Introduction to Algebraic Geometry Lecture Notes

Lecturer: Sándor Kovács; transcribed by Josh Swanson

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Abstract

The following notes were taking during a pair of graduate courses on introductory Algebraic Geometry at the University of Washington in Winter and Spring 2016. Please send any corrections to `jps314@uw.edu`. Thanks!

Contents

January 4th, 2016: Zariski toplogy, algebraic sets and radical ideals

1 Remark

This course will be a fair amount of work. Homework questions will be given throughout the lecture, like finishing proofs. They are in principle due during the next lecture, though they will not be collected. There may also be more formal homeworks, with more details to come.

2 Notation

Let k be a field. Most of the time it will be of arbitrary characteristic and algebraically closed. The best way to deal with non-algebraically closed fields is roughly to work over the algebraic closure and then restrict to the original subfield.

3 Definition

$\boxed{\mathbb{A}^n} := \mathbb{A}^n_k := \{(x_1, \ldots, x_n) : x_i \in k\}$ is $\boxed{\text{affine } n\text{-space}}$ over the field k. Here we imagine there is no distinguished origin point, so we think of this initially as just a set.

Let $A := k[x_1, \ldots, x_n]$ and pick $f \in A$. We may consider f as a function $f : \mathbb{A}^n \to k$ by $P \mapsto f(P)$. Define the $\boxed{\text{vanishing set of } f}$ as

$$\boxed{Z(f)} := \{P \in \mathbb{A}^n : f(P) = 0\}.$$

Note that we may "change base points" by linear substitutions of the variables. More generally, if $T \subset A$, define the $\boxed{\text{vanishing set of } T}$ as

$$\boxed{Z(T)} := \{P \in \mathbb{A}^n : f(P) = 0, \forall f \in T\}.$$

4 Remark

For all $T \subset A$, there exist *finitely many* $f_1, \ldots, f_r \in A$ such that $Z(T) = Z(f_1, \ldots, f_r)$. Why? It's easy to see that $Z(T) = Z((T))$, and by Hilbert's basis theorem, $(T) \subset A$ is finitely generated.

5 Notation

We'll write $\boxed{I \lhd A}$ to mean that I is an ideal in A.

6 Definition

An $\boxed{\text{algebraic set}}$ $X \subset \mathbb{A}^n$ is one of the form $X = Z(T)$ for some $T \subset A$. The above observation says that we may restrict to finite T in this definition.

7 Proposition

Let X_α be a collection of algebraic sets.

(1) If X_1, X_2 are algebraic sets, then $X_1 \cup X_2$ is an algebraic set.

(2) $\cap_\alpha X_\alpha$ is an algebraic set.

(3) $\varnothing, \mathbb{A}^n$ are algebraic sets.

Proof Homework.

8 Definition

The $\boxed{\text{Zariski topology}}$ on $\mathbb{A}^n$ is the topology whose closed sets are the algebraic sets. (The preceding proposition assures us this is in fact a topology.)

9 Remark

From manifolds and real analysis, we're used to topological spaces being Hausdorff, meaning one can separate points by disjoint open sets. By contrast, the Zariski topology is almost never Hausdorff.

10 Example

Let $\mathbb{A}^1 \supset X = Z(T)$, $(T) \lhd A = k[x]$. Since $k[x]$ is a PID, we have $X = Z(f)$. Hence if $T \neq \{0\}$, then X is finite, since f has finitely many roots. Indeed, the closed sets are precisely the finite sets, except for $\mathbb{A}^1$ itself.

11 Definition

Let X be a topological space. We say X is irreducible if $X \neq \varnothing$ and whenever $X = X_1 \cup X_2$ for $X_1, X_2 \subset X$ *closed*, then $X = X_1$ or $X = X_2$.

12 Remark

If X is irreducible and Hausdorff, then X is a point. For if X is Hausdorff and $x \neq y \in X$, take $U \ni x, V \ni y$ open and $U \cap V = \varnothing$. It follows that $X = (X - U) \cup (X - V)$, meaning X is not irreducible.

13 Example

$\mathbb{A}^1$ is irreducible when $|k| = \infty$, since the union of two proper closed sets is finite while $\mathbb{A}^1$ is infinite.

14 Homework

1. Show that $\mathbb{A}^n$ is irreducible when $|k| = \infty$. [This is harder than the $n = 1$ case.]
2. Suppose X is an irreducible topological space and that $\varnothing \neq U \subset X$ is open. Show that U is irreducible and dense.
3. Suppose X is irreducible and $X \subset Z$ for a topological space Z. Show that $\overline{X} \subset Z$ is also irreducible.
4. If $f\colon X \to Z$ is continuous and X is irreducible, show that $f(X)$ is irreducible.

15 Definition

An affine (algebraic) variety is an irreducible algebraic set in $\mathbb{A}^n$.

16 Aside

The literature is not entirely consistent. Some people call algebraic sets algebraic varieties and irreducible algebraic sets irreducible algebraic varieties. These are "algebraic" in the sense that, for instance, one could do the same thing with analytic functions, or more general functions, whereas we're restricting to polynomial functions.

A quasi-affine variety is an open set of an affine variety. We don't yet have the tools to show (or precisely define) this, but there are in fact quasi-affine varieties which are not isomorphic to affine varieties.

17 Example

Consider $\mathbb{A}^1 - \{0\} \subset \mathbb{A}^1$. This is an open subset of $\mathbb{A}^1$. Projecting the hyperbola in $\mathbb{A}^2$ onto the origin ends up giving an isomorphism onto $\mathbb{A}^1 - \{0\}$. As it turns out, the punctured plane is quasi-affine but not affine; more on this hopefully next week.

18 Definition

Let $S \subset \mathbb{A}^n$. Define the functions that vanish on S by

$$\boxed{I(S)} := \{f \in A : f(P) = 0, \forall P \in S\}.$$

Note that $I(S) \lhd A$.

19 Proposition

We now relate the two operations "$Z\colon A \to \mathbb{A}^n$, $I\colon \mathbb{A}^n \to A$". Take $k = \overline{k}$ for (v) below.

(i) *If* $T_1 \subset T_2 \subset A$*, then* $Z(T_1) \supset Z(T_2)$*.*

(ii) *If* $S_1 \subset S_2 \subset \mathbb{A}^n$*, then* $I(S_1) \supset I(S_2)$*.*

(iii) *If* $S_1, S_2 \subset \mathbb{A}^n$*, then* $I(S_1 \cup S_2) = I(S_1) \cap I(S_2)$*.*

(iv) *If* $S \subset \mathbb{A}^n$*, then* $Z(I(S)) = \overline{S}$*.*

(v) *If* $I \trianglelefteq A$*, then* $I(Z(I)) = \sqrt{I}$ *where* $\sqrt{I}$ *is the radical of the ideal* I*, which is the intersection of all prime ideals containing* A*. This is Hilbert's Nullstellensatz.*

20 Aside

Sándor recommends Miles Reid's *Undergraduate Commutative Algebra* as background for this course. It's an easy read and the author comes from the geometric perspective; don't let the "undergraduate" scare you off.

Proof (i)-(iii) are self-evident. (iv) is homework. (v) is very famous; it is in Reid, 5.6.

21 Remark

If $I \triangleleft A$ is *proper*, then $Z(I) \neq \varnothing$ using (v), since then $I(Z(I)) = \sqrt{I} \neq A$, so $Z(I) \neq \varnothing$. It is essential that k be algebraically closed here and in (v), since roughly otherwise the zeros of the polynomials might not lie in the subfield we've chosen. In this way, we recover the assumption that k is algebraically closed, so (v) is essentially a more precise condition for a field to be algebraically closed.

More generally, if $\mathfrak{m} \triangleleft_{\max} A$, then $Z(\mathfrak{m}) = \{P\}$ as follows. If $P \in Z(\mathfrak{m})$, then $\mathfrak{m} \subset I(P) \neq A$, so $\mathfrak{m} = I(P)$. Then $Z(\mathfrak{m}) = Z(I(P)) = \overline{\{P\}} = \{P\}$ since if $P = (a_1, \ldots, a_n)$, then $Z(x_1 - a_1, \ldots, x_n - a_n) = \{P\}$. In fact, then $I(P) = (x_1 - a_1, \ldots, x_n - a_n) = \mathfrak{m}$.

We've just shown that there is a one-to-one correspondence between maximal ideals and points. This is a key ingredient for generalizing our reasoning to schemes, which we'll discuss further later.

22 Corollary

If $k = \overline{k}$*, there is a one-to-one, inclusion-reversing correspondence*

$$\{\textit{Algebraic sets } X \textit{ in } \mathbb{A}^n\} \leftrightarrow \{\textit{Radical ideals } I = \sqrt{I} \textit{ in } A\}$$
$$X \mapsto I(X)$$
$$Z(I) \mapsfrom I$$

where $X = Z(I(X))$ *and* $I = I(Z(I))$*. Furthermore,* X *is irreducible if and only if* $I(X)$ *is prime.*

Proof Everything up to the "furthermore" follows from the above proposition. The rest is homework. Note that (1) in the above homework follows from the furthermore clause since $I(\mathbb{A}^n) = 0$ is prime.

23 Example

Let $f \in k[x, y] =: A$ be an irreducible polynomial. Then $I := (f) \triangleleft A$ is a prime ideal, so $X := Z(f)$ is irreducible. This is an irreducible affine plane curve.

More generally, if $g \in k[x, y]$ is arbitrary, then $Z(g)$ is an affine plane curve. If $g \in k[x_1, \ldots, x_n]$, then $Z(g) \subset \mathbb{A}^n$ is called an affine hypersurface.

January 6th, 2016: Coordinate rings, irreducible decompositions, and dimensions

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24 Remark

Homework policy is still in the works; more details by next week.

Our next main topic is the coordinate ring of an affine variety.

25 Definition

Let $X \subset \mathbb{A}^n$ be an affine algebraic set, $A \coloneqq k[x_1, \ldots, x_n]$. The coordinate ring of X is

$$\boxed{A(X)} \coloneqq A/I(X).$$

Here we imagine two "algebraic" functions are equal if they agree on X.

26 Remark

The coordinate ring is a finitely generated k-algebra with a trivial nilradical since $\mathrm{Nil}(R/I) = \sqrt{I}/I$ and $I(X)$ is a radical ideal. Now suppose B is a finitely generated k-algebra with trivial nilradical, so we have

$$k[x_1, \ldots, x_m] \twoheadrightarrow k[b_1, \ldots, b_m] = B$$

meaning $B \cong A/I$ for some I. Since B has trivial nilradical, it follows that I is radical. Hence we can set $X \coloneqq Z(I) \subset \mathbb{A}^m$, and using the Nullstellensatz we find $I(X) = I$. Hence $B \cong A(X)$.

In summary, we've just shown that the finitely generated k-algebras with trivial nilradical are precisely the coordinate rings of algebraic sets.

27 Aside

In the scheme-theoretic setting, we'll include all finitely generated k-algebras and not just those with trivial nilradical, so we'll get a strict generalization, which of course has the "drawback" of being "less constrained".

28 Definition

Let X be a topological space. Going from rings to topological spaces is inclusion-reversing, which motivates the following definition: we say that X is a noetherian topological space if any descending chain of closed subsets terminates after finitely many steps. More precisely, if

$$X \supset X_1 \supset X_2 \supset \cdots$$

is a chain of closed sets, then there is some r such that $X_r = X_{r+1} = X_{r+2} = \cdots$. An equivalent condition is that any set of closed subsets contains a minimal element.

It is clear that a closed subset of a noetherian topological space is itself noetherian. It is also easy to see that the Zariski topology coming from a noetherian ring is a noetherian topological space.

29 Theorem

Let X be a noetherian topological space. If $\varnothing \neq Y \subset X$ is a closed subset, then there exists a decomposition $Y = Y_1 \cup \cdots \cup Y_r$ where each $Y_i \subset Y$ is irreducible and closed and for all $i \neq j$, $Y_i \not\supset Y_j$. Moreover, this decomposition is unique up to reordering. It is called the irreducible decomposition of X.

Proof Define

$$\mathcal{S} \coloneqq \{Y \subset X \text{ closed} : Y \text{ does not have such a decomposition}\}.$$

If $\mathcal{S}$ is non-empty, then it has a minimal element $Y \in \mathcal{S}$. If Y were irreducible, then it would form such a decomposition, so it must have a decomposition $Y = Y' \cup Y''$ where $Y', Y'' \subsetneq Y$ are closed. But then $Y', Y'' \notin \mathcal{S}$ by minimality, so they have decompositions of the above form and one may check that concatenating the two decompositions yields one for Y, a contradiction.

For uniqueness, suppose $Y = Y_1 \cup \cdots \cup Y_r = Y_1' \cup \cdots Y_{r'}'$. Now $Y_1 = (Y_1' \cap Y_1) \cup \cdots \cup (Y_{r'}' \cap Y_1)$, but since Y_1 is irreducible, it follows that $Y_1 = Y_i' \cap Y_1 \subset Y_i'$ for some i. By symmetry of this argument, $Y_i' \subset Y_j$ for some j, so $Y_1 \subset Y_j$ forcing $j = 1$. But then $Y_1 \subset Y_i' \subset Y_1$ forces $Y_i' = Y_1$, and we may induct.

30 Corollary

Let $X \subset \mathbb{A}^n$ be an affine algebraic set. Then there is a unique decomposition of X into affine varieties.

We now turn to basic dimension theory. This is not contained in Reid, though it is in Atiyah-Macdonald. The topological notion of dimension is largely motivated by the example of a plane containing a line which contains a point. This is a chain of irreducible sets, and the only way we can "increase dimension" to satisfy our intuition is by adding a new irreducible subset.

31 Definition

Let X be a topological space. The dimension of X is

$$\boxed{\dim(X)} := \sup\{m : Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_m \subseteq X, Z_i \text{ irreducible and closed}\}.$$

Note that the disjoint union of a plane and a line has dimension 2 by this definition. One can refine this notion to give the dimension at a point $\boxed{\dim_x(X)}$ by requiring that $x \in Z_0$ in the above definition.

Note that if $Y \subset X$ is closed, then $\dim Y \leq \dim X$.

32 Example

Let $X \subset \mathbb{A}^n$ be an algebraic set. Consider $\dim X$ as a topological space for the Zariski topology. We have $\dim \mathbb{A}^1 = 1$ since we know the topology of $\mathbb{A}^1$. The only irreducible sets are points and $\mathbb{A}^1$ itself.

33 Definition

Let B be a ring. The Krull dimension of B is defined analogously:

$$\boxed{\dim(B)} := \sup\{m : \mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \cdots \supsetneq \mathfrak{p}_m, \mathfrak{p}_i \text{ prime}\}.$$

34 Definition

Let $I \lhd B$. Define the height of I as

$$\boxed{\mathrm{ht}(I)} := \sup\{m : I \supset \mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \cdots \supsetneq \mathfrak{p}_m, \mathfrak{p}_i \text{ prime}\}.$$

Then

$$\dim B = \sup\{\mathrm{ht}\, I : I \lhd B\} = \sup\{\mathrm{ht}\, \mathfrak{p} : \mathfrak{p} \lhd_{\mathrm{pr}} B\} = \sup\{\mathrm{ht}\, \mathfrak{m} : \mathfrak{m} \lhd_{\max} B\}$$

where $\mathfrak{p} \lhd_{\mathrm{pr}} B$ means that $\mathfrak{p}$ is a prime ideal in B.

35 Homework

Find B such that for all $I \lhd B$, $\mathrm{ht}\, I < \infty$, but $\dim B = \infty$.

36 Proposition

Let $X \subset \mathbb{A}^n$ be an algebraic set. Then $\dim X = \dim A(X)$.

PROOF Homework.

37 Corollary

$\dim \mathbb{A}^n = n$.

PROOF We have $X(\mathbb{A}^n) = k[x_1, \ldots, x_n]$, which has Krull dimension n. Homework: prove that $\dim k[x_1, \ldots, x_n] = n$. More generally, suppose R is a commutative noetherian ring with unit, and show that $\dim R[x] = \dim R + 1$.

38 Remark

Algebraic sets have finite dimension, since roughly $\dim Y \leq \dim X = n$.

39 Theorem

Let B be a finitely generated k-algebra which is also an integral domain. Then

(i) $\dim B = \mathrm{trdeg}_k \mathrm{Frac}(B)$

(ii) For any $\mathfrak{p} \lhd_{\text{pr}} B$, $\dim B = \text{ht}(\mathfrak{p}) + \dim B/\mathfrak{p}$.

PROOF This is in Atiyah-Macdonald, Chapter 11.

January 8th, 2016: Codimension 1 affine varieties; homogeneous ideals and zero sets

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40 Proposition

Let $X \subset \mathbb{A}^n$ *be a quasi-affine variety. Then* $\dim X = \dim \overline{X}$.

41 Remark

Note that we do not have a straightforward coordinate ring for such an X.

As an exercise, consider $\mathbb{A}^2$, so $A(\mathbb{A}^2) = k[x,y]$. Now form X by puncturing $\mathbb{A}^2$ by taking out a point. Try to find a reasonable definition of the regular functions on the resulting quasi-affine–we'll come back to this later.

PROOF Since $X \subset \overline{X}$, we expect $\dim X \leq \dim \overline{X}$. Given a chain of closed, irreducible subsets of X

$$\varnothing \neq Z_0 \subsetneq Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_r \subset X.$$

Of course, $\overline{Z_i} \subset \overline{X}$. Can it happen that $\overline{Z_i} = \overline{Z_{i+1}}$? No, essentially since $Z_i = \overline{Z_i} \cap X$. Hence $\dim X \leq \dim \overline{X}$. This reasoning works on the topological level with no further assumptions on X.

For the other direction, we need to leverage the fact that X is quasi-affine. We showed last time that $\dim \overline{X} = \dim A(\overline{X})$, and the theorem mentioned at the end of class last time says that every prime is part of a maximal sequence of primes giving the appropriate dimension. That is, pick $P \in X \subset \overline{X}$, and suppose P corresponds to $\mathfrak{m}_p \lhd_{\max} A(\overline{X})$ together with a sequence

$$\varnothing \neq W_0 \subsetneq \cdots \subsetneq W_r \subset \overline{X}$$

where the W_i are closed, irreducible. Now for all i, $W_i \cap X \subset W_i$ is a non-empty open subset of W_i, which is then dense. If $W_i \cap X = W_{i+1} \cap X$, then their closures in $\overline{X}$ would be equal, so $W_i = W_{i+1}$, so $W_i \cap X \neq W_{i+1} \cap X$. This completes the proof.

We next recall some results from commutative algebra and interpret them in terms of hypersurfaces:

42 Theorem

Let B *be a noetherian ring. Suppose* $f \in B$ *is neither a zero-divisor nor a unit. Then for all minimal* $\mathfrak{p} \lhd_{pr} B$*, if* $f \in \mathfrak{p}$*, then* $\text{ht}\,\mathfrak{p} = 1$.

43 Remark

In some sense, zero-divisors and prime ideals are not all that interesting when it comes to which primes contain them, so the assumption is relatively mild. This is in Atiyah-Macdonald, page 122.

44 Theorem

Suppose B *is an integral domain. Then* B *is a UFD if and only if every prime ideal of height* 1 *is principal.*

PROOF Homework; follows from considering when irreducibles and primes coincide.

45 Theorem

Let $X \subset \mathbb{A}^n$ *be an affine algebraic variety. Then* $\dim X = n - 1$ *if and only if there is some irreducible* $f \in A$ *such that* $I(X) = (f)$.

46 Remark

Sometimes the conclusion of this theorem is stated as $X = Z(f)$, though $I(X) = (f)$ is a stronger statement. For instance, there exists a curve in $\mathbb{A}^3$ which is the zero set of two polynomials, but its ideal cannot be generated by less than 3 elements. Roughly, $I(X) = (f)$ is telling you about the scheme structure, whereas $X = Z(f)$ is telling you about points.

Proof Consider $I(X) \lhd A$, which is prime. By the theorem from the end of last class, $\operatorname{ht} I(X) = \dim A - \dim A(X) = n - (n-1) = 1$. By the previous theorem, $I(X)$ is principal, and since $I(X)$ is prime, its generator must be irreducible.

In the other direction, let $I(X) = (f)$ as above. We must show $\dim A(X) = n - 1$, but now Krull's theorem says $1 = \operatorname{ht} I(X)$, so the theorem from the end of last class again gives the result.

We next turn to projective varieties. One major difference from the affine case is that we cannot define functions on projective varieties so easily.

47 Definition

Define projective space over a field k to be the set

$$\boxed{\mathbb{P}^n_k} := \{(a_0, \ldots, a_n) \in k^{n+1} - \{0\}\}/\sim$$

where $(a_0, \ldots, a_n) \sim (\lambda a_0, \ldots, \lambda a_n)$ for $\lambda \in k^\times$. Pictorially, this is the set of lines through the origin in $\mathbb{A}^{n+1}$. This is "even less" of a vector space than affine space.

48 Definition

Let $k = \overline{k}$. $f \in k[x_0, \ldots, x_n]$ is a homogeneous polynomial of degree d if for any $\lambda \in k$, $f(\lambda x) = \lambda^d f(x)$. Equivalently, when writing f as a sum of monomials, every monomial has degree d.

Homogeneous polynomials don't give well-defined functions on projective space, but they do give well-defined zero sets, which we next define.

49 Definition

Write $S := k[x_0, \ldots, x_n]$. Conceptually S differs from $A = k[x_1, \ldots, x_n]$ in that S is more explicitly graded. Recall that a graded ring T is an object such that

1. T is a ring,
2. $T = \oplus_{d \in \mathbb{N}_{\geq 0}} T_d$ as an abelian group, and
3. $T_d T_e \subset T_{d+e}$.

Here T_d is the dth homogeneous component of T. Now $S = \oplus_{d \in \mathbb{N}_{\geq 0}} S_d$ where S_d is defined to be the homogeneous polynomials of degree d. This is the prototypical graded ring. The second condition says that any $f \in T$ has a unique decomposition $f = f_0 + \cdots + f_d$ for $f_i \in T_i$ homogeneous.

Write $S^h := \cup_{d \in \mathbb{N}_{\geq 0}} S_d$ for the set of homogeneous polynomials of arbitrary degree. This differs from S in that it is not closed under general sums (when the degrees differ).

