}}$ and the like to denote these versions.

16 Theorem (2.1)

$\mathrm{QSYM}^*_{\mathbb{Q}}$ *is canonically isomorphic as a bialgebra to the concatenation Hopf algebra* $\mathbb{Q}\langle T \rangle$, *where* $T = \{t_n : n \geq 1\}$ *and* $\deg t_n = n$.

Explicitly, the isomorphism is given by

$$t_n \mapsto \sum_{|C|=n} \frac{(-1)^{\ell(C)-1}}{\ell(C)} M^*_C =: \boxed{P_{(n)}},$$

where $\ell(C) = k$ *for* $C = (c_1, \ldots, c_k)$.

17 Remark

Note that $\mathbb{Q}\langle T \rangle$ is finitely generated in each component, and indeed $\dim(\mathbb{Q}\langle T \rangle)_n$ is the number of (strong) compositions of size n.

PROOF There are three main steps:

(1) $M^*_A M^*_B = M^*_{AB}$.

(2) $\Delta(M^*_{(n)}) = \sum_{p+q=n} M^*_{(p)} \otimes M^*_{(q)}$ for $n \geq 1$.

(3) The elements $P^*_{(n)} \in \mathrm{QSYM}^*_{\mathbb{Q}}$, $n \geq 1$ from the theorem statement are primitive.

Note that from (1) we may define an isomorphism of algebras $\mathrm{NSYM}_{\mathbb{Q}} \to \mathrm{QSYM}^*_{\mathbb{Q}}$ given by $t_n \mapsto M^*_{(n)}$. Jose asserted this is an isomorphism of Hopf algebras. (2) is a weak form of the coalgebra isomorphism. Roughly, (3) goes from the rock-breaking coproduct on NSYM to the primitive coproduct on $\mathbb{Q}\langle T\rangle$.

(1) Follows since

$$\begin{aligned}
\langle M^*_A M^*_B, M_C\rangle &= \langle M^*_A \otimes M^*_B, \gamma(M_C)\rangle \\
&= \sum_{C=A'B'} \langle M^*_A \otimes M^*_B, M_{A'} \otimes M_{B'}\rangle \\
&= \sum_{C=A'B'} \delta_{AA'}\delta_{BB'} = \delta_{AB,C} \\
&= \langle M^*_{AB}, M_C\rangle.
\end{aligned}$$

Thus $\mathrm{QSYM}^*_{\mathbb{Q}}$ is freely generated as a $\mathbb{Z}$-algebra by $\{M^*_{(i)} : i \geq 1\}$.

(2) is similar to (1) and not worth the time to discuss.

(3) begins by defining the $P^*_{(n)}$ through a generating function,

$$\sum_{n\geq 1} P^*_{(n)} t^n = \log(1 + M^*_{(1)}t + M^*_{(2)}t^2 + \cdots),$$

where these expressions live in $\mathrm{QSYM}^*[[t]]$. By comparing coefficients and using (1), the formula from the theorem statement follows, so in particular P_n is homogeneous of degree n. These elements are primitive:

$$\begin{aligned}
\sum_{n\geq 1} \Delta(P^*_{(n)})t^n &= \Delta(\log \sum_{i\geq 0} M^*_{(i)}t^i) = \log\left(\sum_{i\geq 0} \Delta(M^*_{(i)})t^i\right) \\
&= \log\left(\sum_{p,q\geq 0} M^*_{(p)}t^p \otimes M^*_{(q)}t^q\right) \\
&= \log\left((\sum_{p\geq 0}(M^*_{(p)}t^p \otimes 1)(1 \otimes \sum_{q\geq 0} M^*_{(q)}t^q)\right) \\
&= \log\left(\sum_{p\geq 0} M^*_{(p)}t^p \otimes 1\right) + \log\left(1 \otimes \sum_{q\geq 0} M^*_{(q)}t^q\right) \\
&= \log\left(\sum_{p\geq 0} M^*_{(p)}t^p\right) \otimes 1 + 1 \otimes \log\left(\sum_{q\geq 0} M^*_{(q)}t^q\right) \\
&= \sum_{n\geq 1} P^*_{(n)}t^n \otimes 1 + 1 \otimes \sum_{n\geq 1} P^*_{(n)}t^n.
\end{aligned}$$

(The fifth equality uses the fact that $\log(ab) = \log(a) + \log(b)$ when a, b commute.)

Applying exp to the generating function defining P_n, we find

$$M^*_{(n)} = \sum_{|C|=n} \frac{1}{\ell(C)!} P^*_C.$$

Hence each $M^*_{(i)}$ is a polynomial combination of P_n's, so we've found a primitive generating set. Apparently it's free; they say this follows from the formula in the theorem statement, but I don't see it immediately.