Given $f \in S^h$, define the zero set of f by

$$\boxed{Z(f)} := \{P \in \mathbb{P}^n : f(P) = 0\}.$$

Likewise if $T \subset S^h$, we may define

$$\boxed{Z(T)} := \{P \in \mathbb{P}^n : f(P) = 0, \forall f \in T\}.$$

50 Definition

Let T be an arbitrary graded ring. An ideal $I \lhd T$ is a homogeneous ideal if $T = \oplus_{d \in \mathbb{N}_{\geq 0}}(I \cap T_d)$. Equivalently, I is generated by homogeneous elements, which follows in part because $(I \cap T_d)(I \cap T_e) \subset I \cap T_{d+e}$; the rest of the verification is homework.

51 Homework

For homogeneous ideals, the sum, product, intersection, and radical are all homogeneous. Moreover, testing whether a homogeneous ideal is prime can be done using homogeneous elements. Also, a finitely generated homogeneous ideal is generated by finitely many homogeneous elements.

52 Proposition

For any $T \subset S^h$, there exist finitely many $f_1, \ldots, f_r \in S^h$ such that $Z(T) = Z(f_1, \ldots, f_r)$.

Proof Consider $I = (T)$, which is homogeneous by definition, so by the preceding exercise and the fact that S is noetherian, the result follows.

53 Definition

If $I \lhd S$ is a homogeneous ideal, we define

$$Z(I) := Z(\cup_{d \in \mathbb{N}_{\geq 0}}(I \cap S_d)) = Z(I \cap S^h).$$

January 11th, 2016: Projective Zariski topology, homogeneous coordinate rings, etc.; regular functions

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54 Remark

The homework policy is as follows; there is a document on Canvas which discusses this more fully and explains the grading policy. Homework is envisioned to be posted and critiqued by each other in the discussion portion of Canvas. You have a choice of what to work on. There are groups of three problems and you have to pick one from each group, but you're encouraged to look at all of them and to at least think about how to do them all.

Deadlines for homework begin next Monday. (They are quite flexible—ask if you need an extension.)

55 Definition

A set $X \subset \mathbb{P}^n$ is a projective algebraic set if it is the zero set of homogeneous polynomials, i.e. $X = Z(T)$ for $T \subset S^h$. It is a quasi-projective algebraic set if it is an open subset of a projective algebraic set.

56 Proposition

The projective algebraic sets satisfy the axioms for the closed sets of a topological space.

Proof Homework.

Hence we have a projective Zariski topology. Similarly, the dimension of a projective algebraic set is defined to be its dimension as a topological space.

57 Corollary

Every projective algebraic set is a union of finitely many irreducible subsets, and this set is unique up to reordering.

58 Definition

For $Y \subset \mathbb{P}^n$, define the ideal of functions that vanish on Y as

$$\boxed{I(Y)} := (\{f \in S^h : f(P) = 0, \forall P \in Y\}) \lhd S.$$

59 Proposition

For $Y \subset \mathbb{P}^n$, we have $\overline{Y} = Z(I(Y))$.

60 Homework

State and prove a homogeneous Nullstellensatz.

61 Definition

Let $X \subset \mathbb{P}^n$ be a projective algebraic set. Define the $\boxed{\text{homogeneous coordinate ring}}$ of X as

$$\boxed{S(X)} := S/I(X).$$

This differs slightly from the affine case in the sense that these are *not* functions, i.e. we do *not* have $x_i: \mathbb{P}^n \to k$.

62 Remark

We will see later that two affine algebraic sets are isomorphic if and only if their coordinate rings are isomorphic. This is quite false for projective algebraic sets. As a vague example, take a conic in $\mathbb{P}^2$ and use stereographic projection from some fixed point to some fixed line $\mathbb{P}^1$. Say the conic is $x_0x_1 = x_2^2$, which has homogenoeus coordinate ring $k[x_0, x_1, x_2]/(x_0x_1 - x_2^2)$, while $\mathbb{P}^1$ has homogeneous coordinate ring $k[x_0, x_1]$. Homework: these two coordinate rings are not isomorphic.

The homogenoeus coordinate ring does tell you something about X, but it really tells you about its embedding into projective space. Very likely different embeddings will result in different coordinate rings.

63 Remark

We may think of $S(X)$ as the affine coordinate ring of the "cone" over X, i.e. the inverse image of X under the natural map $\mathbb{A}^{n+1} - \{0\} \to \mathbb{P}^n$, together with 0. In the previous example, the cone over the conic is a literal cone, while the cone over a line is a plane. The first is singular and the second is not, so they cannot be isomorphic. This can be detected from the ring—localizing at the maximal ideal corresponding to the singular point yields a non-(regular local) ring. (This is a rather algebraic way to say a fundamentally geometric ideal, since we have the following principle: "algebra = geometry".)

Indeed, we may map $k[x_0, x_1, x_2]$ into $k[t, u]$ via $x_0 = t^2, x_1 = u^2, x_2 = ut$. The kernel will be precisely $(x_0x_1 - x_2^2)$. This gives a scheme-theoretic morphism, even, but it is not an isomorphism of rings.

This discussion roughly culminates in the observation that we don't have a great notion of globally defined functions for projective algebraic sets. However, we do have a good notion for affine varieties. So, we'll soon take a page out of manifold theory and consider affine varieties as "local neighborhoods" and build larger spaces out of them.

64 Aside

Here is a test for whether or not you've mastered schemes: do you think of schemes in the same way you think of varieties and topological spaces? Do you flinch when someone says, "Let X be a circle, scheme-theoretically"? This will matter later.

65 Definition

In $\mathbb{P}^n$, consider the $\boxed{\text{homogeneous coordinates}}$ $[x_0 : \cdots : x_n]$. Let

$$U_i := \{[x_0 : \cdots : x_n] \in \mathbb{P}^n : x_i \neq 0\} = \mathbb{P}^n - Z(x_0).$$

This is an open subset of $\mathbb{P}^n$. It is affine in the following sense: define bijections

$$\begin{aligned} \mathbb{A}^n_{y_1, \ldots, y_n} &\leftrightarrow U_i \\ (y_1, \ldots, y_n) &\overset{\psi_i}{\mapsto} [y_1 : \cdots : y_{i-1} : 1 : y_i : \cdots : y_n] \\ (x_0/x_i, \ldots, x_{i-1}/x_i, x_{i+1}/x_i, \ldots, x_n/x_i) &\overset{\phi_i}{\leftarrow\!\shortmid} [x_0 : \cdots : x_n]. \end{aligned}$$

Hence we call U_i the $\boxed{\text{standard open affines}}$ in $\mathbb{P}^n$.

66 Homework

Show that

(1) $\psi_i = \phi_i^{-1}$.

(2) These functions define homeomorphisms.

67 Remark

We may map a quasi-affine algebraic set through the map ψ_i to show that it is also quasi-projective.

68 Definition

Let X be a topological space. A subset $W \subset X$ is $\boxed{\text{locally closed}}$ if for all $P \in W$, there exists $U \subset X$ with $P \in U$ for which $W \cap U \subset U$ is closed in U.

69 Homework

Prove that this is equivalent to $W = Z \cap U$ where Z is closed and U is open (both in X). Furthermore, prove that W is closed if and only if the above condition holds upon replacing "$P \in W$" with "$P \in X$".

70 Proposition

Let $X \subset \mathbb{P}^n$ be quasi-projective. Hence we may write $X = \cup_{i=0}^n (X \cap U_i)$, which we call the $\boxed{\textit{standard open affine cover}}$.

71 Homework

Look up localization and local rings from commutative algebra if you don't remember it well. This will be important in a lecture or two.

72 Definition

Let $X \subset \mathbb{A}^n$ be a quasi-affine variety, i.e. a locally closed subset of $\mathbb{A}^n$. Let $f : X \to k$ be a set-theoretic function, so we think of k as $\mathbb{A}^1$. For $P \in X$, we say f is $\boxed{\text{regular at } P}$ if there exists $g, h \in A := k[x_1, \ldots, x_n]$ and an open set $p \in U \subset X$ such that

(1) $h(Q) \neq 0$ for all $Q \in U$

(2) $f = g/h$ as functions on U, i.e. for all $Q \in U$, $f(Q) = g(Q)/h(Q)$.

We say f is $\boxed{\text{regular}}$ if it is regular for all $P \in X$.

73 Remark

This is not exactly the classical version of the definition, but is instead closer to the modern version which emphasizes local behavior.

74 Example

Let $Z := Z(xy - zt) \subset \mathbb{A}^4$, $W := Z(y, t) \subset \mathbb{A}^4$. It's easy to see $W \subset Z$. Set $X := Z - W$, which is hence quasi-affine. Define $f : X \to k$ by

$$f = \begin{cases} x/t & \text{if } t \neq 0 \\ z/y & \text{if } y \neq 0. \end{cases}$$

This is well-defined since it agrees on the overlap and the two cases exhaust the complement. This is regular in the sense of the above local definition, but it is not regular in the more classical sense that this function is not the ratio of two polynomial functions on all of X.

75 Lemma

Let $f : X \to k$ be regular. Then f is continuous, where k has the topology of $\mathbb{A}^1$, i.e. the finite complement topology.

Proof We need to show that if $Z \subset \mathbb{A}^1$ is closed, then $f^{-1}(Z) \subset X$ is closed. If $Z = \mathbb{A}^1$, then $f^{-1}(Z) = X$ is closed. Otherwise, $Z = \{P_1, \ldots, P_r\}$ is a finite set and $f^{-1}(Z) = \cup_{i=1}^r f^{-1}(P_i)$. Hence it's enough to prove the $r = 1$ case. For that we use the "locally closed" criterion above. By regularity, for all $P \in f^{-1}(P_1)$, there exists a neighborhood $P \in U \subset X$ such that $f = g/h$ on U. Now

$$U \cap f^{-1}(P_i) = \{Q \in U : g(Q) = P_1 h(Q)\} = U \cap Z(g_i - P_1 h_i),$$

which is closed in U.

We can roughly summarize this argument by saying it shows that f is "locally continuous", hence continuous.

76 Remark

The converse of this statement is quite false in general. Take a regular function from $\mathbb{A}^1$ to $\mathbb{A}^1$; we may mutate the function on any one point, which will not matter topologically, but will break regularity.

January 13th, 2016: Projective regular functions, morphisms and isomorphisms of varieties

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77 Remark

We've been talking about regular functions on quasi-affine varieties, which are roughly locally rational functions. We'll next do this in the projective case and introduce morphisms of varieties.

78 Definition

Let $X \subset \mathbb{P}^n$ be a quasi-projective variety. A map $X \to k$ (of sets) is regular at P if there exists an open $P \in U \subset X$ and $g, h \in S^h$ such that

1. $\deg g = \deg h$
2. $h(Q) \neq 0$ for all $Q \in U$
3. $f = g/h$ on Q

Intuitively, f is locally a rational function which is well-defined on lines through the origin. We say f is regular on X if it is regular at all $P \in X$.

79 Example

The example from last time works in this context, namely $Z := Z(xy - zt) \subset \mathbb{P}^3$ and $W = Z(y, t) \subset Z \subset \mathbb{P}^3$, $X := Z - W$, and

$$f = \begin{cases} x/t & \text{when } t \neq 0 \\ z/y & \text{when } y \neq 0 \end{cases}$$

is a regular function which is not "globally rational" on X.

80 Homework

Show that regular functions on quasi-projective varieties are continuous. Also show that f is regular (in the projective sense) if and only if the restrictions of f to the standard affine opens of X are all regular (in the affine sense).

81 Remark

In much the same way that globally defined holomorphic functions on compact Riemann surfaces are just constants, the regular functions on projective varieties are often just the constants.

82 Corollary

Let X be a quasi-projective variety and let $f, g\colon X \to k$ be regular. If $\varnothing \neq U \subset X$ is open, and if $f|_U = g|_U$, then $f = g$.

83 Remark

This is not true if f and g are merely continuous, essentially because we can sometimes mutate f on finitely many points without affecting continuity. As an explicit example, let $f, g\colon \mathbb{A}^1 \to k$ by $x \mapsto x^2$, except declare $f(0) = 1$. Then $f - g$ is zero everywhere except that it's 1 at 0, which is not continuous since the fiber of 0 is missing a point.

Proof Consider $f - g$, which is evidently regular, hence continuous, and $(f-g)|_U = 0$, so $(f-g)^{-1}(0) \supset U$ is a closed set containing U. Now $U \subset X$ is dense since X is a variety, so $(f-g)^{-1}(0) = X$.

84 Definition

We define a **variety** to be an affine, quasi-affine, projective, or quasi-projective variety.

85 Remark

Indeed, we have the implications

$$\begin{array}{ccc} \text{affine} & \Longrightarrow & \text{quasi-affine} \\ & & \Downarrow (\cong) \\ \text{projective} & \Longrightarrow & \text{quasi-projective} \end{array}$$

Here the horizontal arrows are trivial from the definitions and the vertical arrow is only true up to the notion of isomorphism that we next define. As the course progresses, we will get further away from the explicit embeddings we have been using so far.

These implications allow us to consider regularity entirely in the quasi-projective sense. That is, if X is affine or quasi-affine, a map $X \to k$ is regular in the affine sense if and only if the induced map $\psi_i X \to k$ is regular in the projective sense, where ψ_i was defined last lecture and $\psi_i X$ is quasi-projective.

86 Definition

Let X, Y be varieties. We say a continuous function $\phi\colon X \to Y$ is a **morphism of varieties** if for all $V \subset Y$ open and for all regular functions $f\colon V \to k$, the induced map $f \circ \phi\colon \phi^{-1}V \to k$ is a regular function.

We say that a morphism $\phi\colon X \to Y$ is an **isomorphism of varieties** if there exists a morphism $\psi\colon Y \to X$ such that $\phi \circ \psi = \mathrm{id}_Y$, $\psi \circ \phi = \mathrm{id}_X$.

87 Corollary

The composite of two morphisms of varieties is a morphism. The restriction of a morphism to a subvariety is a morphism. An open embedding is a morphism. These are immediate consequences of the above "abstract" description. They imply that varieties and morphisms form a category.

88 Homework

If X is a variety and $f\colon X \to k$ is a map, show that f is a regular function if and only if $f\colon X \to \mathbb{A}^1$ is a morphism.

89 Definition

We say that a variety X is affine, quasi-affine, projective, or quasi-projective if it is isomorphic to an affine, quasi-affine, projective, or quasi-projective variety, respectively.

90 Example

In this sense, we may be given a variety that appears "strictly quasi-affine", but it may secretly be affine. For example, let $Z \subset \mathbb{A}^n$ be a closed affine variety, pick $f \in A$, and set $X = Z - Z(f)$.

Here we may imagine that we're taking out a curve from a surface to form X. Written this way, X is quasi-affine but not affine in our original sense as a subset of $\mathbb{A}^n$. However it is actually affine in the sense of our new definition!

To see this, we introduce a new variable/dimension, consider the cylinder determined by Z, and show that X is isomorphic to a certain affine subset of that cylinder. Precisely, let $p: \mathbb{A}^{n+1} \to \mathbb{A}^n$ be the map (morphism) which projects onto the first n coordinates. Now set $W := Z(1 - x_{n+1} f) \subset \mathbb{A}^{n+1}$. We claim that $p: p^{-1} Z \cap W \xrightarrow{\sim} X$ is an isomorphism.

We check that this map indeed maps into X. We have $(a_1, \ldots, a_n, a_{n+1}) \mapsto (a_1, \ldots, a_n)$, so $1 - a_{n+1} f(a_1, \ldots, a_n) = 0$ and $(a_1, \ldots, a_n) \in Z$. Evidently $f(a_1, \ldots, a_n) \neq 0$ by the first condition, as required. The inverse is given essentially by $(a_1, \ldots, a_n) \mapsto (a_1, \ldots, a_n, 1/f(a_1, \ldots, a_n))$.

Note that $p^{-1} Z \cap W$ is the intersection of two closed sets, and the result follows. A variant on this construction shows that a punctured line is isomorphic to a hyperbola.

91 Example

Consider $\mathbb{A}^1 \to Z(x^3 - y^2) \subset \mathbb{A}^2$ given by $t \mapsto (t^2, t^3)$. This is one-to-one since it has an inverse $y/x \leftarrow (x, y)$ except $0 \leftarrow (0, 0)$. One may check that the inverse is continuous, but one may also find an explicit example showing the inverse is not regular. Hence a morphism with a continuous inverse need not be an isomorphism.

January 15th, 2016: Presheaves, sheaves, $\mathcal{O}_X$, regular and rational functions

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92 Remark

Today we'll talk about the structure sheaf and hopefully sheaves in general. In a sense it's nothing new; we're just organizing our regular functions.

93 Definition

Let X be a variety (so affine, quasi-affine, projective, or quasi-projective, but in particular irreducible). Let $U \subset X$ be a non-empty open set. Define the $\boxed{\text{regular functions}}$ on to be

$$\boxed{\mathcal{O}(U)} := \{f: U \to k : f \text{ is a regular function}\}.$$

Indeed, $\mathcal{O}(U)$ has a natural ring structure given by composition of rational functions. If $U \supset V$, we have a $\boxed{\text{restriction map}}$

$$\boxed{\rho_{UV}}: \mathcal{O}(U) \to \mathcal{O}(V)$$
$$f \mapsto f|_V.$$

which is a ring homomorphism. The system of rings and restriction maps forms a directed system, and we can take direct limitss at points to get the $\boxed{\text{stalk of } \mathcal{O}_X \text{ at } P}$

$$\boxed{\mathcal{O}_{X,P}} := \varinjlim_{P \in U} \mathcal{O}(U).$$

More explicitly, we may define stalks in terms of "germs" as follows. If $P \in X$, suppose $f \in \mathcal{O}(U)$, $g \in \mathcal{O}(V)$ for neighborhoods U, V of P. We define an equivalence relation

$$(U, f) \sim (V, g) \Leftrightarrow f|_{U \cap V} = g|_{U \cap V}.$$

The resulting equivalence classes $[U, f]_P$ are $\boxed{\text{germs of regular functions at } P}$, which inherit a natural ring structure which is in fact the direct limit $\mathcal{O}_{X,P}$.

94 Claim

$\mathcal{O}_{X,P}$ is a local ring.

Proof Let $\alpha_P: \mathcal{O}_{X,P} \to k$ by $f \mapsto f(P)$. Note that this is well-defined. Evidently the image of α_P is k since $\mathcal{O}_{X,P}$ contains the constant functions, so quotienting by the kernel gives a field, so we have a maximal ideal

$$\mathfrak{m}_{X,P} := \mathfrak{m}_P := \ker \alpha_P.$$

Also, $f \in \mathcal{O}_{X,P} - \mathfrak{m}_{X,P}$ implies $f^{-1} \in \mathcal{O}_{X,P}$, since (U, f) with $f(Q) \neq 0$ for all $Q \in U$ implies that (U, f^{-1}) is a regular function in $\mathcal{O}(U)$. Hence $\mathcal{O}_{X,P}$ is local.

95 Definition

We may perform the above germ construction without respect to a fixed base point as follows. Let X be a variety. Define an equivalence relation on pairs (U, f) where $U \neq \varnothing$ is open and $f \in \mathcal{O}(U)$ by

$$(U, f) \sim (V, g) \Leftrightarrow f|_{U \cap V} = g|_{U \cap V}.$$

Call the resulting equivalence class $[U, f]$. Note that since X is irreducible, any two non-empty open sets intersect non-trivially, so $(U \cap V, f|_{U \cap V})$ is a well-defined regular function.

The function field of X, $K(X)$, is the ring of equivalence classes of such pairs.

96 Claim

$K(X)$ is a field.

Proof If $(U, f) \sim (V, 0)$, then by the proposition from last class, $f = 0$. Otherwise, $(U, f)^{-1} = (U - Z(f), f^{-1})$ where $Z(f) := f^{-1}(0)$ is a closed, proper subset of U.

97 Remark

Hence $K(X)$ is different ring-theoretically from the $\mathcal{O}_{X,P}$. In a sense, $K(X)$ is the union of the $\mathcal{O}_{X,P}$'s. Indeed, we claim we have the following natural inclusions:

$$\begin{aligned} \mathcal{O}(X) \hookrightarrow \mathcal{O}(U) \hookrightarrow \mathcal{O}_{X,P} \hookrightarrow K(X) \\ f \mapsto f|_U \mapsto [U, f|_U]_p \mapsto [U, f|_]. \end{aligned}$$

and that under these embeddings $K(X) = \cup_{P \in X} \mathcal{O}_{X,P}$.

Verifying this is an exercise.

98 Remark

We may write $f \in K(X)$ to mean there is some non-empty open U such that $[U,] \in K(X)$. Such an f is called a rational function, so $K(X)$ is the ring of rational functions. $\mathcal{O}(X)$ is the ring of regular functions on X, $\mathcal{O}(U)$ is the ring of regular functions on U, and $\mathcal{O}_{X,P}$ is the ring of rational functions which are regular at P.

Indeed, given a rational function f, there is a largest open set on which it is regular. In particular, we take the union over all domains in the equivalence class of f, and these "glue": define $W := \cup V$ where the union is over all V such that $(V, g) \sim (U, f)$, and define $h: W \to k$ by $h|_V = g$. One must check that this is well-defined, but this is true as usual. This construction is perhaps more useful psychologically than mathematically. Note that we do *not* have that h is a ratio of two polynomials on all of W, but instead around any point of W there is an open set on which h is the ratio of two polynomials.

99 Remark

We also have $\mathcal{O}(U) = \cap_{P \in U} \mathcal{O}_{X,P}$ under the above inclusions. In words, $\mathcal{O}(U)$ is the set of rational functions which are regular at all points of U, so this is nearly tautological.

These rings are in some sense very concrete, since they consist of actual functions. For instance, they have no nilpotents, since if an element is zero at every point, it is zero, and k is a field. More generally, if $f \in \mathcal{O}(U)$, and restricting f to every open in an open cover of U is zero, then f was zero.

The preceding definitions do not depend on whether or not X is affine or projective, though in the projective case $\mathcal{O}(X)$ will be k, whereas in the affine case $\mathcal{O}(X)$ will typically be much more interesting.

100 Homework

For any non-empty open $U \subset X$, show that $K(U) = K(X)$.

101 Definition

Let X be a topological space. A $\boxed{\text{presheaf}}$ of abelian groups $\mathcal{F}$ on X is the following collection of data:

(a) For all $U \subset X$ open, we have a fixed abelian group $\mathcal{F}(U)$.

(b) For all $U \supset V$, there is a fixed homomorphism of abelian groups $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$.

subject to the following conditions:

(1) $\mathcal{F}(\varnothing) = 0$.

(2) $\rho_{UU} = \mathrm{id}_U$.

(3) If $U \supset V \supset W$, then

$$\rho_{VW} \circ \rho_{UV} = \rho_{UW}.$$

We can replace "abelian groups" with many other things, such as sets, commutative rings, R-modules, etc. In more generality, 0 is replaced by a terminal object. In more generality, 0 is replaced by a terminal object. In more generality, 0 is replaced by a terminal object. We model the above definition on the case where $\mathcal{F} = \mathcal{O}$ above.

If $s \in \mathcal{F}(U)$, we call s a $\boxed{\text{section}}$ and we write $\boxed{s|_V} := \rho_{UV}(s)$, which we call the $\boxed{\text{restriction}}$ of s to V. Note that this need not be the restriction of an actual function in any concrete sense.

102 Definition

A presheaf $\mathcal{F}$ is a $\boxed{\text{sheaf}}$ if it satisfies the following two additional conditions. Let $U \subset X$ be an arbitrary open set and $U = \cup_\alpha U_\alpha$ be an arbitrary open cover.

(1) If $s_{U_\alpha} = 0$ for all α, then $s = 0$.

(2) Given a collection $\{s_\alpha \in \mathcal{F}(U_\alpha)\}_\alpha$ such that for all α, β we have $s_\alpha|_{U_\alpha \cap U_\beta} = s_\beta|_{U_\alpha \cap U_\beta}$, then there exists an $s \in \mathcal{F}(U)$ such that $s|_{U_\alpha} = s_\alpha$.

103 Homework

Try to find examples of sheaves and presheaves. In particular, try to make a sheaf out of $\mathcal{O}$, and try to find presheaves which are not sheaves because they violate either of the two additional conditions above.

January 20th, 2016: Examples of sheaves, sheafification

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104 Remark

Some analytical statements may use the language of sheaf theory. For instance, analytic continuation is often phrased in terms of germs of holomorphic functions, though that wasn't the original motivation for sheaves. The sheaf conditions are very natural when analyzing local and global properties of functions. Intuitively, the first one says that a function which is locally 0 is globally 0, and the second one says that a function which is locally defined can be globally defined.

105 Notation

Let X be a topological space, $P \in X$ a point, $U, V \subset X$ open, and A an abelian group.

106 Example

We consider the sheaf of continuous functions on X defined by

$$\mathcal{F}(U) := \{f : U \to \mathbb{R} : f \text{ is continuous}\}.$$

In a sense this is the prototypical sheaf (of abelian groups). Presheaves of literal functions always satisfy the first sheaf axiom, and continuity is local, so the second condition holds as well. Here the restriction maps are literally restriction. When they're obvious, we often won't write them.

We next consider the constant presheaf associated to A,

$$\mathcal{A}(U) := A, \rho_{UV} := \text{id}, \qquad U \supset V \neq \varnothing$$

where $\mathcal{A}(\varnothing) := 0$. We will tacitly define our presheaves to be 0 on $\varnothing$ throughout the course, often without further comment. This is trivially a presheaf, and it also satisfies the first sheaf axiom immediately. The gluing axiom will hold if and only if X is irreducible. We could replace $\mathcal{A}(U)$ with the group of locally constant functions on U and get an honest sheaf, but then $\mathcal{A}(U)$ would no longer simply be A.

Now we define the skyscraper sheaf,

$$A_P(U) := \begin{cases} A & \text{if } P \in U \\ 0 & \text{if } P \notin U \end{cases}$$

$$\rho_{UV} := \begin{cases} \text{id}_A & \text{if } U \supset V \ni P \\ (A \to 0) & \text{if } U \supset V \not\ni P \end{cases}$$

This is a sheaf in general. We'll motivate the name in a minute.

Finally, if X is a variety, then $\mathcal{O}_X$ is the sheaf of regular functions on X defined last time. It is a sheaf of *rings*, so for instance the restriction maps are ring homomorphisms.

107 Homework

Verify that $\mathcal{O}_X$ is actually a sheaf of rings.

If X is a variety, then define the sheaf of rational functions on X,

$$\mathcal{K}_X(U) := K(U).$$

108 Homework

Show that $\mathcal{K}_X$ is a constant sheaf associated to $K(X)$.

109 Aside

As usual it is important that X be irreducible for the above definition to work. We will shortly give a general procedure to go from a presheaf to a sheaf. Given a *reducible* algebraic set X with two irreducible components X_1, X_2, we can imagine cooking up functions f and g where $Z(f) = X_1 \cap X_2 = Z(g)$. Then define

$$\overline{f} := \begin{cases} f & \text{on } X_1 \\ 0 & \text{on } X_2 \end{cases}$$

$$\overline{g} := \begin{cases} 0 & \text{on } X_1 \\ g & \text{on } X_2 \end{cases}$$

Then $\overline{f} \cdot \overline{g} = 0$, so we've found zero divisors in what we would like to be a subset of $K(X)$. One may get around this issue using the total ring of quotients; more on this later.

110 Definition

Let $\mathcal{F}$ be a presheaf. Define an equivalence relation on pairs

$$(U, s), \qquad P \in U \subset X \text{ open}, s \in \mathcal{F}(U)$$

by

$$(U, s) \sim (V, t) \Leftrightarrow \exists W \text{ s.t. } P \in W \subset U \cap V, s|_W = t|_W.$$

(For regular functions, our varieties were irreducible, so we were able to use $U \cap V$ directly instead of passing to an intermediate W, but in full generality we must use W.) Then define the $\boxed{\text{stalk of } \mathcal{F} \text{ at } P}$ by

$$\begin{aligned} \boxed{\mathcal{F}_P} &:= \{(U, s) : s \in \mathcal{F}(U), P \in U\}/\sim \\ &= \varinjlim_{U \ni P} \mathcal{F}(U). \end{aligned}$$

(In generality, we must use a category containing direct limits, but for most categories we're familiar with, such as abelian groups or commutative rings, the equivalence class description is perfectly fine.) If $s \in \mathcal{F}(U)$, we define the $\boxed{\text{germ of } s \text{ as } P}$ by

$$\begin{aligned} \mathcal{F}(U) &\to \mathcal{F}_P \\ s &\mapsto [(U, s)] =: \boxed{s_P} \end{aligned}$$

111 Example

What are the stalks of the skyscraper sheaf? One may check

$$(A_P)_Q = \begin{cases} 0 & Q \notin \overline{\{P\}} \\ A & Q \in \overline{\{P\}} \end{cases}$$

In particular, if P is a closed point, then the stalks of the skyscraper are 0 everywhere except at P, where they're A, which is very reminiscent of the Dirac delta function.

112 Definition

Let $\mathcal{F}, \mathcal{G}$ be presheaves on X. A $\boxed{\text{morphism of presheaves}}$ $\phi \colon \mathcal{F} \to \mathcal{G}$ is a collection of morphisms $\phi_U \colon \mathcal{F}(U) \to \mathcal{G}(U)$ such that for any $U \supset V$ the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\phi(U)} & \mathcal{G}(U) \\ {\scriptstyle \rho^{\mathcal{F}}_{UV}} \downarrow & & \downarrow {\scriptstyle \rho^{\mathcal{G}}_{UV}} \\ \mathcal{F}(V) & \xrightarrow[\phi(V)]{} & \mathcal{G}(V) \end{array}$$

This is a natural transformation of the underlying functors. Since sheaves are in particular presheaves, this allows us to define morphisms between sheaves and/or presheaves.

113 Definition

Our next goal is to describe a construction which "builds a minimal sheaf from a presheaf". Formally, let $\mathcal{F}$ be a presheaf. We claim there exists a sheaf $\mathcal{F}^+$ and a morphism of presheaves $\theta \colon \mathcal{F} \to \mathcal{F}^+$ such that for any sheaf $\mathcal{G}$ and morphism $\phi \colon \mathcal{F} \to \mathcal{G}$ there exists a unique morphism $\psi \colon \mathcal{F}^+ \to \mathcal{G}$ such that $\phi = \psi \circ \theta$, i.e.

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\theta} & \mathcal{F}^+ \\ & {\scriptstyle \phi} \searrow & \downarrow {\scriptstyle \exists! \psi} \\ & & \mathcal{G} \end{array}$$

More succinctly, we may say

$$\begin{gathered}\mathrm{Hom}(\mathcal{F}^+, \mathcal{G}) \xrightarrow{\sim} \mathrm{Hom}(\mathcal{F}, \mathcal{G}) \\ \psi \mapsto \psi \circ \theta\end{gathered}$$

is a bijection. We call $\mathcal{F}^+$ the $\boxed{\text{sheaf associated to } \mathcal{F}}$. The map θ is usually left unwritten.

Note that θ is not always an injection in any sense; roughly, if there are sections which are locally 0 but not globally 0, then we must annihilate those bad global sections. We'll continue this next lecture.

January 22nd, 2016: Draft

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114 Remark

Last time we discussed examples of sheaves and presheaves. We defined morphisms of sheaves and began to discuss sheafification. We'll continue those discussions today.

115 Homework

Let $\mathcal{F}, \mathcal{G}$ be sheaves on a topological space X, say of abelian groups. Suppose $\phi: \mathcal{F} to \mathcal{G}$ is a morphism. Show:

(a) For any $P \in X$, show that there is an induced morphism on stalks $\phi_P: \mathcal{F}_P \to \mathcal{G}_P$.

(b) ϕ is an isomorphism if and only if ϕ_P is an isomorphism for all $P \in X$.

(c) (b) fails in general for presheaves.

(a) is essentially obvious, so the real work is doing (b).

116 Definition

Let $\phi: \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves on X. We wish to define $\ker\phi, \operatorname{im}\phi, \operatorname{coker}\phi$. We do this "pointwise" in the naive way: if $U \subset X$ is open,

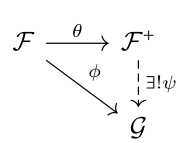
- $(\boxed{\ker\phi})(U) \coloneqq \ker(\phi(U))$
- $(\boxed{\operatorname{im}\phi})(U) \coloneqq \operatorname{im}(\phi(U))$
- $(\boxed{\operatorname{coker}\phi})(U) \coloneqq \operatorname{coker}(\phi(U))$.

117 Homework

Prove that all three of these give presheaves (with the natural induced restriction maps). Moreover, show that if $\mathcal{F}$ and $\mathcal{G}$ are sheaves, then $\ker\phi$ is a sheaf, but that $\operatorname{im}\phi$ and $\operatorname{coker}\phi$ need not be sheaves.

118 Theorem

Let $\mathcal{F}$ be a presheaf. Then there exists a pair $(\mathcal{F}^+, \theta)$ where $\mathcal{F}^+$ is a sheaf, $\theta: \mathcal{F} \to \mathcal{F}^+$ is a morphism of presheaves, such that for any sheaf $\mathcal{G}$ and morphism $\phi: \mathcal{F} \to \mathcal{G}$, there exists a unique morphism $\psi: \mathcal{F}^+ \to \mathcal{G}$ such that



PROOF We define $\mathcal{F}^+$ as a (pre)sheaf of functions,

$$\mathcal{F}^+(U) := \{s: U \to \coprod_{P\in X} \mathcal{F}_P \mid {}_{\substack{\forall Q\in U, s(Q)\in\mathcal{F}_Q, \\ \forall Q\in U \exists Q\in V\subset U, \\ t\in\mathcal{F}(V) \text{ s.t. } s(Q)=t_Q\in\mathcal{F}_Q}}\}.$$

Here we consider $\coprod_{P\in X}\mathcal{F}_P$ as just a set. The elements of $\mathcal{F}^+(U)$ are essentially sections with local compatibility constraints forcing the points of the stalks to have come from the stalks of sections.

The restriction maps are the obvious ones. We see $\mathcal{F}^+(U)$ is a singleton, so 0. The two other presheaf axioms are easy. The two sheaf axioms are also straightforward verifications. The map $\theta: \mathcal{F} \to \mathcal{F}^+$ is defined naturally, and one may verify it has the tated universal property.

119 Definition

Let $\phi: \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. We defined $\boxed{\ker\phi}$ as a sheaf above. We define $\boxed{\operatorname{im}\phi}$, $\boxed{\operatorname{coker}\phi}$ to be the sheaves associated to the presheaf versions defined above. This is a slight abuse of notation, but we will almost always be interested in the sheaf rather than pre-sheaf versions of these concepts, so these symbols will be interpreted in this way.

We say that ϕ is an $\boxed{\text{injective sheaf morphism}}$ if $\ker\phi = 0$ and is a $\boxed{\text{surjective sheaf morphism}}$ if $\operatorname{coker}\phi = 0$, or equivalently if $\operatorname{im}\phi = \mathcal{G}$. Note that $\operatorname{coker}\phi$ requires passing to the associated sheaf, so surjectivity is not as straightforward as injectivity.

120 Definition

Let $\mathcal{F}, \mathcal{G}$ be (pre)sheaves on a topological space X. Define the $\boxed{\text{direct sum of (pre)sheaves}}$ as

$$(\boxed{\mathcal{F}\oplus\mathcal{G}})(U) := \mathcal{F}(U) \oplus \mathcal{G}(V).$$

This is a (pre)sheaf.

121 Example

Let $P \in \mathbb{P}^1$. Let k_P denote the skyscraper sheaf at P with group k. There is a morphism of sheaves $\alpha_P: \mathcal{O}_{\mathbb{P}^1} \to k_P$ given essentially by $f \mapsto f(P)$. Define $\mathcal{O}_{\mathbb{P}^1}(-P) := \ker\alpha_p \hookrightarrow \mathcal{O}_{\mathbb{P}^1}$. We then have a map

$$\mathcal{O}_{\mathbb{P}^1}(-P) \oplus \mathcal{O}_{\mathbb{P}^1}(-Q) \to \mathcal{O}_{\mathbb{P}^1}$$

defined on $U \subset X$ by

$$(f, g) \mapsto f + g.$$

Here we assume $P \neq Q$. This will not be injective, since we will have functions that have the same value at P and Q.

Claim: ϕ is surjective, but the presheaf cokernel is non-zero. Proof of the second part: $\mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1)$ are in fact just the constant functions, whereas the global sections of $\mathcal{O}_{\mathbb{P}^1}(-P)$ are the global regular functions which have a zero at P, which is only 0. Hence the induced morphism on global sections is $0 \oplus 0 \to k$, which is not surjective. Hence $\operatorname{coker}(\mathbb{P}^1) = k$ as presheaves.

Proof of the first part: we first give a criterion for surjectivity of a sheaf morphism $\mathcal{F} \to \mathcal{G}$. For every open U and $s \in \mathcal{O}_{\mathbb{P}^1}(U)$, there exists an open cover $\cup_\alpha U_\alpha = U$ and $t_\alpha \in \mathcal{F}(U_\alpha)$ such that $\phi(t_\alpha) = s|_{U_\alpha}$. To motivate this, these elements will glue together to given an element of the sheaf associated to the presheaf image. Returning to the example above, for any open set U, we have an open cover $U = (U - \{P\}) \cup (U - \{Q\})$. On either of these open sets, the sections will be unrestricted, and it follows that we may apply the criterion.

122 Definition

Let $\mathcal{F}$ be a (pre)sheaf. A $\boxed{\text{sub(pre)sheaf}}$ $\mathcal{F}' \subset \mathcal{F}$ is a (pre)sheaf such that for all $U \subset X$ open, $\mathcal{F}(U) \subset \mathcal{F}(V)$ is a subgroup and the restriction maps are induced by inclusions. We sometimes define sub(pre)sheaves as injective sheaf morphisms, which is the same up to isomorphism.

If $\mathcal{F}' \subset \mathcal{F}$ is a subpresheaf, then the quotient presheaf is

$$(\boxed{\mathcal{F}/\mathcal{F}'})(U) := \mathcal{F}(U)/\mathcal{F}'(U).$$

If $\mathcal{F}' \subset \mathcal{F}$ is a subsheaf, then the quotient sheaf is the sheaf associated to the quotient presheaf.

123 Homework

If $\phi: \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves, then $\operatorname{coker} \phi \cong \mathcal{G}/\operatorname{im} \phi$. This is trivial for presheaves, so the content of the homework is that passing to the associated sheaf in the appropriate places still gives an isomorphism.

Show that a morphism ϕ of sheaves is surjective if and only if ϕ_P is surjective for all P, and it is injective if and only if ϕ_P is injective for all P. Injectivity works for presheaves as well. The punchline is that surjectivity and injectivity of sheaf maps can be checked on stalks.

With all of these definitions, we can define exact sequences of sheaves in the obvious way, namely the image of one map is the kernel of the next. That is,

$$\mathcal{F} \xrightarrow{\psi} \mathcal{F}' \xrightarrow{\phi} \mathcal{F}''$$

is exact when $\operatorname{im} \psi = \ker \phi$. In fact, one may show that if $\phi \circ \psi = 0$, then there exists a natural map $\operatorname{im} \psi \to \ker \phi$. Exactness is then saying this natural map is an isomorphism.

January 25th, 2016: Draft

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124 Remark

"Universal homework:" check any statement we make for sheaves to see if it holds for presheaves. The ones that aren't obviously nonsense are often true.

125 Definition

Let $f: X \to Y$ be a continuous map of topological spaces. Let $\mathcal{F}$ be a sheaf on X and define the direct image sheaf or pushforward sheaf of $\mathcal{F}$ on Y by

$$(\boxed{f_* \mathcal{F}})(V) := \mathcal{F}(f^{-1}V)$$

on any $V \subset Y$ open. It is an exercise to verify this preserves the sheaf structure, though it's quite straightforward.

126 Remark

We like to think of $\mathcal{F}$ as a collection of functions on X, and $f_*\mathcal{F}$ as a collection of functions on Y. There is a natural way to get functions from X to whatever $f_*\mathcal{F}$ maps to, namely precompose with f; this is *not* what the pushforward construction does, but our next construction does. Interestingly, it is rather more complicated.

Let $\mathcal{G}$ be a sheaf on Y. Define the inverse image sheaf on X as follows: for any $U \subset X$ open, set

$$(f^{-1}\mathcal{G})^{\text{pre}}(U) := \lim_{V \supset f(U)} \mathcal{G}(V)$$

where more explicitly we may define the limit as equivalence classes via

$$(V, t) : t \in \mathcal{G}(V), V \text{ open}, V \supset f(U)$$
$$(V, t) \sim (V', t'), \exists W \text{ s.t. } V \cap V' \supset W \supset f(U), t|_W = t'|_{W'}.$$

Now $\boxed{f^{-1}\mathcal{G}}$ is the sheaf associated to $(f^{-1}\mathcal{G})^{\text{pre}}$.

127 Example

Let $Z \overset{i}{\hookrightarrow} X$ be an open embedding, both of the previous constructions are easy. If i is injective (an embedding?), we define

$$\boxed{\mathcal{F}|_Z} := i^{-1}\mathcal{F}.$$

If X is a variety and $Z \subset X$ is a subvariety, define the $\boxed{\text{ideal sheaf}}$ of Z as the subsheaf $\mathcal{I}_Z \subset \mathcal{O}_X$ where

$$\mathcal{I}_Z(U) := \{f \in \mathcal{O}_X(U) : f(P) = 0, \forall P \in Z \cap U\}.$$

More on this later. It is related to the (dual of the) normal bundle. We actually have a short exact sequence of sheaves

$$0 \to \mathcal{I}_Z \to \mathcal{O}_X \to i_*\mathcal{O}_Z \to 0.$$

Roughly, for each point not in Z, we can find an open set V around it which does not intersect Z, and $\mathcal{I}_Z(V) \to \mathcal{O}_X(V)$ is an isomorphism, so $\mathcal{O}_X(V)/\mathcal{I}_Z(V) = 0$. This implies that for all $P \in U - Z$, $(\mathcal{O}_X/\mathcal{I}_X)_P = 0$. Hence $\mathcal{O}_X/\mathcal{I}_Z$ is supported on Z in the sense that non-zero stalks occur only in Z. This discussion shows that the above sequence is reasonable. A more careful and complete argument will prove it:

128 Homework

Show that we have a natural map $\mathcal{O}_X \to i_*\mathcal{O}_Z$, show that this is in fact surjective in the sense that $\mathcal{O}_X \to i_*\mathcal{O}_Z \to 0$ is exact, and show that its kernel is $\mathcal{I}_Z$.

129 Remark

In the previous example, we used the pushforward instead of the inverse image. We could have tried the same sort of thing using inverse images. As it turns out, f_* and f^{-1} are adjoint functors, with $\mathcal{G} \mapsto f_*\mathcal{F}$, $f^{-1}\mathcal{G} \mapsto \mathcal{F}$.

130 Definition

A sheaf $\mathcal{F}$ is an $\mathcal{O}_X$-module if for all $U \subset X$, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module, and the restriction maps ρ_{UV} are $\mathcal{O}_X(U)$-module homomorphisms (U here?).

131 Remark

If $\mathcal{F}$ is an $\mathcal{O}_X$-module, then $f_*\mathcal{F}$ is an $\mathcal{O}_Y$-module. As it turns out, even if $\mathcal{G}$ is an $\mathcal{O}_Y$-module, $f^{-1}\mathcal{G}$ need not be an $\mathcal{O}_X$-module. It will naturally be a $f^{-1}\mathcal{O}_Y$-module instead. Indeed, $f_*\mathcal{F}$ is naturally an $f_*\mathcal{O}_X$-module, and there is always a natural map $\mathcal{O}_Y \to f_*\mathcal{O}_X$, which allows us to turn $f_*\mathcal{O}_X$ into an $\mathcal{O}_Y$-module. By adjointness, we have a corresponding morphism $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$.

We may fix the issue with f^{-1} using tensor products to "extend scalars" along $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. This will define

$$f^*\mathcal{G} := f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X.$$

This will automatically be an $\mathcal{O}_X$-module. Using $\mathcal{O}_X$-module morphisms, f^* is in fact adjoint of f^{-1}. One could define f_* as above and declare f^{-1} is the left adjoint on the category of sheaves and that f^* is the left adjoint on the category of $\mathcal{O}_X$-modules. One may relate f_*, $f^!$, derived functors, and adjoints—more on this later in the course.

132 Remark

The following theorems don't require sheaves to state, but are really about sheaves.

133 Theorem

Let $X \subset \mathbb{A}^n$ be an affine variety. Recall $A(X) := A/I(X)$ where $A := k[x_1, \ldots, x_n]$, and $\mathcal{O}_X$ is the structure sheaf.