18 Corollary
$\mathrm{QSYM}_{\mathbb{Q}}$ *has a free generating set containing a free generating set of* $\mathrm{SYM}_{\mathbb{Q}}$. *In particular,* $\mathrm{QSYM}_{\mathbb{Q}}$ *is a free commutative algebra and a free* $\mathrm{SYM}_{\mathbb{Q}}$*-module.*

Proof Identify $\mathrm{QSYM}^*_{\mathbb{Q}}$ and $\mathbb{Q}\langle T\rangle$ as in the theorem. Define $t_C := t_{c_1} \cdots t_{c_k}$ for $C = (c_1, \ldots, c_k)$. Note $\{t_C\}$ is a $\mathbb{Q}$-basis, essentially by definition. Let $\{P_C\}$ denote its dual basis in $\mathrm{QSYM}_{\mathbb{Q}}$. Take L to be the set of Lyndon compositions, which is to say, the set of compositions which are Lyndon words (with respect to the natural ordering on $\mathbb{P}$), which is to say the set of compositions which are lexicographically smaller than all of their rotations. Then $\{P_\ell : \ell \in L\}$ is a free generating set for QSYM as an algebra: see Reutenauer, "Free Lie Algbras", Theorem 6.1(i).

We claim $P_{(n)} = M_{(n)}$. Since this is just the usual power symmetric functions of degree n, these are free generators of $\mathrm{SYM}_{\mathbb{Q}}$, giving the first part of the corollary. To prove the claim, roughly, use the formula for $M^*_{(n)}$ in terms of P^*_C to write M^*_D as a certain sum of P^*_C, where the sum is over $C \leq D$ (ordered by refinement). Hence the transition matrix from M^*_D to P^*_C is upper triangular. Taking duals just transposes the matrix, whence P_C is a sum over $C \leq D$ of M_D. Since $C = (n)$ is as coarse as possible for n-compositions, the sum has one term, and in fact the coefficient is 1.

The second part of the corollary is just saying

$$\mathrm{QSYM}_{\mathbb{Q}} = \mathbb{Q}[\{P_\ell\}] = \mathbb{Q}[\{P_{(n)}\}][\{P_\ell\} - \{P_{(n)}\}] = \mathrm{SYM}_{\mathbb{Q}}[\{P_\ell\} - \{P_{(n)}\}],$$

so $\mathrm{QSYM}_{\mathbb{Q}}$ is a free $\mathrm{SYM}_{\mathbb{Q}}$-algebra, and in particular a free $\mathrm{SYM}_{\mathbb{Q}}$-module.

Definition 19. Let F_C denote the fundamental quasisymmetric function indexed by C, let I be the usual stars and bars bijection between (strong) compositions of n and subsets of $[n-1]$. Courting ambiguity, we denote both I and its inverse by the same letter I, relying on context to disambiguate. Let $\overline{C}$ denote the reverse of the composition C. Define ω to be the involution on n-compositions given by applying I, taking the complement in $[n-1]$, applying I, and reversing the resulting composition.

20 Example
$\omega((2, 1, 3, 2, 1)) = (2, 2, 1, 3, 1)$:

$$\begin{aligned} 2+1+3+2+1 &\mapsto **\,|\,*\,|\,***\,|\,**|* \\ &\mapsto *|\,***\,|\,*\,|\,**|\,** \\ &\mapsto 1+3+1+2+2 \\ &\mapsto 2+2+1+3+1. \end{aligned}$$

21 Corollary
$\mathrm{QSYM}_{\mathbb{Q}}$ *is a Hopf algebra with antipode* S *equivalently defined by either*

$$S(M_C) := \sum_{C \leq D} (-1)^{\ell(C)} M_{\overline{D}} \qquad \text{or} \qquad S(F_C) := (-1)^{|C|} F_{\omega(C)}.$$

PROOF The concatenation Hopf algebra $\mathbb{Q}\langle T\rangle$ from the theorem has antipode S^* given by $S^*(t_1\cdots t_k) = (-1)^k t_k\cdots t_1$, which is the unique anti-automorphism of $\mathbb{Q}\langle T\rangle$ such that $S^*(t_n) = -t_n$. Hence $\mathrm{QSYM}^*_{\mathbb{Q}}$ is a Hopf algebra with antipode S^* determined by $S^*(P^*_{(n)}) = -P^*_{(n)}$. Since $\mathrm{QSYM}^{**}_{\mathbb{Q}}$ is canonically isomorphic to $\mathrm{QSYM}_{\mathbb{Q}}$ as a bialgebra, $\mathrm{QSYM}_{\mathbb{Q}}$ is a Hopf algebra with antipode $S^{**} = S$, which we now compute.