(i) $\mathcal{O}_X(X) = \mathcal{O}(X) \cong A(X)$

(ii) For any $P \in X$, the correspondence $P \leftrightarrow \mathfrak{m}_{X,P} := I(P)/I(X) \triangleleft_{max} A(X)$ is a bijection.

(iii) *For all* $P \in X$, $\mathcal{O}_{X,P} \cong A(X)_{\mathfrak{m}_{X,P}}$. *Moreover,* $\dim \mathcal{O}_{X,P} = \dim X$.

(iv) $K(X) \cong \operatorname{Frac}(A(X))$. *In particular,* $K(X)$ *is a finitely generated extension of the base field* $k = \overline{k}$. *Moreover,* $\operatorname{trdeg}_k K(X) = \dim X$.

(Part (ii) certainly requires $k = \overline{k}$*, though we are not restricting the characteristic.)*

Proof This philosophy of the proof is very sheaf-like. The idea is to prove things on stalks and then globalize. Part (ii) is an easy consequence of the Nullstellensatz. Next, we have a natural map $A \to \mathcal{O}(X)$ by considering a polynomial to be a regular function on $\mathbb{A}^n$, hence on X. By definition, the kernel of $A \to \mathcal{O}(X)$ is $I(X)$, so we have a natural injection $\alpha\colon A(X) \to \mathcal{O}(X)$. This induces injections of localizations

$$A(X)_{\mathfrak{p}_{X,P}} \overset{\alpha_P}{\to} \mathcal{O}_{X,P} \frac{f}{g} \qquad \mapsto \frac{\alpha(f)}{\alpha(g)}.$$

Since $A(X)$ is a domain, we can consider $A(X)_{\mathfrak{p}_{X,P}} \subset \operatorname{Frac}(A(X))$. (This is injective since localization is exact, or more explicitly since $\ker \alpha_P = 0$ directly.) For each $h \in \mathcal{O}_{X,P}$, there is an open set $p \in U \subset X$ such that there exists $f, g \in A$ such that $h = [U, f/g]_P$, so α_P is also surjective, which proves (iii). We will continue next time.

January 27th, 2016: Draft

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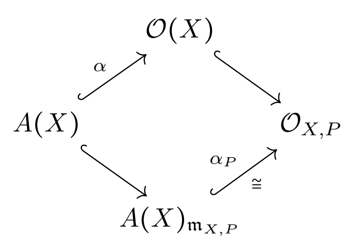
134 Remark

We continue the proof from the end of last class.

Proof We had an affine variety $X \subset \mathbb{A}^n$. We showed that $\mathcal{O}_{X,P} \cong A(X)_{\mathfrak{m}_{X,P}}$. From our usual height/dimension theorem, this says $\dim \mathcal{O}_{X,P} = dim A(X)_{\mathfrak{m}_{X,P}} = \dim A(X) = \dim X$. All that's left is to show that $K(X) \cong \operatorname{Frac} A(X)$, $\mathcal{O}(X) \cong A(X)$.

Recall that for $P \in X$ we have injections



where $\mathcal{O}_{X,P} \subset \operatorname{Frac} A(X) \subset K(X) = \cup_{Q \in X} \mathcal{O}_{X,Q}$. Since $\operatorname{Frac} A(X) = \cup_{Q \in X} A(X)_{\mathfrak{m}_{X,Q}}$, we have $K(X) \cong \operatorname{Frac} A(X)$.

On the other hand, by definition $\mathcal{O}(X) = \cap_{Q \in X} \mathcal{O}_{X,Q}$. It is a standard fact that $A(X)$ in $\operatorname{Frac} A(X)$ is $\cap_{Q \in X} A(X)_{\mathfrak{m}_{X,Q}}$, which finishes the theorem. More precisely, we have:

135 Lemma

Let B *be an integral domain. Then* $B \subset B_{\mathfrak{m}} \subset \operatorname{Frac} B$ *for all* $\mathfrak{m} \lhd_{max} B$. *Then*

$$B = \cap_{m \lhd_{max}} B_{\mathfrak{m}} \subset \operatorname{Frac} B.$$

Proof The geometric intuition behind this is that a regular function is one which is regular at all points. We have $\subseteq$ immediately. For the other containment, first define for $x \in B$ an ideal

$$I_x := \{c \in B : cx \in B\} \subset B.$$

If $I_x \neq B$, then we would have a maximal ideal $\mathfrak{m}$ in B such that $I_x \subseteq \mathfrak{m}$. But $x \in B_{\mathfrak{m}}$, so $x = a/b$ for $a, b \in B$, $b \notin \mathfrak{m}$, so $bx = a \in B$, so $b \in I_x \subset \mathfrak{m}$, a contradiction. Hence $1 \in I_x$, so $x \in B$.

(There is an analogous but more involved result concerning integrally closed domains and height 1 primes instead of maximal height primes.)

136 Remark

The conclusion $A(X) \cong \mathcal{O}(X)$ is perhaps surprising: a locally defined condition which holds everywhere is equivalent to a globally defined condition. This perhaps suggests that the "global" behavior of affine varieties is in some sense straightforward, and that they might be similar to, say, Euclidean spaces. Another analogue is that in, say, the complex plane, a function which is locally a polynomial everywhere is globally a polynomial. A different analogue is Cho's theorem (spelling?) which roughly says the only complex compact submanifolds of $\mathbb{P}^n$ are complex varieties. (The key idea is to look at the corresponding cones; very roughtly, the analytic vanishing turns out to be polynomial because the cones contain lines through the origin.)

This observation is essentially the key ingredient to making schemes "work."

137 Remark

We next get the analogue of the previous theorem for projective spaces. In the Zariski topology, open sets are "huge," so it's not set up to follow our usual intuition behind "compactness." For complex projective varieties, say, we can consider the Euclidean topology as well, obtained by just using the subspace topology coming from $\mathbb{P}^n_{\mathbb{C}}$. It turns out they are automatically compact in the Euclidean topology. (Anything that's closed in the Zariski topology is closed in the Zariski topology.)

As it happens, if Y is an open subset of $\overline{Y}$ all in the Zariski topology, then the Euclidean and Zariski closures of Y agree.

138 Remark

Recall that in $\mathbb{P}^n$ we had the standard open sets $U_i := (x_i \neq 0) = \mathbb{P}^n - Z(x_i)$. We had maps

$$\begin{aligned} \phi_i \colon U_i &\to \mathbb{A}^n \\ (y_1, \ldots, y_n) &\mapsto [y_1, \ldots, y_i, 1, y_{i+1}, \ldots, y_n] \\ \psi_i \colon \mathbb{A}^n &\to U_i \\ [x_0 : \cdots : x : n] &\mapsto (x_0/x_i, \ldots, \widehat{x_i/x_i}, \ldots, x_n/x_i). \end{aligned}$$

These give isomorphisms of varieties. If $X \subset \mathbb{P}^n$ is a quasiprojective variety, set $X_i := X \cap U_i$, so $X = \cup X_i$.

139 Definition

We discuss "graded localization" (a term Sandor made up). Let $S = \oplus_{d \geq 0} S_d$ be a graded ring. Recall that $S^h := \coprod_{d \geq 0} S_d$ is the set of homogeneous elements. Let $T \subset S^h$ be a multiplicative subset. It makes sense to consider the usual localization $T^{-1}S$. Since we've chosen homogeneous elements, $T^{-1}S$ has a natural $\mathbb{Z}$-grading. More explicitly, $f/g \in T^{-1}S$ for $f \in S_d$, $g \in T$ has degree

$$\deg(f/g) := \deg f - \deg g.$$

Hence $T^{-1}S =: \oplus_{d \in \mathbb{Z}} (T^{-1}S)_d$. Note that $(T^{-1}S)_0 \subset T^{-1}S$ is a subring.

If $\mathfrak{p}$ is a homogeneous prime in S, define

$$S_{(\mathfrak{p})} := (S_{\mathfrak{p}})_0 = ((S - \mathfrak{p})^{-1}S)_0.$$

(The notation is a bit unfortunate; using different typefaces to distinguish between localization and homogeneous components helps some.) Similarly, if $f \in S^h$, define

$$S_{(f)} := (\{1, f, f^2, \ldots\}^{-1}S)_0.$$

Be careful not to name your prime ideal f, and write the zero ideal as (0) instead of 0. As an exercise, ponder the possible meanings of $S_0, S_{(0)}, S_{((0))}$.

140 Homework
Let $X \subset \mathbb{P}^n$. Show that $S(X)_{(x_i)} \cong A(X_i)$. (Hint: do it for $X = \mathbb{P}^n$, and deduce it works when appropriately quotienting.)

January 29th, 2016: Draft

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141 Remark
Last time we considered projective varieties $X \subset \mathbb{P}^n$ with standard affine opens $\mathbb{A}^n \cong U_i := (x_i \neq 0) \subset \mathbb{P}^n$, $X_i := X \cap_i$. We had ended with a homework problem, namely show $A(X_i) \cong S(X)_{(x_i)}$.

142 Theorem
Let $X \subset \mathbb{P}^n$ be a projective variety.

(i) $\mathcal{O}_X(X) = \mathcal{O}(X) = k$.

(ii) $\mathcal{O}_{X,p} \cong S(X)_{(\mathfrak{n}_{X,P})}$ *where* $\mathfrak{n}_{X,P} := (f \in S^h : f(P) = 0)$.

(iii) $K(X) \cong (S(X)_{(0)})_0$.

143 Remark
In (i), we might write $\mathcal{O}(X) \cong k$, but there's a canonical injection $k \hookrightarrow \mathcal{O}(X)$, which is in fact an isomorphism. So, it's equal in this sense.

For $P \in X$, we defined $I(P) \subset S := k[x_0, \ldots, x_n]$ by $I(P) := (f \in S^h : f(P) = 0)$. In contrast to the affine case, this is not a maximal ideal in S. However, this is maximal among non-maximal ideals. More precisely, there is the irrelevant maximal ideal $\mathfrak{m} := (x_0, \ldots, x_n) = \oplus_{d>0} S_d$, which is a homogeneous prime ideal corresponding to no projective variety. If I is a homogeneous, proper ideal in S, then it contains no constants other 0, so $I \subset \mathfrak{m}$.

The proof of (i) and (ii) is a bit formal; everyone is invited to ponder it on their own to see that it's all actually easy. Doing (i) is a bit harder, heuristically because it's a global statement rather than a local one.

144 Corollary
Let $X \subset \mathbb{P}^n$ be a projective variety, $Y \subset \mathbb{A}^n$ be a (quasi)affine variety. Then there exists $Q \in Y$ such that for all $P \in X$, $\phi(P) = Q$, i.e. ϕ is constant. In particular, the only (non-empty?) variety which is both affine and projective is a point.

Proof Let $x_i : \mathbb{A}^n \to \mathbb{A}^1 \cong k$ be the ith coordinate function. Now $x_i \circ \phi \in \mathcal{O}(X) = k$ is constant.

Proof Pick $P \in X$. Certainly $P \in X_i$ for some i. We have

$$\mathcal{O}_{X,P} \cong \mathcal{O}_{X_i,P} \cong A(X_i)_{\mathfrak{m}_{x_i,P}}.$$

We mentioned a correspondence between homogeneous polynomials in $\mathbb{P}^n$ and not-necessarily-homogeneous polynomials in $\mathbb{A}^n$, roughly given by $y_j := x_j/x_i$. Under this correspondence, $\mathfrak{m}_{X,P}$ corresponds to $\mathfrak{m}_{X_i,P}$. Now since $x_i \notin \mathfrak{n}_{X,P}$,

$$\mathcal{O}_{X,P} \cong (S(X)_{(x_i)})_{\mathfrak{n}_{X,P}} \cong (S(X)_{\mathfrak{n}_{X,P}})_0$$

where we've used the abstract fact $(S_{(f)})_{(\mathfrak{p})_0} \cong S_{(\mathfrak{p})}$. (Double check when we're taking degree 0 parts and when we're not....)

We have $K(X) \cong \operatorname{Frac} A(X_i)$ since we may compute $K(X)$ on any non-empty open subset, and X_i is affine. This is

$$\operatorname{Frac} A(X_i) \cong \operatorname{Frac} S(X)_{(x_i)} \cong S(X)_{((0))}$$

where the last equality follows quickly by considering the fractions involved.

Having proved (ii) and (iii), we turn to (i). Let $f \in \mathcal{O}(X)$; we wish to show f is constant. We have the natural restriction map $\mathcal{O}(X) \to \mathcal{O}(A_i) \cong A(X_i) \cong S(X)_{(x_i)}$. For convenience, we can consider each $S(X)_{(x_i)}$ as a subset of $\operatorname{Frac} S(X)$. That is, for every i, f can be written as a fraction $g_i/x_i^{N_i}$ where $g_i \in S(X)_{N_i}$ is a homogeneous polynomial of degree N_i. Equivalently, $x_i^{N_i} f \in S(X)_{N_i}$. Now consider $S(X)_N f$ where $N \geq \sum N_i$. Every homogeneous polynomial is a sum of monomials; if the total degree is N, then at least one of the x_i's have degree N_i, so it follows that $S(X)_N f \subset S(X)_N$, therefore $S(X)_N f^q \subset S(X)_N$ for all $q > 0$. Then in particular, $x_0^N f^q \in S(X)_N$ for every $q > 0$, so $f^q \in S(X)_N x_0^{-N}$. Now $S(X)[f] \subset \operatorname{Frac} S(X)$ is then contained in $S(X)x_0^{-N}$. We have a ring extension $S(X) \subset S(X)[f]$ where $S(X)[f]$ is contained in the finitely generated extension $S(X)x_0^{-N}$. Hence f is integral over $S(X)$, i.e. it satisfies a monic polynomial with coefficients in $S(X)$, say for some $a_1, \ldots, a_m \in S(X)$

$$f^m + a_1 f^{m-1} + \cdots + a_m = 0.$$

We may break this equation up into homogeneous components, which each must be zero independently. Since f is degree 0, the degree 0 part is

$$f^m + a_1^0 f^{m-1} + \cdots + a_m^0 = 0$$

where a_i^0 denotes the degree 0 part of a_i. Since the degree 0 part of $S(X)$ consists only of constants, each $a_i \in k$. But then f is algebraic over k and not just integral over $S(X)$. Since $k = \overline{k}$, $f \in k$.

145 Remark
If $k \neq \overline{k}$, the very last line does not work, among other things. There is a natural map $\mathbb{P}_k^n \to \mathbb{P}_{\overline{k}}^n$ given by $X \mapsto \overline{X}$ where $\overline{X}$ means to consider the solutions of the defining equations over the algebraic closure. Defining functions "in the right way" allows one to define extensions of functions from X to $\overline{X}$. If we define regular functions as above, the global regular functions can live in $\overline{k} - k$ (probably, he seemed to backpeddle a bit), though if one does it "correctly" then the above map can be used to immediately show that functions remain constant.

"I withdraw every statement I made about regular functions", so take the above remark with a grain of salt.

146 Lemma
Let X be a variety, $Y \subset \mathbb{A}^n$ an affine variety. Say $y_1, \ldots, y_n$ are coordinates in $\mathbb{A}^n$. Given a set-theoretic map $\psi: X \to Y$, ψ is a morphism if and only if $y_i \circ \psi \in \mathcal{O}(X)$.

Proof The $\Rightarrow$ direction is essentially trivial. For $\Leftarrow$, we first show ψ is continuous. Let $W \subset Y$ be closed, so $W = Z(f_1, \ldots, f_r)$ for $f_i \in A$. Now $\psi^{-1}W = \cap_i (f_i \circ \psi = 0)$. Now $f_i \circ \psi$ is a "polynomial" of the $y_i \circ \psi$, which is continuous, so it follows that $\psi^{-1}W$ is the intersection of closed sets, so is closed.

Now, if g is regular on an open set $V \subset Y$, we must show $g \circ \psi$ is regular on $\psi^{-1}V \subset X$. Regularity means that we are locally a fraction of polynomials, from which it follows that $g \circ \psi$ is locally a fraction of regular functions, which themselves are locally fractions of polynomials, which gives the result.

147 Remark

Our next theorem will classify the morphisms into affine varieties, namely it will say they are in bijective correspondence with k-algebra homomorphisms between rings of regular functions (contravariantly). It will follow that there is an equivalence of categories between affine varieties over k and certain k-algebras given essentially by sending an affine variety to its coordinate ring. For other varieties, it's a bit more complicated.

February 1st, 2016: Draft

(Missed.)

February 3rd, 2016: Draft

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148 Definition

Fix n, d and set $N \coloneqq \binom{n+d}{d} - 1$. Note that $N+1$ is the number of degree d homogeneous monomials in variables $x_0, \ldots, x_n$, call them $\{M_0, \ldots, M_n\}$. Define the d-uple embedding

$$\mathbb{P}^n \overset{\rho_{n,d}}{\to} \mathbb{P}^N$$
$$\rho_{n,d}(P) \coloneqq [M_0(P) : \cdots : M_N(P)].$$

149 Homework

Show this is an injective morphism whose image in $\mathbb{P}^N$ is closed.

150 Remark

Projective varieties have a nice property: they're "universally closed." That is, they always map under morphisms to closed subsets. This is similar to compact sets getting mapped to compact sets under continuous maps (in Hausdorff space?).

151 Example

Let $n = 1, d = 2$, so

$$P^1_{a:b} \to \mathbb{P}^2_{u:v:w}$$
$$[a : b] \mapsto [a^2 : ab : b^2]$$

The image is defined by $uw = v^2$, which is a conic. Another famous case is $n = 1, d = 3$, so

$$\mathbb{P}^1 \to \mathbb{P}^3$$
$$[a : b] \mapsto [a^3 : a^2 b : ab^2 : b^3].$$

This is in fact the twisted cubic. It is a one-dimensional object in three-dimensional space. Attempt to write its vanishing ideal with two generators. More geenrally, if $n = 1$, d is arbitrary, the the image of the d-uple embedding $\mathbb{P}^1 \to \mathbb{P}^d$ is the rational normal curve of degree d.

Indeed, in $\mathbb{P}^2$ any degree 2 curve fits into a plane, and in arbitrary dimension the rational normal curve is "the most twisted" curve since the only linear space it lies in is all of $\mathbb{P}^d$.

For another example, take $n = 2, d = 2$. Then $\mathbb{P}^2 \to \mathbb{P}^5$ is called the Veronese embedding, and $\operatorname{im}\rho_{2,2}$ is called the Veronese surface, often denoted by V. (A curve is a variety of dimension 1. A surface is a variety of dimension 2. An n-fold is a variety of dimension n.) $V \subset \mathbb{P}^5$ has an interesting property: given a closed curve $C \subset V$, there exists a hypersurface $H \subset \mathbb{P}^5$ (i.e. $H = Z(f)$, f irreducible) such that $C = V \cap H$.

152 Homework

Prove the last sentence.

Note: confusingly, sometimes the d-uple embedding is called the Veronese embedding, so "Veronese embeddings" may mean the d-uple embeddings above.

153 Definition

We next define projection from a point. Let $P \in \mathbb{P}^n$. Choose homogeneous coordinates $[x_0 : \cdots : x_n]$ so that $P = [0 : \cdots : 0 : 1]$ for convenience. Define

$$\pi\colon \mathbb{P}^n - \{P\} \to \mathbb{P}^{n-1}$$
$$[a_0 : \cdots : a_{n-1} : a_n] \mapsto [a_0 : \cdots : a_{n-1}].$$

This is a well-defined map, and it is very straightforward to check this is a morphism. (It is mildly instructive to do this from an arbitrary point instead of P as above. Also, imagine how this is actually projection through P, say on the Riemann sphere. It's a simple example, but "just do it once in your life.")

154 Definition

We next discuss the Segre embedding. (There were in fact a couple of Segre's in this context. This one is apparently Corrado.) Pick n, m and set $N := nm + n + m = (n+1)(m+1) - 1$. Define a map by "pairwise multiplication"

$$\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$$
$$[a_0 : \cdots : a_n] \times [b_0 : \cdots : b_m] \mapsto [a_0b_0 : a_0b_1 : \cdots : a_0b_m : a_1b_0 : \cdots : a_nb_m].$$

(This is essentially the outer product of two row vectors.)

155 Homework

Show that the Segre embedding $\sigma_{n,m}$ above is injective and its image in $\mathbb{P}^N$ is a closed, irreducible subset.

We may then *define* the variety structure on $\mathbb{P}^n \times \mathbb{P}^m$ as the one it inherits under this embedding. Importantly, the inherited topology is *not* the product of the Zariski topologies on each factor. (One may do the same thing for $\mathbb{A}^n \times \mathbb{A}^m \to \mathbb{A}^{n+m}$. Indeed, the Zariski topology on $\mathbb{A}^2$ is not the product of the Zariski topologies on $\mathbb{A}^1$.)

156 Example

Consider $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ by $[a : b] \times [c : d] \mapsto [ac : ad : bc : bd]$. Letting $\mathbb{P}^3$ have coordinates $[x : y : z : t]$, one finds $xt - yz$ both defines the image of this Segre embedding and in fact generates the corresponding vanishing ideal. This is the "cooling tower," a hyperbola of one sheet. Given any point on it, consider the tangent plane at that point (say in $\mathbb{C}$). It will intersect the surface in two lines, which arise from the horizontal and vertical line passing through the preimage of that point in $\mathbb{P}^1 \times \mathbb{P}^1$.

However, this surface does not have the Veronese surface property, namely there are curves in $\mathbb{P}^3$ that cannot be cut out by a hyperplane and the quadric above. Indeed, the smallest degree you can get as a hyperplane section is 2, so lines are not hit.

To given another example, on $\mathbb{P}^2$, any two curves always intersect. However, the parallel lines on the Segre quadric above do not intersect.

157 Definition

Given (quasiprojective) varieties $X \subset \mathbb{P}^n$, $Y \subset \mathbb{P}^m$, define the $\boxed{\text{product variety}}$ as

$$X \times Y := \{(x, y) : x \in X, y \in Y\} \subset \mathbb{P}^n \times \mathbb{P}^m$$

with the algebraic variety structure being the one inherited by the Segre embedding of $X \times Y \hookrightarrow \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$.

158 Homework

We may do the same definition above in the affine case. Show that this gives the same as defining $\mathbb{A}^n \times \mathbb{A}^m := \mathbb{A}^{n+m}$. Ideally, also check that different embeddings give rise to isomorphic products in the projective case.

Moreover, show that if X, Y are affine varieties over k, then $A(X \times Y) \cong A(X) \otimes_k A(Y)$.

Indeed, one could define $X \times Y$ in the affine case by the coordinate ring property above. Moreover, one could generalize this program to arbitrary varieties—given affine covers, one can compute products "locally" and then glue together the coordinate rings above. This is how scheme-theoretic products are defined.

159 Lemma

Let $\phi: X \to Y$ be a morphism of varieties such that $\varnothing \neq U \subset X$ is an open set and $\phi|_U = \psi|_U$. Then $\phi = \psi$.

Proof We may assume $Y = \mathbb{P}^n$.

160 Homework

Show that $X \times Y$ yields a product in the category of varieties. For instance, one must check that $\phi \times \psi: X \to \mathbb{P}^n \times \mathbb{P}^n$ given by $x \mapsto (\phi(x), \psi(x))$ is a morphism.

Consider the diagonal $\Delta \subset \mathbb{P}^n_x \times \mathbb{P}^n_y$ given by $\Delta := Z(x_i y_j - x_j y_i \mid i, j)$. Now $\phi|_U = \psi|_U$ says $(\phi \times \psi)(U) \subset \Delta$, but then $(\phi \times \psi)(X) = \overline{\phi \times \psi(U)} = \Delta$, which says precisely that $\phi = \psi$.

February 5th, 2016: Draft

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161 Remark

Today we'll begin with rational maps. We had a lemma from last time which said that given two morphisms $\phi, \psi: X \to Y$ and $\varnothing \neq U \subset X$ open, then $\phi|_U = \psi|_U$ implies $\phi = \psi$.

162 Definition

Let X, Y be varieties. A $\boxed{\text{rational map}}$ $\boxed{\phi: X \dashrightarrow Y}$ is an equivalence class of pairs (U, ϕ_U) where $\varnothing \neq U \subset X$ is open, $\phi|_U: U \to Y$ is a morphism, and $(U, \phi_U) \sim (V, \phi_V)$ iff $\phi_U|_{U \cap V} = \phi_V|_{U \cap V}$.

(The lemma from last time shows that this is a well-defined equivalence relation.)

163 Example

A rational map need not be a function in the classical sense. Let $\varnothing \neq U \subset X$ be an arbitrary open. We have the inclusion map $U \to X$. We can define $\phi: X \dashrightarrow Y$ by $\phi_V := \mathrm{id}_V$ for $V \subset U$ and don't define it for $V \not\subset U$.

164 Remark

There is a largest open set on which a rational map is defined, namely the union of all U's appearing in a given equivalence class. This is perhaps more helpful conceptually than mathematically.

165 Remark

Warning: composition of rational maps need not always be defined. Intuitively, it could be that the image of one does not hit the largest domain of the next.

166 Definition

As a rule, we call a map of topological spaces dominant if the image is dense. In particular, a dominant rational map is one where the image of the largest open set on which it is defined is a dense open subset of the codomain.

167 Remark

We may compose dominant rational maps. That is, suppose

$$X \overset{\phi}{\dashrightarrow} Y \overset{\psi}{\dashrightarrow} Z$$
$$U \overset{\phi_U}{\to} V \overset{\psi|_V}{\to} Z$$

If ϕ is dominant, then the image of ϕ is dense in Y, so it intersects V, so $\phi_U^{-1}V$ is a non-empty open set. Hence $\psi \circ \phi$ may be defined as the equivalence class of $(\phi_U^{-1}V, \psi_V \circ \phi_U)$.

168 Example

In the previous example involving the inclusion $U \overset{i}{\hookrightarrow} X$ and $X \overset{\phi}{\dashrightarrow}$, we have $\phi \circ i = \mathrm{id}_U$ trivially, and also $i \circ \phi$ on U is $\mathrm{id}|_U$, so $i \circ \phi \sim \mathrm{id}_X$.

169 Definition

If for X, Y varieties there exists $\phi\colon X \dashrightarrow Y$ and $\psi\colon Y \dashrightarrow X$ such that $\psi \circ \phi = \mathrm{id}_X$ and $\phi \circ \psi = \mathrm{id}_Y$ as rational maps, then X and Y are called birationally equivalent or just birational. We write $X \sim_{\text{bir.}} Y$.

170 Remark

For some people, the aim of the study of varieties is to classify them all in some sense. Doing this up to rational equivalence is sometimes done. Note that rational equivalence is in some sense very special to the case of varieties. In a Hausdorff topological space, because open sets are generally not so "huge," the corresponding notion of rational equivalence is silly.

171 Definition

Given a dominant rational map $\phi\colon X \dashrightarrow Y$, there is a corresponding k-algebra homomorphism

$$\phi^*\colon K(Y) \to K(X)$$
$$f \mapsto f \circ \phi.$$

172 Remark

We could restrict the domain of ϕ^* to $\mathcal{O}(Y) \subset K(Y)$, but it would still map into $K(X)$, so we may as well define it on $K(Y)$ in the first place.

173 Theorem

Let X, Y be varieties. There is a bijection

$$\{\text{dominant rational maps } \phi\colon X \dashrightarrow Y\} \leftrightarrow \{k\text{-algebra homomorphisms } K(Y) \to K(X)\}$$
$$\phi \mapsto \phi^*.$$

More precisely, there is a (contravariant) equivalence of categories between the category of varieties over k with dominant rational maps and the category of finitely generated field extensions of k with k-algebra homomorphisms.

Proof First, observe that for $\varnothing \neq U \subset X$ open, $K(X) = K(U)$. For all varieties X, take $\varnothing U \subset X$ to be an affine open. Then $K(U) = K(X)$ is the fraction field of $\mathcal{O}(U)$, which is a finitely generated k-algebra. On the other hand, say $K = k(\overline{x_1}, \ldots, \overline{x_n}) \supset k[\overline{x_1}, \ldots, \overline{x_n}] =: B$. Now B is an integral domain which is a finitely generated k-algebra, and we've seen such objects are precisely the coordinate rings of affine varieties. Hence we have some $Y \subset \mathbb{A}^n$ such that $B \cong A(Y)$, so that $K = \operatorname{Frac}(B) \cong K(Y)$. Hence the right-hand side category is indeed as claimed.

The main thing we still need to prove is that there is an inverse to $\phi \mapsto \phi^*$. Suppose $\theta \colon K(Y) \to K(X)$. We may assume Y is affine (X as well), so $K(Y) \cong \operatorname{Frac}(A(Y)) \supset A(Y)$. Say $A(Y) = k[y_1, \ldots, y_n]$ (where the y_i's may have some relations). Consider the images $\theta(y_i) \in K(X)$. Let U be the intersection of open sets on which each y_i is a regular function. This is non-empty since X is a variety. Then $\theta \colon A(Y) \to \mathcal{O}(U)$. By an earlier theorem, these maps are in one-to-one correspondence with morphisms $\phi_U \colon U \to Y$. Moreover, $\theta = \phi_U^*$. It is easy to check this is indeed an inverse to $\phi \mapsto \phi^*$.

174 Corollary

Let X, Y be varieties. The following are equivalent:

(i) $X \sim_{\text{bir.}} Y$

(ii) There exists $\varnothing \subset U \subset X$, $\varnothing \subset V \subset Y$ open such that $U \sim_{\text{bir.}} V$

(iii) $K(X) \cong K(Y)$.

175 Remark

Isomorphism of affine varieties was equivalent to k-algebra isomorphism of their coordinate rings. Similarly, birational isomorphism of varieties is equivalent to k-algebra isomorphism of their function fields.

Proof All the implications except (i) ⇒ (ii) are immediate from the theorem. For the remaining implication, say $\phi \colon X \dashrightarrow Y$, $\psi \colon Y \dashrightarrow X$ give the rational equivalence, and say $\phi \sim (U, \phi_U)$, $\psi \sim (V, \psi_V)$. Now $\psi_V^{-1} U \subset V$ as before. By definition $\phi \circ \psi \sim \mathrm{id}_Y$ and $\psi \circ \phi \sim \mathrm{id}_X$. Now $(\psi_V^{-1} U, \phi_U \circ \psi_V|_{\psi_V^{-1} U})$ is the identity. This is completely symmetric in that we may interchange ϕ, ψ, U, V. Now $U' := \phi_U^{-1} \psi_V^{-1} U$ has the property that

$$U' = \phi_U^{-1} \psi_V^{-1} U' \subset \phi_U^{-1} V \subset \phi_U^{-1} \psi_V^{-1} \phi_U^{-1} V =: \phi_U^{-1}(V') \subset \phi_U^{-1} \subset U'.$$

Hence equality must have held at every step, so $\phi_U|_{V'}$ and $\psi_V|_{U'}$ are inverse maps. Hence $U' \cong V'$.

176 Proposition

Every variety of dimension n is birational to a hypersurface (say in $\mathbb{P}^{n+1}$).

177 Remark

Given the proposition and a desire to only study varieties up to birational equivalence, why not just study hypersurfaces? Sometimes it's a good idea, but sometimes we sacrifice nice properties in the translation, e.g. roughly we may start with a smooth manifold and passing to a hypersurface may introduce many bad singularities.

Proof $K := K(X)$ is a finitely generated field extension of K. Hence there is a transcendence basis $x_1, \ldots, x_n \in K$ such that $K/k(x_1, \ldots, x_n)$ is a separable, finite algebraic extension. By the theorem of the primitive element, we have $f \in k(x_1, \ldots, x_n)[x_{n+1}]$ such that $K = k(x_1, \ldots, x_n, y)$ where $f(x_1, \ldots, x_n, y) = 0$. Clearing the denominator of f yields $g \in k[x_!, \ldots, x_{n+1}]$ such that $g(x_1, \ldots, x_n, y) = 0$. Now let $Y := Z(g) \subset \mathbb{A}^{n+1}$. Then $K(Y) \cong K(X)$ by construction, so $X \sim_{\text{bir.}} Y$. This hypersurface in $\mathbb{A}^{n+1}$ is birational to its projective closure in $\mathbb{P}^{n+1}$, which is a hypersurface.

The key fact about separable algebraic extensions above is in, for instance, Zariski-Samuel, Chapter 2, Theorem 31. The primitive element theorem is in Chapter 2, Theorem 19.

February 8th, 2016: Draft

(First 17 minutes missed.)

178 Definition

We say a rational map is **defined at a point** if there is an element of its equivalence class defined at that point.

179 Proposition

Let $\phi: X \to Y$ be a morphism of varieties. Define

$$\Gamma_\phi := \{(x, \phi(x)) : x \in X\} \subset X \times Y.$$

Then the restriction of projection $\pi: X \times Y \to X$, $\pi: \Gamma_\phi \xrightarrow{\sim} X$ with inverse $X \to X \times Y$ given by $x \mapsto (x, \phi(x))$.

Proof Say $X \subset \mathbb{P}^n, Y \subset \mathbb{P}^m, X \times Y \subset \mathbb{P}^n \times \mathbb{P}^m$. Now

$$X \times \mathbb{P}^m too\psi \mathbb{P}^m \times \mathbb{P}^m \supset \Delta := Z(x_i y_j - x_j y_i : i, j)$$
$$(x, y) \mapsto (\phi(x), y).$$

We have $\Gamma_\phi = \psi^{-1}\Delta$, which is closed. Now $\phi: X \dashrightarrow Y$ given by $\phi_U: U \to Y$, $\Gamma_{\phi_U} \subset U \times Y \subset X \times Y$, $\Gamma_\phi := \overline{\Gamma_{\phi_U}} \subset X \times Y$.

180 Proposition

Γ_ϕ is independent of (U, ϕ_U).

Proof Suppose $(U, \phi_U), (V, \phi_V)$ are representatives of ϕ, $\Gamma_{\phi_U}, \Gamma_{\phi_V} \subset X \times Y$, $\phi_U|_{U\cap V} = \phi_V|_{U \cap V}$. We have

$$\Gamma_{\phi_U} \cap ((U \cap V) \times Y) = \Gamma_{\phi_U|_{U\cap V}} \subset \Gamma_{\phi_U}, \Gamma_{\phi_V}.$$

181 Corollary

Let $\phi X: \dashrightarrow Y$ be a rational map. Then $\pi: \Gamma_\phi \overset{\sim_{bir.}}{\to} X$ ($\Gamma_\phi \supset \Gamma_{\phi_U} \overset{\pi}{\to} U \subset X$ where $\Gamma_\phi \sim_{bir.} \Gamma_{\phi_U}$, $U \sim bir. X$). $\pi^{-1}: X \dashrightarrow \Gamma_\phi$.

Proof Moreover, π^{-1} is defined at P if and only if ϕ is defined at P. The $\Leftarrow$ implication is immediate. For $\Rightarrow$, there exists V such that π^{-1} is defined, $\phi_V = \pi_2 \circ \pi_V^{-1}$ where π_2 is projection onto the second factor.

182 Theorem (The main theorem of elimination theory.)

For $i = 1, \ldots, r$, suppose f_i is a homogeneous polynomial in $x_0, \ldots, x_n$ with indeterminate coefficients a_{ij} for $j = 1, \ldots, d_i$ where d_i is the number of coefficients of f_i. Then there exists polynomials $G_1, \ldots, G_q \in \mathbb{Z}[y_{ij} : 1 \le i \le r, 1 \le j \le d_i]$ such that for any field k and any choices $a_{ij} \in k$,

$$\mathbb{P}^n \supset Z(f_1, \ldots, f_r) \neq \varnothing \Leftrightarrow (a_{ij}) \in Z(g_1, \ldots, g_q) \subset \mathbb{A}^{\sum_{i=1}^r d_i}$$

where $g_i := \overline{G_i} \in k[y_{ij}]$ where $\mathbb{Z} \to k$.

Proof We will not prove it. Stated without proof in Hartshorne, referencing van der Waarder. Also in Lang's Algebra, Theorem 9.3.2. (Aside: search for Himne Mendelbableko.)

183 Example

Consider $Z = Z(xy - 1) \subset \mathbb{A}^2$. Define the usual "projection onto the x-axis from a hyperbola" map $\phi: \mathbb{A}^2 \to \mathbb{A}^1$. Now $\phi(Z) = \mathbb{A}^1 - \{0\}$ is not closed. Contrast this with the following projective case.

184 Theorem

Let $\pi\colon \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^m$ be projection onto the second factor. Then π is closed. (That is, $Z \subset \mathbb{P}^n \times \mathbb{P}^m$ closed implies $\pi(Z) \subset \mathbb{P}^m$ is closed.)

Proof We have $\mathbb{P}^n_{x_0,\ldots,x_n}$, $\mathbb{P}^m_{y_0,\ldots,y_m}$. We will use the preceding theorem. We begin with the Segre embedding $\mathbb{P}^n \times \mathbb{P}^m \hookrightarrow \mathbb{P}^N$ where $\mathbb{P}^N_{z_{ij}}$ where $z_{ij} = x_i y_j$. We have a homogeneous ideal $I(Z) \subset k[z_{ij}]$ generated by, say, $F_1, \ldots, F_r$. We can test closedness locally. Pick $P \in \mathbb{P}^m$, and say $y_0(P) \neq 0$, so we may take $P = [1 : \cdots]$. Hence $P \in U_0 = (y_0 \neq 0) \cong \mathbb{A}^m_{y_j}$. Now $A := k[y_1, \ldots, y_m] = A(\mathbb{A}^m_{y_j})$. Let $f_t \in A[x_0, \ldots, x_n]$ be the image of F_t after substituting $z_{i0} \mapsto x_i, z_{ij} \mapsto x_i y_j$.

Let $\{b_{ts}\}_{\substack{1\le t\le r\\ 1\le s\le d_t}} \subset A[y_0, \ldots, y_m](= A)$ be the coefficients of the f_t. By the previous theorem, we have certain $g_1, \ldots, g_q \in k[y_{ts}]$. Set $a_{ts} := b_{ts}(P) \in k$. One can think of this as using a map $\beta\colon (b_{ts})\colon \mathbb{A}^m_{y_j} \to \mathbb{A}^{\sum_{t=1}^r d_t}_{y_{ts}}$).

Consider $\mathbb{P}^n \times \{P\} \supset Z(\overline{f_1}, \ldots, \overline{f_t})$ (where the bar denotes replacing the b_{ts} with the a_{ts}). Now $P \in \pi(Z) \cap U_0$ if and only if this is non-empty, and by the theorem, that occurs if and only if $(a_{ts} \in Z(g_1, \ldots, g_q) \subset \mathbb{A}^{\sum d_t}_{y_{ts}}$, which occurs if and only if $P \in \beta^{-1} Z(g_1, \ldots, g_q) \subset \mathbb{A}^m_{y_j}$. That is,

$$\pi(Z) \cap U_0 = \beta^{-1} Z(g_1, \ldots, g_q)$$

is closed, so $\pi(Z)$ is closed.

February 10th, 2016: Draft

Summary Recall that the projection map $\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^m$ is closed. We now give some corollaries.

185 Corollary

Let X be a projective variety, Y an arbitrary variety. Then $X \times Y \to Y$ is closed.

Proof If X is projective, then $X \subset \mathbb{P}^n$ is closed, and $Y \subset \mathbb{P}^m$ is locally closed. Having fixed these embeddings, we have a map $\pi\colon X \times Y \to Y$ induced from $\pi\colon \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^m$. If $Z \subset X \times Y$ is closed, we need $\pi(Z) \subset Y$ to be closed.

We have the projective closure $\overline{Y} \subset \mathbb{P}^m$, and $\overline{Z} \subset X \times \overline{Y} \subset \mathbb{P}^n \times \mathbb{P}^m$. Hence applying π gives

$$\pi(\overline{Z}) \subset \overline{Y} \subset \mathbb{P}^m$$

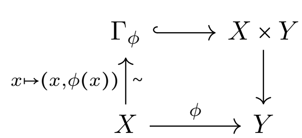
where $\pi(\overline{Z})$ is closed. In fact,

$$\pi(Z) = \pi(\overline{Z} \cap (X \times Y)) = \pi(\overline{Z}) \cap Y.$$

186 Corollary

Let X be a projective variety, Y an arbitrary variety, and $\phi\colon X \to Y$ a morphism. Then ϕ is closed, so in particular $\phi(X) \subset Y$ is closed.

Proof Letting Γ_ϕ be the graph of ϕ, we have



187 Corollary

If X is projective, then $\mathcal{O}(X) = k$.

PROOF Let $f \in \mathcal{O}(X)$ with $f: X \to \mathbb{A}^1 \subsetneq \mathbb{P}^1$. The image is a closed, proper subset of $\mathbb{P}^1$, from which one finds the image must be a finite set, but X is irreducible, so f is a constant.

188 Definition

Let $P \in \mathbb{A}^n$, use variables $x_1, \ldots, x_n$, let $P := (0, \ldots, 0)$, and consider $\pi: \mathbb{A}^n - \{P\} \to \mathbb{P}^{n-1}$ given by **projection from P (along the lines of stereographic projection). Alternatively, consider $\mathbb{P}^{n-1} \subset \mathbb{P}^n$ and take $\mathbb{A}^n$ to be the complement of $\mathbb{P}^{n-1}$ (a standard open set). Pick $P \in \mathbb{A}^n$ and project onto $\mathbb{P}^{n-1}$ in the natural way.**

For comparison with our earlier notation, let $U := \mathbb{A}^n - \{P\}$, write $\pi_U: U \to \mathbb{P}^{n-1}$. Now $\Gamma_{\pi_U} \subset (\mathbb{A}^n - \{P\}) \times \mathbb{P}^{n-1}$. Also, $\pi \mathbb{A}^n \dashrightarrow \mathbb{P}^{n-1}$ and $\overline{\Gamma_{\pi_U}} = \Gamma_\pi \subset \mathbb{A}^n \times \mathbb{P}^{n-1}$. Precisely, $\pi_U: (a_1, \ldots, a_n) \mapsto [a_1 : \cdots : a_n]$.

Define the blowup of $\mathbb{A}^n$ at P to be

$$\boxed{\operatorname{Bl}_P \mathbb{A}^n} := \Gamma_\pi \subset \mathbb{A}^n \times \mathbb{P}^{n-1}.$$

It comes with a natural map induced from $\sigma: \mathbb{A}^n \times \mathbb{P}^{n-1} \to \mathbb{A}^n$.

189 Proposition

We have

(a) $\operatorname{Bl}_P \mathbb{A}^n = Z(x_i y_j - x_j y_i : 1 \le i, j \le n) \subset \mathbb{A}^n \times \mathbb{P}^{n-1}$.

(b) $\sigma: \operatorname{Bl}_P \mathbb{A}^n \to \mathbb{A}^n$ *is an isomorphism over* $\mathbb{A}^n - \{P\}$.

(c) $\operatorname{Bl}_P \mathbb{A}^n$ *is irreducible and* σ *is a birational map to* $\mathbb{A}^n$.

(d) $\sigma^{-1}(P) = \{P\} \times \mathbb{P}^{n-1} \cong \mathbb{P}^{n-1}$. *Moreover, points* $Q \in \sigma^{-1}(P)$ *are in natural bijective correspondence with lines* $\ell \subset \mathbb{A}^n$ *containing* P. *We call* $\sigma^{-1}(P)$ *the* $\boxed{\text{exceptional set}}$ *of* σ.

190 Remark

Strictly speaking we haven't quite defined the vanishing set in (a). Essentially, just remember that the x_i need not be homogeneous, but the y_i do need to be homogeneous. One can embed $\mathbb{A}^n \times \mathbb{P}^{n-1}$ into $\mathbb{P}^n \times \mathbb{P}^{n-1}$ and essentially compute the vanishing locus there. The details work out and are straightforward.

PROOF For (b), π_U is a morphism, so $U \to \Gamma_{\pi_U}$ is an isomorphism, so its inverse σ is an isomorphism. Since U is an open dense subset of $\mathbb{A}^n$, and Γ_{π_U} is dense in Γ_π, σ is birational to $\mathbb{A}^n$. It also follows that $\operatorname{Bl}_P \mathbb{A}^n$ is irreducible.

For (d), fix $(a_1, \ldots, a_n) \neq P$. The corresponding line through that point (minus P) is $\{(\lambda a_1, \ldots, \lambda a_n : \lambda \in k^\times\}$, which maps under π to the class of $(a_1, \ldots, a_n)$, call it $Q \in \mathbb{P}^{n-1}$. Now $\sigma^{-1}(\ell - \{P\})$ contains (P, Q), and the correspondence follows.