Using generating functions, one may show

$$S^*(M^*_{(n)}) = \sum_{|C|=n} (-1)^{\ell(C)} M^*_C.$$

It follows that

$$S^*(M^*_D) = \sum_{C\le D} (-1)^{\ell(C)} M^*_{\overline{C}};$$

the $\overline{C}$ comes from the fact that S^* is an *anti*automorphism; eg. try the $D = (n, m)$ case. Applying duality (and reversing all compositions in sight) gives the first formula. The second formula follows with a little more work; see the paper for details.

Definition 22. Define $\boxed{\omega}$: $\mathrm{QSYM}_{\mathbb{Q}} \to \mathrm{QSYM}_{\mathbb{Q}}$ to be the linear map given by $\omega(F_C) = F_{\omega(C)}$.

23 Corollary

ω is an antiautomorphism of $\mathrm{QSYM}_{\mathbb{Q}}$ which extends the usual conjugation automorphism of $\mathrm{SYM}_{\mathbb{Q}}$.

PROOF Since $\omega(F_C) = (-1)^{|C|}S(F_C)$, ω is an antiautomorphism. Since $\omega(n) = (1^n)$, we have $\omega(F_{(n)}) = F_{1^n}$, which is $\omega(h_n) = e_n$ using the complete homogeneous and elementary symmetric polynomials. This property characterizes the usual conjugation automorphism.

24 Remark

The theorem was useful. Our next goal is to define the Solomon Descent Algebras, recall previous results, and endow it with a new Hopf algebra structure making it isomorphic to $\mathrm{QSYM}^*_{\mathbb{Q}}$ from above.

Definition 25. Let X, Y be countable totally ordered sets. Define XY as $X\times Y$ with lecicographic order. Recall the canonical map $\mathbb{Z}[XY] \to \mathbb{Z}[X\cup Y]$ given by $M^{XY}_C \mapsto M^{X\cup Y}_C$. Suppressing other similar canonical maps, define a second coproduct γ' on QSYM to be the composite

$$\boxed{\gamma'}\colon \mathrm{QSYM}(T) \to \mathrm{QSYM}(T)\otimes \mathrm{QSYM}(T)$$
$$\mathrm{QSYM}(XY) \to \mathrm{QSYM}(X\cup Y) \to \mathrm{QSYM}(X)\otimes\mathrm{QSYM}(Y)$$
$$\to \mathrm{QSYM}(Y)\otimes\mathrm{QSYM}(X).$$

Let $\boxed{\epsilon'}$ be the counit determined by $\epsilon'(F_{(n)}) = 1$ and $\epsilon'(F_C) = 0$ for $\ell(C)\ge 2$.

26 Remark

More concretely, Gessel showed (and Jair said)

$$\gamma'(F_{D(\pi)}) = \sum_{\sigma\tau=\pi} F_{D(\sigma)}\otimes F_{D(\tau)}$$

where $D(\pi) := I(\mathrm{Des}(\pi))$.

These operations give QSYM^* a second bialgebra structure $(\mathrm{QSYM}, m, u, \gamma', \epsilon')$.

Definition 27. Let $n\ge 0$ and $I\subset[n-1]$. Define

$$\boxed{D_I} := \sum_{\substack{\sigma\in S_n\\ \mathrm{Des}(\sigma)=I}} \sigma \in \mathbb{Z}S_n.$$

Say $\deg D_i := n$. Let $\boxed{\Sigma_n} := \mathrm{Span}_{\mathbb{Z}}\{D_I\}$ be the $\boxed{\text{Solomon Descent Algebra}}$. (Note: it is not obvious that it is closed under multiplication.)

28 Theorem (Solomon, 1976)
Σ_n is a subalgebra of $\mathbb{Z}S_n$.

29 Remark
See for instance Schocker 2004, "The Descent Algebra of the Symmetric Group", for a survey of relatively recent work, a formula for the expansion coefficients, and much more. Note he uses $(\mathbb{Z}S_n)^{\mathrm{op}}$. This algebra is also implemented in Sage; see "Descent Algebras". Solomon in fact defined similar algebras for all Coxeter groups.