Finally, consider (a). The map $\pi_U: (a_1, \ldots, a_n) \mapsto [a_1 : \cdots : a_n]$. Give $\mathbb{A}^n$ coordinates $x_1, \ldots, x_n$, and give $\mathbb{P}^{n-1}$ coordinates $y_1, \ldots, y_n$. If we weren't using projective space, we'd have $y_i = x_i$, but over projective space we just need $\lambda y_i = x_i$ for some λ. Hence $x_i/y_i = x_j/y_j$ for all i, j where the denominators are non-zero. It follows that $\operatorname{Bl}_P \mathbb{A}^n$ is contained in $Z := Z(x_i y_j - x_j y_i)$. For the other inclusion, recall that σ is an isomorphism over $\mathbb{A}^n - \{P\}$, so the difference $Z - \operatorname{Bl}_P \mathbb{A}^n$ is a subset of $\sigma^{-1}(P) = \{P\} \times \mathbb{P}^{n-1}$. However, this is a subset of $\operatorname{Bl}_P \mathbb{A}^n$ already, so equality holds.

191 Definition

Let $Z \subset \mathbb{A}^n$. The $\boxed{\text{strict transform}}$ (sometimes called the $\boxed{\text{proper transform}}$) is defined as follows. Let $\sigma: \operatorname{Bl}_P \mathbb{A}^n \to \mathbb{A}^n \supset Z$. Define

$$\boxed{\widetilde{Z}} := \overline{\sigma^{-1} Z - \{P\}}.$$

192 Remark

Of $P \notin Z$, the strict transform is just (isomorphic to) Z. The strict transform is not an abstract notion; it takes place inside a concrete affine space.

193 Definition

Let X be an affine variety, $P \in X \subset \mathbb{A}^n$. Choose coordinates so $P = (0, \ldots, 0)$. Define the $\boxed{\text{blowup of } X \text{ at } P}$ by

$$\begin{array}{ccc} \operatorname{Bl}_P \mathbb{A}^n & \xrightarrow{\sigma} & \mathbb{A}^n \\ \uparrow & & \uparrow \\ \boxed{\operatorname{Bl}_P X} := \widetilde{X} & \xrightarrow{\sigma_X := \sigma|_{\widetilde{X}}} & X \end{array}$$

If $Z \subset X$ is closed, the strict transform of Z is defined by

$$\widetilde{Z} := \overline{\sigma_X^{-1}(Z - \{P\})} \subset \operatorname{Bl}_P X.$$

194 Example

Let $\mathbb{A}^n \subset \mathbb{A}^m$ be a linear subspace. Choose coordinates $x_1, \ldots, x_m$ so that $\mathbb{A}^n = Z(x_{n+1}, \ldots, x_m)$. Set $P = (0, \ldots, 0)$. We have $\operatorname{Bl}_P \mathbb{A}^n \cong \widetilde{\mathbb{A}^{\ltimes}} \subset \operatorname{Bl}_P \mathbb{A}^m$.

Proof The notation is a bit unfortunate, but we'll muddle through. We have $\operatorname{Bl}_P \mathbb{A}^m = Z(x_i y_j - x_j y_i : 1 \le i, j \le m)$. On the other hand, combining these relations with the relations from $Z(x_{n+1}, \ldots, x_m)$ gives $Z(x_i y_j - x_j y_i : 1 \le i, j \le n)$, which gives $\operatorname{Bl}_P \mathbb{A}^n$.

195 Homework

Fill in the details for this argument.

196 Example

Try blowing up the origin in $(x^2 = y^3) \subset \mathbb{A}^2$. See if you can blow up the result in an interesting way and repeat as many times as you like.

February 12th, 2016: Draft

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197 Remark

Last time we discussed blowing up a point on an affine variety. For a projective variety, choose $P \in X \subset \mathbb{P}^n$ and pick a standard affine open $P \in U_0 \subset \mathbb{P}^n$. Then $P \in X_0 := X \cap U_0 \subset \mathbb{A}^n$, and $\operatorname{Bl}_P X_0 \subset \mathbb{A}^n \times \mathbb{P}^{n-1} \subset \mathbb{P}^n \times \mathbb{P}^{n-1}$. Hence we may set

$$\boxed{\operatorname{Bl}_P X} := \overline{\operatorname{Bl}_P X_0} \subset \mathbb{P}^n \times \mathbb{P}^{n-1}.$$

We can do the same for quasi-projective varieties. Precisely, let $P \in Y \subset \mathbb{P}^n$ for Y quasi-projective. Set $X := \overline{Y} \subset \mathbb{P}^n$. We have $\operatorname{Bl}_P X \subset \mathbb{P}^n \times \mathbb{P}^{n-1}$ with $\operatorname{Bl}_P X \overset{sigma}{\to} X$ and $P \in Y \subset X \subset \mathbb{P}^n$. Hence we may define $\boxed{\operatorname{Bl}_P(Y)} := \sigma^{-1} Y \subset \operatorname{Bl}_P X$.

198 Example

Consider $\mathbb{A}^3_{x,y,z} \supset Z(xy - z^2)$. In $\mathbb{P}^2$, this would be a conic, so this is a cone over that conic. Consider blowing up the origin $P = (0, 0, 0)$. Take $\mathbb{P}^2_{u:v:w}$. We have $\mathbb{A}^3 \times \mathbb{P}^2 \supset \operatorname{Bl}_P \mathbb{A}^3 = Z(xv - yu, xw - zu, yw - zv)$. (One way to remember these relations is that the matrix

$$\begin{pmatrix} x & y & z \\ u & v & w \end{pmatrix}$$

must have rank 1, so the 2×2 minors must all vanish.)

Now

$$\overline{\sigma^{-1}(X-P)} \coloneqq \mathrm{Bl}_P(X) \subset Z(xv-yu, xw-zu, yw-zv, xy-z^2) = \sigma^{-1}X \subset \mathbb{P}^2 \times \mathbb{A}^3.$$

Recall $\sigma\colon \mathrm{Bl}_P \mathbb{A}^3 \to \mathbb{A}^3$ is an isomorphism outside of P. Hence $\mathrm{Bl}_P X - \sigma^{-1}(P) = \sigma^{-1}X - \sigma^{-1}P = \sigma^{-1}(X-P)$.

Consider $U_z = X - Z(z)$. On U_z, we can invert z in each of the three equations above to get $u = xw/z, v = yw/z, uv = xyw^2/z^2$. From the fourth equation, we then have $uv = w^2$. Hence we may add $uv - w^2$ to the three constraints for $\sigma^{-1}X$ in an attempt to get rid of the parts of $\sigma^{-1}P \cong \mathbb{P}^2$ which are not in the closure $\sigma^{-1}(X-P)$.

We could also have begun by considering $\mathbb{P}^3_{x:y:z:t} \supset Z(xy - z^2)$. The preceding computation then corresponds to $t = 1$. Indeed, our earlier version was roughly speaking a cone "missing some points at infinity" and this projective version adds those points back in. To illustrate this, intersect the affine version with a hyperplane; the result is necessarily not the full projective conic (not a circle; really a hyperbola with missing points at infinity).

199 Remark

Given $X \subset \mathbb{P}^n$, consider the projection map $\pi\colon \mathbb{A}^{n+1} - \{P\} \to \mathbb{P}^n$ (where we'll take $P = 0$ for convenience). The cone over X is $C(X) \coloneqq \overline{\pi^{-1}X} \subset \mathbb{A}^{n+1}$, which is $\pi^{-1}(X) \cup \{0\}$.

Now $\mathrm{Bl}_P C(X) \subset \mathbb{A}^{n+1} \times \mathbb{P}^n$ comes with a map $\sigma\colon \mathrm{Bl}_P C(X) \to X$. We have $\sigma^{-1}(P) \subset \mathbb{P}^n$ ($\{P\} \times \mathbb{P}^n$) and $\sigma^{-1}(P) \cong X \subset \mathbb{P}^n$.

200 Remark

Our next goal is to define blow-ups along higher dimensional subvarieties, like lines, planes, surfaces, etc.

201 Definition

Let $L \subset \mathbb{A}^n$ be a linear variety (namely, one defined by linear equations), which is isomorphic to $\mathbb{A}^m$. The $m = 0$ case is a point. Use coordinates $x_1, \ldots, x_n$ and suppose $L = Z(x_1, \ldots, x_{n-m})$. First define a projection map $\pi_L\colon \mathbb{A}^n - L \to \mathbb{P}^{n-m-1}$ by $(a_1, \ldots, a_n) \mapsto [a_1 : \cdots : a_{n-m}]$. This again gives a rational map $\pi_L\colon \mathbb{A}^n \dashrightarrow \mathbb{P}^{n-m-1}$. Define

$$\mathrm{Bl}_L \mathbb{A}^n \coloneqq \Gamma_{\pi_L} \subset \mathbb{A}^n \times \mathbb{P}^{n-m-1}.$$

This comes with the map $\sigma\colon \mathrm{Bl}_L \mathbb{A}^n \to \mathbb{A}^n$ given by projecting onto the first factor.

More generally, given a linear subvariety $L \subset \mathbb{A}^n$ and an affine variety $X \subset \mathbb{A}^n$, using coordinates as above for L, we may define the strict transform of X to be

$$\widetilde{X} \coloneqq \overline{\sigma^{-1}(X-L)}.$$

If $L \subset X$, then we define $\mathrm{Bl}_L X \coloneqq \widetilde{X}$, with $\sigma_X \coloneqq \sigma|_{\mathrm{Bl}_L X}$.

202 Proposition

Given a linear variety $\mathbb{A}^m \cong L \subset \mathbb{A}^n$, the map $\sigma\colon \mathrm{Bl}_L \mathbb{A}^n \to \mathbb{A}^n$ has the following properties.

(i) $\mathrm{Bl}_L \mathbb{A}^n = Z(x_i y_j - x_j y_i | 1 \le i, j \le m)$.

(ii) $\sigma\colon \mathrm{Bl}_L \mathbb{A}^n \to \mathbb{A}^n$ *is birational, is an isomorphism outside of* L*, and in fact* $\mathrm{Bl}_L \mathbb{A}^n - \sigma^{-1}(L) \xrightarrow{\sim} \mathbb{A}^n - L$*. Hence* $\mathrm{Bl}_L \mathbb{A}^n$ *is irreducible.*

(iii) $\sigma^{-1}L = L \times \mathbb{P}^{n-m-1}$*. Moreover, points* $Q \in \sigma^{-1}L$ *are in natural bijective correspondence with* $(m+1)$*-dimensional linear subvarieties* L' *of* $\mathbb{A}^n$ *containing* L*.*

PROOF The argument for the $m = 0$ case from last time goes through essentially verbatim. To get (iii), note that

$$\overline{\sigma^{-1}(L' - \{P\})} = \overline{(L' - \{P\}) \times \{Q\}} = L' \times \{Q\}.$$

203 Remark

Consider the projective version. Let $H \subset \mathbb{P}^n$ be a linear hyperplane. Consider $H \cap U_0 \subset U_0 \subset \mathbb{P}^n$. One may define $\mathrm{Bl}_{H \cap U_0} U_0$ precisely as above, but then one must argue that one can glue the results together as U_0 varies over all the standard open sets. For blowing up a point, we were able to get away with just embedding $\mathbb{A}^n \times \mathbb{P}^{n-1}$ in $\mathbb{P}^n \times \mathbb{P}^n$. Taking the closure of $\mathrm{Bl}_{H \cap \mathbb{A}^n} \mathbb{A}^n$ in $\mathbb{P}^n \times \mathbb{P}^{n-1}$ ends up not necessarily giving you a large enough space. These considerations are rather technical, and we probably won't go through them.

February 17th, 2016: Draft

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204 Remark

Today we'll discuss relative projective varieties.

205 Definition

Let X be an affine variety over k with affine coordinate ring $A(X)$. Consider n-dimensional projective space over X, namely $X \times \mathbb{P}^n_k =: \mathbb{P}^n_X$. This space comes with a projection $\mathbb{P}^n_X \overset{\pi_X}{\to} X$. Pick $Z \subset \mathbb{P}^n_X$ closed; we call Z projective over X or X-projective. If further $U \subset Z$ is locally closed, then we call U quasi-projective over X.

Suppose $Z, Y \subset \mathbb{P}^n_X$. An X-morphism $Z \to Y$ is a morphism which is compatible with the projection map,

$$\begin{array}{ccc} Z & \longrightarrow & Y \\ & \searrow^{\pi_X|_Z} & \downarrow \pi_X|_Y \\ & & X \end{array}$$

This allows us to define the notion of X-isomorphism in the obvious way. Isomorphic varieties over X need not be X-isomorphic, but the converse does hold.

We may define the homogeneous coordinate ring relative to X as the graded ring

$$S(\mathbb{P}^n_X) := A(X)[y_0, \ldots, y_n]$$

where the zero-degree part is $A(X)$ and the y_i each have degree 1. Homogeneous ideals, primes, etc. all continue to make sense precisely as before. The homogeneous elements (a subset, not usually closed under addition) continue to be denoted by $S(\mathbb{P}^n_X)^h$. Given $f \in S(\mathbb{P}^n_X)^h$, we define $Z(f) \subset \mathbb{P}^n_X$ to be the set of points (P, Q) for which f vanishes at (P, Q), where Q is substituted in for the y_i's. More generally, if I is a homogeneous ideal, we define

$$Z(I) = \{(P, Q) : f(P, Q) = 0, \forall f \in I \cap S^h\}.$$

We may also define homogeneous vanishing sets and coordinate rings, namely if $Z \subset \mathbb{P}^n_X$ then then $I(Z)$ is the homogeneous ideal in $S(\mathbb{P}^n_X)$ generated by homogeneous elements which vanish at all points of Z. Then

$$S(Z/X) := S(\mathbb{P}^n_X)/I(Z).$$

206 Remark

A note on the meaning of $f(P,Q)$: we may write $f = \sum_\alpha f_\alpha y^\alpha$ for multi-indexes α, so that $f(P,Q) = \sum_\alpha f_\alpha(P) y^\alpha(Q)$. Assuming f is homogeneous, scaling Q multiplies this sum by an overall factor, so whether or not $f(P,Q) = 0$ is well-defined.

207 Remark

Note that $X = \{\star\}$ (a point) recovers our earlier definition of projective space. In the language of schemes, we would use $X = \operatorname{Spec} k$.

208 Remark

It is possible that Z and Y are isomorphic as varieties over X, but their homogeneous coordinate rings are not isomorphic. We've already seen examples of this over a field. "Almost any" two projective embeddings will give different coordinate rings. Nonetheless, we have the following proposition.

209 Proposition

Let $Z \subset \mathbb{P}^n_X$ and $Y \subset \mathbb{P}^m_X$ be projective over X such that $S(Z/X) \cong (Y/X)$ as graded $A(X)$-algebras. Then $Z/X \cong Y/X$.

Proof Suppose we have an isomorphism $S(Z/X) \to S(Y/X)$. We'll actually show that if we have a map of this form which is surjective, then we have a morphism $Y/X \to Z/X$. We have

$$\begin{array}{ccc} S(Z/X) & \twoheadrightarrow & S(Y/X) = A(X)[\overline{y}_0, \ldots, \overline{y}_r] \\ \uparrow\!\!\uparrow & \nearrow_{\alpha} & \\ S(\mathbb{P}^n_X) = A(X)[z_0, \ldots, z_n] & & \end{array}$$

Let $\zeta_i \coloneqq \alpha(z_i) \in S(Y/X)$, which by assumption is homogeneous of degree 1. Now $Z(\zeta_1 0, \ldots, \zeta_n) = \varnothing$ because α is surjective. Now define

$$Y \xrightarrow{\phi} \mathbb{P}^n_X$$
$$(P,Q) \mapsto (P, [\zeta_0(P,Q), \ldots, \zeta_n(P,Q)]).$$

Is is left as homework to check that this is a well-defined morphism. One must check $\operatorname{im}\phi \subset Z$. Pick $f \in S(\mathbb{P}^n_X)$ with $f \in I(Z)$. We must show $f(\phi(P,Q)) = 0$. We have

$$f(\phi(P,Q)) = (\alpha(f))(P,Q) = 0(P,Q) = 0$$

since $f \in \ker(S(\mathbb{P}^n_X) \to S(Z/X)) \subset \ker\alpha$.

By symmetry, if our original map is an isomorphism, we get a map the other way. Checking that the resulting two maps are mutual inverses is left as homework.

210 Remark

We next turn to blowing up ideals rather than just blowing up points and linear varieties.

211 Definition

Let $A = A(\mathbb{A}^n_k)$ be the usual affine coordinate ring. Let $I \subset A$ be an ideal. We'll define a rational map $\pi_I : \mathbb{A}^n \dashrightarrow \mathbb{P}^{m-1}$. Picking generators $I = (f_1, \ldots, f_m)$, set

$$\pi_I : \mathbb{A}^n - Z(I) \to \mathbb{P}^{m-1}$$
$$P \mapsto [f_1(P) : \cdots : f_m(P)] \in \mathbb{P}^{m-1}.$$

In the linear case, given two choices of generators (of the same length) they are related by a change of basis, so the resulting rational map is obviously essentially the same. The analogue in this context is no so clear.

We define the blowup of $\mathbb{A}^n$ at I as

$$\mathrm{Bl}_I \mathbb{A}^n := \Gamma_{\pi_I} \subset \mathbb{A}^n \times \mathbb{P}^{m-1} = \mathbb{P}^{m-1}_{\mathbb{A}^n}.$$

At the moment we should really write $\mathrm{Bl}_{\{f_1,\ldots,f_m\}} \mathbb{A}^n$. We will show later that the choice of generators amounts to a different choice of embedding. This blow-up comes with a projection $\mathrm{Bl}_I \mathbb{A}^n \overset{\sigma_I}{\to} \mathbb{A}^n$.

If $Z \subset \mathbb{A}^n$ is a closed subset, the strict transform of Z is defined precisely as before,

$$\widetilde{Z} := \overline{\sigma^{-1}(Z - Z(I))} \subset \mathrm{Bl}_I \mathbb{A}^n.$$

If $X \subset \mathbb{A}^n$ is an arbitrary affine variety such that $X \supset Z(I)$, we define

$$\mathrm{Bl}_I X := \widetilde{X} \subset \mathrm{Bl}_I A$$

with the structural morphism $\mathrm{Bl}_I X \overset{\sigma|_X}{\to} X$.

212 Remark

Since $X \supset Z(I)$, $I(X) \subset I$. We may consider $I/I(X) \subset A(X) = A/I(X)$, and we may write $\mathrm{Bl}_{I/I(X)} X$ instead of $\mathrm{Bl}_I X$. It shouldn't really matter.

213 Homework

The following is nice practice. Let $A = k[x,y]$. Let $I = (x^2, y^2)$. Compute $\mathrm{Bl}_I \mathbb{A}^2$. Then consider $\sigma^{-1} Z(I) \subset \mathrm{Bl}_I \mathbb{A}^n$, look at the "worst point you can imagine" on it, and try blowing it up there.

February 19th, 2016: Draft

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214 Remark

Last time we talked about blowing up an ideal. (Nomenclature note: the plural is "blowing ups" and not "blowings up.") Recall that X was affine (at least for now).

215 Proposition

$\mathrm{Bl}_I X \overset{\sigma}{\to} X$ has the following properties:

(i) σ is an isomorphism over $X - Z(I)$

(ii) σ is birational

(iii) $\mathrm{Bl}_I X$ is irreducible

Proof The first is by construction, and the other two are consequences of the first, essentially exactly as before.

216 Remark

Last time we had an explicit description of the ideal of the blowup. In this generality it's a little more complicated; see the next proposition. We'll deduce that $\mathrm{Bl}_I X$ is essentially independent of the choice of generators of I.

217 Proposition

Suppose $I = (f_1, \ldots, f_m)$, $\mathrm{Bl}_I X \subset \mathbb{P}^{m-1}_X$. Let

$$\mathfrak{a} := (f_i y_j - f_j y_i) \lhd S(\mathbb{P}^{m-1}_X) = A(X)[y_1, \ldots, y_m].$$

Then (i)

$$I(\mathrm{Bl}_I X) = \{g \in S(\mathbb{P}^{m-1}_X : \exists r \in \mathbb{N}, g \cdot I^r \subset \mathfrak{a}\}$$

and (ii)

$$S(\mathrm{Bl}_I X) = S(\mathbb{P}^{m-1}_X)/I(B) \cong \oplus_{d \geq 0} I^d \cong A(X)[f_1 t, \ldots, f_m t] \subset A(X)[t]$$

where the right-hand side is the image of the morphism $A(X)[y_1, \ldots, y_m] \overset{\alpha}{\to} A(X)[t]$ given by $y_i \mapsto f_i t$.

218 Remark

The t is essentially just "remembering the degree."

Proof Since $\pi: X - Z(I) \to \mathbb{P}^{m-1})$ is given by $P \mapsto [f_1(P) : \cdots : f_m(P)]$, $f_i y_j - f_j y_i \in I(\mathrm{Bl}_I X)$ for all i, j. In general, we may define "distinguished opens" as follows: if $f \in A(\mathbb{A}^n)$, then $Z(f) \subset \mathbb{A}^n$ and $D(f) := \mathbb{A}^n - Z(f)$ is open; if $f \in A(X)$, then $Z(f) := Z(\overline{f}) \subset X$, $D(f) := X - Z(f)$. It is a fact that if X is affine, then $D(f)$ is affine, and $A(D(f)) \cong A(X)_f$, which follows by considering $Z(1 - x_{n+1} f) \cap (X \times \mathbb{A}^1)$.

Now, $\cup_i D(f_i) = X - Z(I)$, since the complement of this union is the set where all the f_i are zero, namely $Z(I)$. By definition, $\mathrm{Bl}_I X := \overline{\sigma^{-1}(X - Z(I))}$. Write $B_i := \sigma^{-1} D(f_i) \subset \mathbb{P}^{m-1}_X$. Indeed, $B_i \cong D(f_i)$ since σ is an isomorphism away from $Z(I)$. Hence $A(B_i) \cong A(X)_{f_i}$. We can explicitly write down the isomorphism $D(f_i) \to B_i$ using the fact that $f_i(P) \neq 0$, namely $y_j := f_j(P)/f_i(P) y_i$.

Now $\sigma^{-1}(X - Z(I)) = \sigma^{-1}(\cup_i B_i) = \cup_i B_i$, so that

$$I(\mathrm{Bl}_I X) = I(\cup_i B_i)$$

and I of a union is the intersection of the I's. Now

$$\begin{aligned} S(\mathbb{P}^{m-1}_X) &= A(X)[y_1, \ldots, y_m] \\ S(\mathbb{P}^{m-1}_{D(f_i)}) &= A(X)_{f_i}[y_1, \ldots, y_m] \end{aligned}$$

and we have a natural map from the first to the second. Now we find $I(B_i) = \mathfrak{a}_{f_i}$ under this inclusion. Now suppose we have g and r as in (i). Then $\overline{g} \in \mathfrak{a}_{f_i}$, so $g \in I(B_i)$, for all i. On the other hand, $g \in I(B)$ implies $\overline{g} \in \mathfrak{a}_{f_i}$, so for some r, $g \cdot f_i^r \in \mathfrak{a}$. We may vary i and pick R large enough that in any monomial of degree R involving the f_i's, at least one of the exponents is large enough so that $g \cdot f_i^r \in \mathfrak{a}$. Hence $g \cdot I^R \subset \mathfrak{a}$. This completes (i).

For (ii), consider the map α from the problem statement. We must show $\ker \alpha = I(\mathrm{Bl}_I X)$. Consider the part where $f_i \neq 0$. Since localization is an exact functor, $(\ker \alpha)_{f_i} = \ker \alpha_{f_i}$. Localizing at f_i says we may say $t = 1/f_i$. It follows quickly that $(\ker \mathfrak{a})_{f_i} = \mathfrak{a}_{f_i}$; write the details down as homework. The same argument as in the end of (i) shows that $\ker \alpha = I(\mathrm{Bl}_I X)$.

219 Corollary

Let $I = (f_1, \ldots, f_m)$, $J = (g_1, \ldots, g_r)$. *Then if* $I = J$,

$$\begin{array}{ccc} \mathrm{Bl}_I X & \xrightarrow{\cong} & \mathrm{Bl}_J X \\ & \searrow & \downarrow \\ & & X \end{array}$$

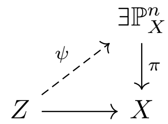
Proof Part (ii) of the proposition says that $S(\mathrm{Bl}_I X/X) \cong S(\mathrm{Bl}_J X/X)$, which we showed last time implies the above isomorphism.

220 Remark

The ideal in the preceding proposition is related to the ideal quotient, $(I : J) := \{g \in A : g \cdot J \subset I\}$. It is essentially $\cup_r(\mathfrak{a} : I^r) = (\mathfrak{a} : \cup_r I^r)$, which should be a direct limit.

221 Remark

Throughout our discussion of blowups, X has been affine. For defining $\mathbb{P}^m_X := X \times \mathbb{P}^n_k$, we don't actually need X affine. We may declare any closed subset of this $\mathbb{P}^m_X$ projective over X, or X-projective. A morphism $\phi: Z \to X$ is then called a projective morphism if it factors through projective space over X, i.e. $\exists n, \exists \psi: Z \to \mathbb{P}_X$ such that $\phi = \pi \circ \psi$,



A **$\mathbb{P}^n$-bundle over X** is a variety P with a morphism $P \xrightarrow{\pi} X$ such that there exists an affine open cover $X = \cup_i U_i$ with the following properties. Set $V_i := \pi^{-1}U_i$ and consider V_i as a U_i-variety. We require isomorphisms $\alpha_i\colon V_i/U_i \to \mathbb{P}^n_{U_i}$ such that

$$(V_i \cap V_j)/(U_i \cap U_j) \cong \mathbb{P}^n_{U_i \cap U_j}$$

where the map $\alpha_j \circ \alpha_i^{-1} \in \operatorname{Aut}(\mathbb{P}^n_{U_i \cap U_j})$ is $A(U_i \cap U_j)$-linear.

There is a notion of blowing up ideal sheaves of $\mathcal{O}_X$, though we won't do it quite yet.

222 Homework

In the above, we took $A(U_i \cap U_j)$, which requires $U_i \cap U_j$ to be affine. Show that if X is a variety, and $U, V \subset X$ are open affine, then $U \cap V \subset X$ is also affine.

February 22nd, 2016: Draft

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223 Remark

We won't prove the next theorem; it is for our "amusement." It says blowups are very "frequent."

224 Definition

An $\mathcal{O}_X$-module $\mathcal{F}$ is a **coherent sheaf** if there exists an open cover $X = \cup_\alpha U_\alpha$ such that $\mathcal{F}|_{U_\alpha}$ is the cokernel of a map of free $\mathcal{O}_X$-modules of finite rank,

$$\oplus^{r_2}\mathcal{O}_{U_\alpha} \to \oplus^{r_1}\mathcal{O}_{U_\alpha} \to \mathcal{F}|_{U_\alpha} \to 0$$

is exact. We call $\mathcal{F}$ a **quasicoherent sheaf** if the same condition holds except without the finiteness requirements. As it turns out, there is a notion of rank for coherent sheaves, which can be computed by tensoring the above sequence with the function field. Figuring out the invariant is left as an exercise.

225 Theorem

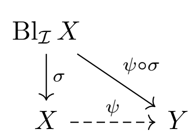
Let X be a variety, $\phi\colon Z \to X$ be a projective birational morphism (meaning ϕ factors through the map to projective space over X). Then there exists $\mathcal{I} \subset \mathcal{O}_X$ such that $Z/X \cong \operatorname{Bl}_{\mathcal{I}} X/X$, where $\mathcal{I}$ is a (coherent) ideal sheaf.

226 Homework

Work out the blowing up of an ideal sheaf for an arbitrary variety, $\operatorname{Bl}_{\mathcal{I}} X/X$. Write down your favorite birational morphism between projective varieties, and try to find the ideal sheaf you need to blow up. If your favorite birational morphism is a blowup already, use your second favorite one, or try to find a different ideal sheaf. Also, take $\mathbb{A}^n$ and pick a hyperplane H with some ideal $I(H) = (x_n) \lhd k[x_1, \ldots, x_n]$. Then the blowup lives in $\mathbb{A}^n \times \mathbb{P}^0$, so any ideal like this will give the same blowup.

227 Corollary

Let X be a variety, $\psi\colon X \dashrightarrow Y$ a rational map. Then there exists a (coherent) ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$ such that the composite



$\psi \circ \sigma$ *is a morphism.*

228 Remark

This is called "resolving the indeterminacy of a rational map." This is extremely useful. The previous theorem is also very useful; it often allows one to reduce to showing properties for blowups. There are issues, like that the ideal sheaves might be quite complicated or non-unique.

229 Example

Consider $Y := (x^2 t = y^2 z) \subset \mathbb{A}^2_{x,y} \times \mathbb{P}^1_{z,t}$, $X := \mathbb{A}^2$. Now $Y = \mathrm{Bl}_{(x^2,y^2)} X$ has support (x,y). Note though that $\mathrm{Bl}_{(x,y)} X$ does not map to Y as a morphism. (Indeed, $\mathrm{Bl}_{(x,y)} X \cong \mathrm{Bl}_{(x^2,xy,y^2)} X$; these don't live in the same space; making sense of this isomorphism is an exercise.)

230 Remark

We'll now put blowups aside for a while and discuss (non)singularity of varieties.

231 Notation

We write $(A, \mathfrak{m}, k)$ to denote a local ring R with unique maximal ideal $\mathfrak{m}$ and residue field $k := A/\mathfrak{m}$. Sometimes this is abbreviated to $(A, \mathfrak{m})$. We will usually assume our local rings are noetherian.

232 Proposition

If A is a noetherian local ring, then $\dim_k \mathfrak{m}/\mathfrak{m}^2 \geq \dim A$.

233 Remark

By Nakayama's lemma, the left-hand side is the minimal number of generators of $\mathfrak{m}$, which by a standard result in dimension theory for noetherian local rings is greater than or equal to $\dim A$. (The minimal number of generators of an $\mathfrak{m}$-primary ideal in this situation is at least as large as $\dim A$.) This proposition is Corollary 11.15 in Atiyah-Macdonald.

234 Homework

If you haven't seen Nakayama's lemma, (1) look it up and (2) prove the first sentence of the preceding remark. Note that $\mathfrak{m}/\mathfrak{m}^2$ is an A-module annihilated by $\mathfrak{m}$, hence an $(A/\mathfrak{m} = k)$-module.

235 Definition

A noetherian local ring $(A, \mathfrak{m}, k)$ is called **regular** if $\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim A$.

236 Remark

While this is a purely algebraic condition, it has a natural geometric interpretation. Suppose X is a variety, $P \in X$ is a point. A "typical" local ring is $\mathcal{O}_{X,P}$ with maximal ideal $\mathfrak{m}_{X,P}$ of functions which are zero at that point. Then $\mathfrak{m}_{X,P}/\mathfrak{m}^2_{X,P}$ are intuitively the differentials of functions that are 0 at P. If we dualize, $(\mathfrak{m}_{X,P}/\mathfrak{m}^2_{X,P})^\vee$ is naturally defined to be the tangent space (in smooth manifold theory) $T_{X,P}$

Geometrically, $\dim T_{X,P} \geq \dim X$, and X is smooth at P when equality holds. The preceding two definitions are modeled on this intuition and match it nicely.

237 Definition

Let X be a variety. We say X is **non-singular at a point** $P \in X$ if $\mathcal{O}_{X,P}$ is a regular local ring. X is a **non-singular variety** if it is non-singular at all points. X is a **singular variety** if it is not non-singular. A point $P \in X$ is a **singular point** if X is not non-singular at P.

The **singular set** of X **$\mathrm{Sing}\, X$** $:= \{P \in X : P \text{ is a singular point}\}$.

238 Theorem

Let $X \subset \mathbb{A}^n$ be an affine variety of dimension d. Suppose $I(X) = (f_1, \ldots, f_r) \lhd A := k[x_1, \ldots, x_n]$. Form the **Jacobian matrix** *at P by*

$$J(P) := \left[\frac{\partial f_i}{\partial x_j}(P)\right].$$

Then X is non-singular at P if and only if $\operatorname{rank} J(P) = n - d$.

239 Remark

Note that $\partial f/\partial x_i = 0$ for some x_i does not necessarily imply that f is constant in x_i. In characteristic p, $d/dx x^p = 0$. This is the only exceptional case in the following sense: $\partial f/\partial x_i = 0$ implies f is a polynomial in x_i^p.

PROOF Let $\mathfrak{m}_P \coloneqq I(P) \lhd A$. Define a k-linear morphism $\theta \colon A \to k^n$ by $f \mapsto (\ldots, \partial f_i/x_i(P), \ldots)$. Now $\mathfrak{m}_P = (x_1 - a_1, \ldots, x_n - a_n)$ where $P = (a_1, \ldots, a_n)$. Observe:

(i) $\theta(x_i - a_i) = (0, \ldots, 0, 1, 0, \ldots, 0)$, so θ is surjective. Moreover, $x_i - a_i \notin \ker\theta$.

(ii) $\mathfrak{m}_P^2 \subset \ker\theta$ by the product rule. By counting dimensions and using (i), it follows that $\mathfrak{m}_P^2 = \ker\theta$, so θ induces $\mathfrak{m}_P/\mathfrak{m}_P^2 \cong k^n$.

Note that $\mathfrak{m}_{X,P}$ corresponds to $I(P)/I(X)$. Now

$$\mathfrak{m}_{X,P}/\mathfrak{m}_{X,P}^2 \cong \mathfrak{m}_P/(\mathfrak{m}_P^2 + I(X))$$

since the left-hand side is

$$(\mathfrak{m}_P/I(X))/((\mathfrak{m}_P^2 + I(X))/I(X)).$$

By construction, the rank of $J(P)$ is precisely the dimension of $\theta(I(X))$, which is the dimension of $(\mathfrak{m}_P^2 + I(X))/\mathfrak{m}_P^2$, which is the dimension of $\mathfrak{m}_{X,P}/\mathfrak{m}_{X,P}^2$. X is non-singular at P if and only if this latter dimension is $\dim \mathcal{O}_{X,P} = \dim X = d$. Finally, note that

$$\dim \mathfrak{m}_{X,P}/\mathfrak{m}_{X,P}^2 + \operatorname{rank} J(P) = n$$

which completes the proof.

240 Remark

Next time we'll show that the singular set is closed. The preceding proof actually showed $\operatorname{rank} J(P) \leq n - d$ in general, so a singular point occurs only for strict inequality.

241 Corollary

$\operatorname{Sing}(X) \subset X$ *is closed.*

242 Remark

Interpret the defining inequality in terms of vanishing of minors of the Jacobian matrix to get closure.

February 24th, 2016: Draft

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243 Remark

Last time we talked about non-singularity and proved the Jacobian criterion.

244 Theorem

$\operatorname{Sing}(X) \subset X$ *is a proper closed subset.*

PROOF We proved the first half last time. Suppose $\operatorname{Sing} X = X$. Then for any $U \subset X$ open, $\operatorname{Sing} U = U$ since $\operatorname{Sing}(X \cap U) = \operatorname{Sing} U$ because singularity is a local condition. Hence if $X \sim_{\text{bir.}} Y$, then $U \cong V$ for $\varnothing \neq U \subset X$, $\varnothing \neq V \subset Y$, with $\operatorname{Sing} Y = Y$, $\operatorname{Sing} V = V$. In summary, we can take any birational model for X and derive a contradiction from there to prove the result.

Recall that any X is birational to a hypersurface $H \subset \mathbb{P}^n$ where $n = \dim X + 1$, which is birational to a hypersurface $H_0 \subset \mathbb{A}^n$. So, suppose $X \subset \mathbb{A}^n$ is an affine hypersurface. Take $X = Z(f)$ with $I(X) = (f)$ for f irreducible. The Jacobian criterion requires

$$\left(\tfrac{\partial f}{\partial x_1}(P) \quad \cdots \quad \tfrac{\partial f}{\partial x_n}(P)\right).$$

to be zero at singular points, so $\operatorname{Sing} X = Z(\ldots, \partial f/\partial x_i, \ldots, f)$. Now since $\operatorname{Sing} X = X$, $\partial f/\partial x_i \in (f)$ for all i. But $\deg_{x_i} \partial f/\partial x_i < \deg_{x_i} f$ then forces $\partial f/\partial x_i = 0$ for all i.

Now, if $\operatorname{char} k = 0$, then $f = 0$, a contradiction. If $\operatorname{char} k = p > 0$, then f is a polynomial in $x_1^p, \ldots, x_n^p$. Since $k = \overline{k}$, form a polynomial g by taking a pth root of each coefficient in f and dividing off a factor of p from the exponent of each variable. It follows that $f = g^p$ is not irreducible, again a contradiction.

245 Definition

Let k be a field (perhaps assume $k = \overline{k}$). Let X be an irreducible topological space, $\mathcal{O}_X$ a sheaf on X. We say X is a pre-variety over k if it has a finite open cover $X = \cup_{i=1}^r U_i$ such that

(i) U_i is an affine variety.

(ii) $\mathcal{O}_X|_{U_i}$ is isomorphic to the structure sheaf of U_i.

(This is nothing more than a reduced and irreducible scheme of finite type over k. Mumford uses this definition, so we're at least in good company.)

246 Remark

We will not explicitly use this definition much, though every time we say "variety," Sandor invites you to replace it with this.

Any of our earlier definitions that "depends only on local data" will work with pre-varieties.

There's a little subtlety in the above. We would really have to define ringed spaces $(X, \mathcal{O}_X)$ and one must defined morphisms and isomorphisms of ringed spaces, which is really the notion used in (i) and (ii) above.

247 Remark

There are pre-varieties that are not varieties; this will be an example shortly. On the other hand, every variety is a pre-variety, since the structure sheaf of a variety satisfies this condition. As for the counterexample, let X be a three-dimensional variety, for simplicity say affine and non-singular, e.g. $X = \mathbb{A}^3$. Take two different curves $C_1, C_2 \subset X$ which intersect at P and Q. Consider $I \coloneqq I(C_1), J \coloneqq I(C_2) \lhd A(X)$. Now consider $\operatorname{Bl}_I X$, $\operatorname{Bl}_J X$. From the dimensions, the fibers over each point of C_1 and C_2 are $\mathbb{P}^1$'s.

Now take $U = X - \{P\}$, $V = X - \{Q\}$ and consider $\operatorname{Bl}_{\overline{J}} \operatorname{Bl}_I X$, $\operatorname{Bl}_{\overline{I}} \operatorname{Bl}_J X$. Over $U \cap V$, these two blowups are isomorphisms, so

$$\operatorname{Bl}_{\overline{J}} \operatorname{Bl}_I X \supset \sigma^{-1}(U \cap V) = \sigma_2^{-1}(U \cap V) \subset \operatorname{Bl}_{\overline{I}} \operatorname{Bl}_J X.$$

So, removing both points yields an isomorphism. Removing one point from each says we may glue together $\sigma_1^{-1}(U)$ and $\sigma_2^{-1}(V)$ along $U \cap V$. Hence this satisfies the definition of a pre-variety. However, if one works through the details, one ends up finding curves ℓ_1, m_1, ℓ_2, m_2 such that ℓ_1 is equivalent (up to degenerating the curves) to $\ell_2 + m_2$ and $\ell_1 + m_1$ is equivalent to m_2. It follows that $\ell_2 + m_1 \sim 0$, but in projective space this cannot happen, so this is not a variety. Justifying these last remarks takes machinery we have not developed.

248 Homework

Make an explicit example from the above discussion and at least intuitively understand the degenerations at the end, as practice with blowups at more than a point.

If you want to understand blowups—which is a good idea if you want to understand algebraic geometry—this is one of the best non-trivial examples.

249 Remark

If X is affine, $\mathcal{O}_{X,P} \cong A(X)_{\mathfrak{m}_P}$. However, $A(X)$ has far more localizations than just at maximal ideals. We can localize at elements, e.g. $A(D(f)) = A(X)_f$, where $D(f)$ is the complement of the locus of vanishing of f. We can also take a prime ideal $\mathfrak{p}$ and form $A(X)_\mathfrak{p}$. What is the geometric meaning of $A(X)_\mathfrak{p}$? One observation: $A(X)_\mathfrak{p}/(\mathfrak{p}_\mathfrak{p}) \cong \mathrm{Frac}(A(X)/\mathfrak{p}) \cong K(Z(\mathfrak{p}))$, so these are rational functions whose denominator is not contained in $\mathfrak{p}$, i.e. where the function is defined at some point of $\mathfrak{p}$.

More formally, for $g \in K(X)$, we can say g is defined on some maximal open set $U \subset X$. $K(X) \supset A(X)_\mathfrak{p}$, and we have $g \in A(X)_\mathfrak{p}$ if and only if $U \cap Z(\mathfrak{p}) \neq \varnothing$.

In "nice enough" rings, every prime ideal is the intersection of the maximal ideals which contain it, which works for polynomial rings.

As noted above, the residue field of $A(X)_\mathfrak{p}$ is $K(Z(\mathfrak{p}))$, and unless $Z(\mathfrak{p})$ is a point, this is not algebraically closed. Homework: think about this.

250 Theorem
Let $(A, \mathfrak{m}, k)$ be a regular local ring. Suppose $\mathfrak{p}$ is any prime in A. Then $A_\mathfrak{p}$ is a regular local ring.

251 Remark
It follows that a non-singular affine variety has the property that every localization at a prime is a regular local ring, not just at maximal ideals. As it turns out, a regular local ring is a UFD, so on a non-singular variety, you can always take regular or rational functions and factor them locally.

February 26th, 2016: Draft

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252 Remark
We begin by clarifying the discussion of pre-varieties from last lecture. We'll then discuss "a bunch of stuff that Serre invented."

253 Remark
A pre-variety is a topological space X with a sheaf $\mathcal{O}_X$ and a finite open cover $X = \cup_i U_i$ such that for all i there exists an affine variety $V_i \subset \mathbb{A}^n$ such that U_i is homoeomorphic to V_i and $\mathcal{O}_X|_{V_i}$ corresponds to $\mathcal{O}_{V_i}$ under the homoeomorphism (i.e. using it to translate between open subsets of U_i and V_i).

This is an instance of a ringed space. Formally, it is an object $(X, \mathcal{O}_X)$ where X is a topological space, $\mathcal{O}_X$ is a sheaf of rings on X, and morphisms $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ are given by pairs of morphisms $f\colon X \to Y$ with $\mathcal{O}_Y \to f_*\mathcal{O}_X$. If these sheaves are sheaves of functions, then the morphism of sheaves is of the form $g \mapsto g \circ f$.

254 Definition
Let A be a noetherian ring. We say that A satisfies Serre's R_n condition or that A is R_n if for all prime ideals $\mathfrak{p} \subset A$ such that $\dim A_\mathfrak{p} \leq n$ (i.e. $\mathrm{ht}\,\mathfrak{p} \leq n$), $A_\mathfrak{p}$ is a regular local ring. (One may read "R_n" as "regular in codimension n.")

255 Example
Let $A = A(X)$ for a variety X. Then $\dim A_\mathfrak{p} + \dim A/\mathfrak{p} = \dim A(X)$, i.e. $\dim A_\mathfrak{p}$ is the codimension of $Z(\mathfrak{p})$.

Note that X is R_n if $\mathcal{O}_{X,P}$ is R_n for all $P \in X$. Indeed, this reformulation makes sense even if X is not affine.

256 Proposition
Let X be a variety of dimension d. Suppose $\dim \mathrm{Sing}\, X = t < d$. Then X is R_n for all $n < d - t$.

Proof For any $P \in X$, set $A := \mathcal{O}_{X,P}$. For any prime $\mathfrak{p}$ in A with $\mathrm{ht}\,\mathfrak{p} \leq n$, we must show $A_\mathfrak{p}$ is a regular local ring. We may assume X is affine since this is a local statement. Hence $\mathfrak{p}$ corresponds to a prime ideal $\mathfrak{q}$ in $A(X)$ and P corresponds to a maximal ideal $\mathfrak{m} \coloneqq \mathfrak{m}_{X,\mathfrak{p}}$ in $A(X)$. Hence $A = A(X)_\mathfrak{m}$ and $p = \mathfrak{q}_\mathfrak{m}$ with $\mathfrak{q} \subset \mathfrak{m}$. Now consider $Z(\mathfrak{q}) \subset X$. We have $\operatorname{ht}\mathfrak{p} = \operatorname{ht}\mathfrak{q}$, which can be seen in a variety of ways (either using the usual description of the primes in a localization, or using the "transitivity" of localization).