30 Remark
Malvenuto and Reutenauer define $\Sigma := \oplus_{n\geq 0}\Sigma_n$ with a (non-unital) ring structure given by $\sigma\tau = 0$ if σ, τ do not belong to the same S_n. They claim in Theorem 3.2 that Σ is isomorphic to $\mathrm{QSYM}^*((\gamma')^*)$ as a not-necessarily-unital ring, but since γ' had a counit ϵ', the latter is a unital ring, forcing the former to be as well, a contradiction. More concretely, if you compute the product of two elements from different homogeneous components of QSYM^* using their bijection, you always get 0 on the Σ side, which is nonsense.

They almost surely meant the slight variation below, given by (Gessel, 1984). They do not use this theorem for anything more than motivation.

31 Theorem (Gessel, 1984)
$(\mathrm{QSYM}^*_n, (\gamma')^*)$ is isomorphic as a ring to Σ_n, with $F^*_C \leftrightarrow D_C$.

32 Remark
Letting juxtaposition denote $(\gamma')^*$, as before this product on QSYM^*_n is characterized by

$$\langle \phi\psi, F\rangle = \langle \phi \otimes \psi, \gamma'(F)\rangle.$$

Using $\phi = F^*_A, \psi = F^*_B, F = F_C$, and applying the theorem, the right-hand side is the coefficient of $F_A \otimes F_B$ in $\gamma'(F_C)$ and the left-hand side is the coefficient of D_C in the product $D_A D_B$. Compactly,

$$(\gamma'(F_C))_{A\otimes B} = (D_A D_B)_C.$$

33 Theorem
Let $\mathbb{Z}S := \oplus_{n\geq 0}\mathbb{Z}S_n$ as a $\mathbb{Z}$-module. There is a product $*$ and coproduct Δ on $\mathbb{Z}S$ (with unit and counit) which make $\mathbb{Z}S$ into a Hopf algebra. Indeed, $\Sigma := \oplus_{n\geq 0}\Sigma_n \subset \mathbb{Z}S_\infty$ is a Hopf subalgebra, and $(\mathrm{QSYM}^*, \gamma^*, \epsilon^*, m^*, u^*)$ is isomorphic to Σ as a Hopf algebra via $F^*_C \leftrightarrow D_C$.

34 Remark
Note that Σ no longer has a product induced by the group algebra in any sense.

Proof Lengthy; main tool is the "shuffle Hopf algebra"; they connect $*$ to the convolution in $\mathrm{End}(\mathbb{Z}\langle T\rangle)$ and consider a second (ultimately dual) bialgebra structure on $\mathbb{Z}S$; see their paper for details. We merely define the operations involved.

Definition 35. Let $\boxed{\mathrm{str}}$ denote the $\boxed{\text{straightening}}$ in S_n of a word (of length n) on a totally ordered alphabet. Call $\mathrm{str}(w)$ the $\boxed{\text{standard permutation}}$ of w.

36 Example
$\mathrm{PIAZZA} \mapsto - - 1 - -2 \mapsto -31 - -2 \mapsto 431 - -2 \mapsto 431562 = \mathrm{str}(\mathrm{PIAZZA})$.

Definition 37. For $\sigma \in S_n$ and $I \subset [n]$, let $\boxed{\sigma|I}$ denote the word obtained from σ (viewed as a word on $[n]$ in one-line notation) where only letters in I are kept.

Definition 38. Define a coproduct Δ on $\mathbb{Z}S$ by

$$\boxed{\Delta}(\sigma) := \sum_{i=0}^{n} \sigma|[1,i] \otimes \mathrm{str}(\sigma|[i+1,n]).$$

39 Example

$$\begin{aligned}\Delta(3124) &= \lambda \otimes 3124 + 1 \otimes \mathrm{str}(324) + 12 \otimes \mathrm{str}(34) + 312 \otimes \mathrm{str}(4) + 3124 \otimes \lambda \\ &= \lambda \otimes 3124 + 1 \otimes 213 + 12 \otimes 12 + 312 \otimes 1 + 3124 \otimes \lambda.\end{aligned}$$

Here $\lambda \in S_0$ is the empty word. The associated counit is given by $\lambda \mapsto 1$ and $\sigma \mapsto 0$ for $\sigma \in S_n$ with $n \geq 1$.

Definition 40. Define a product $\boxed{*}$ on $\mathbb{Z}S_\infty$ as follows. For $\sigma \in S_n, \tau \in S_m$, let

$$\sigma * \tau = \sum uv$$

where the sum is over words u, v in $[n+q]$ such that u, v together are a disjoint union of $[n+q]$, $\mathrm{str}(u) = \sigma$, and $\mathrm{str}(v) = \tau$.

41 Example
$12 * 12 = 1234 + 1324 + 1423 + 2314 + 2413 + 3412.$

Evidently λ is the identity.