Now $\operatorname{codim}_X Z(\mathfrak{q}) < d-t = \operatorname{codim}_X \operatorname{Sing} X$, so $Z(\mathfrak{q}) \not\subset \operatorname{Sing} X$ contains a non-singular point $Q \in Z(\mathfrak{q})$. Hence $\mathcal{O}_{X,Q}$ is regular, and $\mathcal{O}_{X,Q} \cong A(X)_{\mathfrak{m}_{X,Q}}$ with $\mathfrak{q} \subset \mathfrak{m}_{X,Q}$. Now $(A(X)_{\mathfrak{m}_{X,Q}})_{\mathfrak{q}_{\mathfrak{m}_{X,Q}}} \cong A(X)_\mathfrak{q}$ is a regular local ring, so $A = \mathcal{O}_{X,P}$ is a further localization of this, which preserves regularity by an earlier remark.

257 Remark

Suppose we have an affine variety X and consider a prime ideal $\mathfrak{p}$ in $A(X)$. This corresponds to picking a subvariety $Z(\mathfrak{p})$. Now $A(X)_\mathfrak{p}$ will be regular if and only if $Z(\mathfrak{p})$ is not contained in $\operatorname{Sing} X$. Since $Z(\mathfrak{p})$ is an irreducible closed subset, $\operatorname{Sing} X \cap Z(\mathfrak{p}) \neq Z(\mathfrak{p})$ must have smaller dimension than $Z(\mathfrak{p})$. Hence $Z(\mathfrak{p})$ is non-singular "almost everywhere."

As an example, take something like a curve which crosses itself; optionally cross it with a line, or take the cone over it. Note that a point may be non-singular in the ambient space but become singular in the subvariety (which occurs in the previous examples).

Note that if $0 \neq f \in A(X)$ in an affine variety X, then if $P \in Z(f) \subset X$ is a non-singular point of $Z(f)$, $P \in X$ is a non-singular point of X.

258 Remark

Our next topic is normal varieties.

259 Definition

Suppose X is a variety, $P \in X$. We say that X is normal at P if $\mathcal{O}_{X,P}$ is integrally closed (in its field of fractions). X is a normal variety if it is normal at P for all $P \in X$.

Recall that a ring A is integrally closed and $A \subset \operatorname{Frac} A$ is integrally closed, meaning for any $f \in \operatorname{Frac} A$, if there exist $a_0, \ldots, a_{n-1} \in A$ such that

$$f^n + a_{n-1} f^{n-1} + \cdots + a_n = 0$$

then $f \in A$.

260 Homework

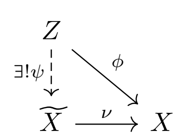
Show that $Z(xy - z^2) \subset \mathbb{A}^3$ is normal. (It's singular at the origin.)

261 Remark

A regular local ring is integrally closed; this may or may not be difficult. Hence a non-singular variety is normal. This is then a weakening of non-singularity. It turns out "normality is a lot more functorial than non-singularity."

262 Theorem

If X is a variety, then there exists a morphism $\nu\colon \widetilde{X} \to X$ with the following universal property. For any normal variety Z and any dominant morphism $\phi\colon Z \to X$, there exists a unique $\psi\colon Z \to \widetilde{X}$ such that $\phi = \nu \circ \psi$.



In particular, ν is unique up to a unique isomorphism. We call $\widetilde{X}$ the normalization of X.

263 Example

Consider $Z(x^2(x+1) - y^2)$. We can remove the singular point and try taking $\widetilde{X}$ to be the result, but this does not help—it's essentially trying to ignore the singular point. The projective morphism constraint is in some sense forcing us to separate points rather than ignore them. If we instead blow up the singularity, everything does work—it's automatically a projective morphism, we have finite fibers, etc. The map can be taken to be the parameterization given by $t \mapsto (t^2 - 1, t(t^2 - 1))$.

Let $C \subset \mathbb{P}^2$ be an arbitrary curve and choose $\mathbb{P}^1 \subset \mathbb{P}^2$ not equal to C. Choose $P \in \mathbb{P}^2 - (C \cup \mathbb{P}^1)$ and project from P, giving $\pi_P : C \to \mathbb{P}^1$. This is a projective morphism with finite fibers, we can choose C to be non-singular so normal. The universal property is forcing a sort of minimality constraint.

Proof Homework–if X is affine, then take the integral closure of $A(X)$ in $\operatorname{Frac} A(X)$. A theorem of Noether says this is a finitely generated k-algebra, so it corresponds to an affine variety $\widetilde{X}$ with coordinate ring $A(\widetilde{X}) \cong \widetilde{A(X)}$. Translate everything to the universal property of integral closures. The homework is to show that this construction can be used to give a pre-variety. (It turns out this is an actual variety, but we don't have the tools to prove this yet.)

February 29th, 2016: Draft

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264 Remark

Last time we discussed the normalization; there was some slight initial confusion, but it was hopefully corrected above. To summarize:

265 Notation

Let X be a variety. Call $P \in X$ $\boxed{\text{normal}}$ if $\mathcal{O}_{X,P}$ is integrally closed (in its field of fractions).

266 Theorem

If X is a variety, then there exists a morphism of varieties $\nu : \widetilde{X} \to X$ with the following universal property. For all normal varieties Z and dominant morphisms $\phi : Z \to X$, there is a unique morphism $\psi : Z \to \widetilde{X}$ such that $\phi = \nu \circ \psi$.

Proof First suppose X is affine. Then $A(X) \subset K(X) = \operatorname{Frac} A(X)$. Let $\widetilde{A}$ be the integral closure of $A(X)$. Then for all integral ring homomorphisms $A(X) \subset B$, we have an inclusion $A(X) \hookrightarrow \widetilde{A} \to B$.

A theorem due to Noether says that $\widetilde{A}$ is a finitely generated $A(X)$-module. Note that $A(X)$ is a finitely generated k-algebra. Hence $A(X)$ is a finitely generated k-algebra and, being a subring of $A(X)$, it is an integral domain. Hence we have some affine variety $\widetilde{X}$ such that $\widetilde{A} \cong A(\widetilde{X})$. Now the inclusion $A(X) \to A(\widetilde{X})$ induces a morphism $\widetilde{X} \to X$. Moreover, given ψ as in the theorem statement, it corresponds to

$$\begin{array}{ccc} A(\widetilde{X}) & \longleftarrow & A(X) \\ \downarrow & \swarrow & \\ \mathcal{O}(Z) & & \end{array}$$

$$\begin{array}{ccc} \widetilde{X} & \longrightarrow & X \\ \uparrow & \nearrow & \\ Z & & \end{array}$$

so the universal property of ν is just a geometric restatement of the universal property of the integral closure.

We next sketch the non-affine case. Let $X = \cup_i U_i$ be an open affine cover. From the first part, we have $\nu_i : \widetilde{U}_i \to U_i$. Now define $\widetilde{X} := \cup_i \widetilde{U}_i$, which requires some explanation. We need to say how to glue these together.

267 Homework

Given a variety and two open affine subvarieties, show that their intersection is affine, in the sense that there is an embedding which makes it closed in an affine variety.

From the homework problem, $U_i \cap U_j$ has a normalization which yields the following diagram:

$$\begin{array}{ccc} \widetilde{U}_{ij} & \longrightarrow & U_i \cap U_j \\ \downarrow & & \downarrow \\ \widetilde{U}_i & \longrightarrow & U_i \end{array}$$

Now glue $\widetilde{U}_i$ and $\widetilde{U}_j$ together and identify points of $\widetilde{U}_{ij}$.

It is relatively easy to see that $\widetilde{X}$ is a pre-variety; further claim: it is actually a variety.

268 Remark

The fibers of the map ν are finite. The proof has some subtlety, so we skip it.

The upshot of this construction is that we may often assume normality. For instance, given a dominant morphism of varieties $X \to Y$, we have an induced square

$$\begin{array}{ccc} \widetilde{X} & \longrightarrow & \widetilde{Y} \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

Moreover, of $X \to Y$ is not itself dominant, we may always factor it as $X \to Z \hookrightarrow Y$.

269 Example

One place where you cannot simply normalize your problems away is when considering families. Roughly, suppose we have a morphism whose codomain is a line and whose fibers are curves. Sadly, in general there is no normalization whose fibers are the normalizations of the original fibers.

For instance, consider the family $(xy = t) \subset \mathbb{A}^2$ over $\mathbb{A}^1$ given by projecting onto t. This family does not have a simultaneous normalization. It is "very rare" that they do.

270 Definition

Let A be a ring, M an A-module. Recall that the annihilator of M is

$$\boxed{\operatorname{Ann} M} := \{a \in A : aM = 0\}.$$

It is an ideal. Similarly, if $x \in M$, we define

$$\operatorname{Ann}(x) := \{a \in A : ax = 0\}.$$

We call $a \in A$ a zero-divisor on M if there exists a non-zero $x \in M$ such that $ax = 0$.

271 Remark

Note that A could be an integral domain but it could still have zero-divisors on some M. Indeed, if $M = A/I$, then all $a \in I$ are zero-divisors on M. Trivially $I \subset \operatorname{Ann} M$, and since $1 \in A$ we have the reverse inclusion as well.

A regular element (or M-regular element) is some $a \in A$ which is not a zero-divisor on M. An element $m \in M$ is a torsion element if there is $a \in A$ which is A-regular (i.e. not a zero-divisor in A) such that $ax = 0$.

A sequence $x_1, \ldots, x_n \in A$ is a regular sequence (or M-regular sequence of length n) if

(i) x_1 is M-regular

(ii) For all i, x_i is $M/(x_1, \ldots, x_{i-1})M$-regular

(iii) $M/(x_1, \ldots, x_n)M \neq 0$.

It's an interesting fact that M-regularity is preserved under permutations of the sequence. One can reinterpret condition (ii) in a symmetric fashion in terms of the solution of a certain equation, though we will not go into details.

Define the dimension of a module as

$$\dim_A M := \dim A/\operatorname{Ann} M.$$

If $A = A(X)$ for X affine, then $Z(\operatorname{Ann} M)$ is called the support of M. Hence $\dim_A M$ is the maximal length of a chain of primes each containing $\operatorname{Ann} M$.

272 Remark

One of the first pages of Miles Reid's Undergraduate Commutative Algebra has a diagram (which we will not reproduce here). The caption is "Let A be a ring and M an A-module." Given a morphism of affine varieties $Y \to X$, we have $A(X) \to A(Y)$. We may call $M := A(Y)$, which is naturally an $A(X)$-module. In this manner, we can get a geometric interpretation of an A-module M. The support of M is a certain closed subset of X in this context.

If M is an A-module, it can be naturally considered an $A/\operatorname{Ann}(M)$-module, in which case M has the same dimension over $A/\operatorname{Ann} M$ as its ring of scalars $A/\operatorname{Ann} M$.

Note that $\dim_{A(X)} A(Y) \neq \dim A(Y)$ in general. The left-hand side is $\dim A(Z)$ where $Z = Z(\operatorname{Ann} M)$ is the support of M.

273 Example

Let $M = A/I^2$. Then $\operatorname{Ann} M = I^2$.

Now suppose $(A, \mathfrak{m})$ is a local ring an M is a non-zero finitely generated A-module. Then the depth of M over A is defined as the maximal length of an M-regular sequence, written $\operatorname{depth}_A M$.

274 Theorem

$\operatorname{depth}_A M \leq \dim_A M$.

In some vague sense, depth is dual to dimension; depth is cohomology, dimension is homology; depth is differentials, dimension is tangent vectors.

March 2nd, 2016: Draft

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275 Remark

Next quarter will include material on subschemes of varieties; divisors; cohomology of sheaves; Riemann-Roch; and possibly others. The next incarnation of this class will be three quarters and more scheme-theoretic material will be reasonable.

276 Remark

Let $(A, \mathfrak{m})$ be a local ring, M a finitely generated A-module (i.e. M is a finite A-module; $k[x, y]$ is a finitely generated k-algebra, but certainly not a finite k-module). Last time we mentioned $\operatorname{depth}_A M \leq \dim_A M := \dim A/\operatorname{Ann} M$.

Recall Serre's property R_n, essentially that a ring is regular in codimension n.

277 Definition

Let A be a noetherian ring, M be a non-empty finitely generated A-module. We say that M satisfies Serre's S_n condition if for all primes $\mathfrak{p}$ in A,

$$\operatorname{depth} M_{\mathfrak{p}} \geq \min(n, \dim_{A_{\mathfrak{p}}} M_{\mathfrak{p}}).$$

278 Remark

For those primes for which $\dim_{A_{\mathfrak{p}}} M_{\mathfrak{p}} \geq n$, we require $\operatorname{depth} M_{\mathfrak{p}} \geq \dim_{A_{\mathfrak{p}}} M_{\mathfrak{p}}$, which forces equality. Otherwise, we require $\operatorname{depth}_{A_{\mathfrak{p}}} M_{\mathfrak{p}} \geq n$. Here $\operatorname{depth} M_{\mathfrak{p}}$ means viewing $M_{\mathfrak{p}}$ as an $A_{\mathfrak{p}}$-module.

279 Definition

Let A be a noetherian ring, M a non-empty finitely generated A-module. We call M Cohen-Macaulay if it is S_n for all n. Equivalently, we require $\operatorname{depth} M_{\mathfrak{p}} = \dim_{A_{\mathfrak{p}}} M_{\mathfrak{p}}$ for all primes $\mathfrak{p}$ in A.

280 Remark

The Cohen-Macaulay condition ends up being very useful, but sometimes it's too much to ask for. Serre's condition S_n essentially says the module is Cohen-Macaulay in codimension n.

If X is a variety, we say that X is S_n at a point $P \in X$ if $\mathcal{O}_{X,P}$ is S_n (as a module over itself), and X is S_n if it is S_n at all $P \in X$. Likewise, we say X is Cohen-Macaulay at a point $P \in X$ if $\mathcal{O}_{X,P}$ is Cohen-Macaulay, and X is Cohen-Macaulay if it is Cohen-Macaulay at all points.

281 Theorem (Serre)

Let X be a variety. Then X is normal if and only if X is R_1 and S_2.

282 Corollary

A surface X is normal if and only if $\dim \operatorname{Sing} X = 0$ and X is Cohen-Macaulay.

283 Remark

Equivalently, a domain is integrally closed if and only if it is R_1 and S_2.

284 Definition

We say a ring A is reduced if it has no nilpotents.

285 Theorem

A is reduced if and only if it is R_0 and S_1.

286 Remark

R_0 means regular in codimension 0, meaning if we localize at minimial primes we get a regular local ring. For example, localizing a domain at 0 gives the field of fractions. S_1 nearly means it contains at least one regular element, i.e. it contains at least one non-zero-divisor which is a non-unit. A dimension zero ring (e.g. a field) can also satisfy the S_1 condition trivially.

In this sense, "reduced" is a weakened version of "integrally closed."

287 Example

Consider $k[x_1, \ldots, x_n]/(x_1, \ldots, x_n)^N$. This typically contains (many) zero-divisors which are non-units. Consider $k[x] \oplus k[y]$ similarly. Explore which S_i and R_j these satisfy.

288 Homework

Try to prove this.

289 Remark

There are several "extension theorems" in complex analysis. Examples include Riemann's theorem on removable singularities; Painleve's (sp?) theorem concerning small Hausdorff measure "holes," and Hartog's theorem concerning codimension two extensions. We have no need to recall these statements precisely; they serve as motivation for the following theorem:

290 Theorem

Let X be a normal variety, $Z \subset X$ a closed subvariety such that the codimension of Z in X is ≥ 2. Let $j: U := X - Z \hookrightarrow X$, which induces $\mathcal{O}_X \to j_* \mathcal{O}_U$. This is an isomorphism. More explicitly, For any open $V \subset X$ and any regular function f on $U \cap V$, f extends to a regular function on V.

291 Remark

More explicitly, recall that

$$(j_* \mathcal{O}_U)(V) = \mathcal{O}_U(j^{-1}V) = \mathcal{O}_X(U \cap V)$$

and that $\mathcal{O}_U = \mathcal{O}_X|_U$, so that we're essentially just considering the restriction map $\mathcal{O}_X(U) \to \mathcal{O}_X(U \cap V)$.

292 Remark

The word "normal" may be replaced by "S_2" in the above statement. This is a stronger result, by Serre's theorem. This property makes being normal very useful, since it means it's regular in codimension 1 and it has this extension property.

If X is normal, then the codimension of $\operatorname{Sing} X$ in X is at least 2, so the theorem says that regular functions on $X_{\text{reg}} := X - \operatorname{Sing} X$ extend to global regular functions. That is, $\mathcal{O}_X \cong j_* \mathcal{O}_{X_{\text{reg}}}$. In this way, one can often prove things for normal varieties by proving them on the regular part and extending.

293 Example

If X is a Cohen-Macaulay variety and $f \in \mathcal{O}_X(X)$, then $Z(f) \subset X$ is also Cohen-Macaulay. Indeed, if X is S_n, then $Z(f)$ is S_{n-1}. To see this, first note all conditions are local, so we may assume X is affine, so we are considering $A(X)_\mathfrak{m}/(f)$. As it happens, in a domain, any non-zero non-unit is part of a maximal-length regular sequence, and for finitely generated k-algebras, any two such sequences have the same length (namely, the depth). Applying this fact to the preceding ring gives the result immediately.

For another example, a regular local ring is Cohen-Macaulay. Hypersurfaces and more generally complete intersections are Cohen-Macaulay. More precisely, an affine variety X is a complete intersection if $I(X) \subset k[x_1, \ldots, x_n]$ is generated by $\operatorname{codim} X$ elements. This is "the number of elements you would expect;" for a hypersurface, it is one element, and we proved that anything in codimension one has one generator, though the analogous statement begins to fail already in codimension two.

294 Example

A standard example of a singular, normal variety is a cone. It is a hypersurface, so it is Cohen-Macaulay. Its singular set is codimension 2, so it is R_1 and S_2, hence normal. This is a nice argument, since otherwise one must show $(k[x,y,z]/(xy - z^2))_{(x,y,z)}$ is integrally closed (which this argument shows).

Another homework is to show $X := \mathbb{A}^2 - (0,0)$ is not affine. Indeed, by the extension theorem, $\mathcal{O}(X) = \mathcal{O}(\mathbb{A}^2) = k[x,y]$ where the "=" is really the restriction map. But if X were affine, then the inclusion $X \hookrightarrow \mathbb{A}^2$ can't induce an isomorphism since it is not an isomorphism, a contradiction. Hence X is not affine.

295 Remark

Next time we'll talk about non-singular curves and desingularization of curves. What we'll do next will be very much like doing schemes while pretending we are not.

March 4th, 2016: Draft

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296 Example

Serre said normal is the same as R_1 and S_2. If $\dim X = 1$, then X is normal if and only if X is non-singular. Hence to get an example of a non-normal variety, we may take any singular curve. More precisely, for all $P \in X$, $\mathcal{O}_{X,P}$ is R_1. By the dimension count, for all primes $\mathfrak{q}$ in $\mathcal{O}_{X,P}$, $\mathfrak{q}$ has height at most 1, so by assumption $(\mathcal{O}_{X,P})_{\mathfrak{q}}$ is a regular local ring. More generally, if X is R_n, then the codimension of $\operatorname{Sing} X$ in X is $> n$. Hence if X is normal, then the codimension of the singular set is at least 2.

297 Aside

Consider $X \coloneqq Z(xz, xt, yz, yt) \subset \mathbb{A}^4$. This is the union of two planes. It is R_1 but not S_2. This surface is not normal and not Cohen-Macaulay, so this is a nice basic test case. Similarly a nice test case is two planes with conics on them arranged so that the conics intersect in two points.

298 Homework

Show that the above surface X is not S_2. Two approaches: try to compute the depth and show the inequality is not satisfied; or, find a point where the local ring is not integrally closed.

And now for something completely different:

299 Definition

Let K be a field, $(G, \leq)$ a totally ordered abelian group. A valuation v with value group G is a map

$$v: K - \{0\} \to G$$

such that

(i) $v(xy) = v(x) + v(y)$

(ii) $v(x + y) \geq \min(v(x), v(y))$

300 Example

Let $K = \mathbb{Q}$, pick $p \in \mathbb{Z}$ prime. Let $v_p(q) \coloneqq n$ where $q = p^n r$ for $n \in \mathbb{Z}$, $r \in \mathbb{Q}$ such that p does not divide the numerator or denominator of r (when written in lowest terms). Hence in $q_1 + q_2$ we can certainly factor out at least $p^{\min(v_p(q_1), v_p(q_2))}$, so (ii) is satisfied.

301 Example

Let $K = k(x)$. Pick $f/g \cdot x^m$ with $f(0) \neq 0, g(0) \neq 0$. Then define $v_x: f/g \cdot x^m \mapsto m$. This example essentially extends the one from the previous example and can be generalized in an obvious way.

If $G = \mathbb{Z}$ with the usual ordering, then v is called a discrete valuation.

302 Homework

Cook up some non-discrete valuations, e.g. using $G = \mathbb{R}, \mathbb{Z} \oplus \mathbb{Z}$. (What partial order?)

303 Definition

Given a valuation, we can define a corresponding subring of K, namely the valuation ring

$$R_v \coloneqq \{x \in K^\times : v(x) \geq 0\} \cup \{0\}.$$

This is a local ring with unique maximal ideal

$$\mathfrak{m}_v \coloneqq \{x \in K^\times : v(x) > 0\} \cup \{0\}.$$

304 Remark

To see this, note $v(1) = 0$ from (i), so

$$0 = v(x \cdot 1/x) = v(x) + v(1/x)$$

implies $v(1/x) = -v(x)$. Hence $x \in R_v - \mathfrak{m}_v$ implies $1/x \in R_v - \mathfrak{m}_v$, so $\mathfrak{m}_v$ is an ideal in which everything outside of it is a unit, which is thus the unique maximal ideal.

Sometimes one replaces G with $G \coprod \{\infty\}$, in which case we require $v(0) := \infty$.

In general, a valuation ring is a ring $R \subset K$ such that there exists a valuation $v: K^\times \to G$ such that $R = R_v$. Note that v is not part of the data of a valuation ring; many v's may work. If we can find v discrete, we call R a discrete valuation ring or DVR.

If $k \subset K$ is a field extension and v is such that $v|_k \equiv 0$, we say v is a valuation of K/k and R_v is a valuation ring of K/k.

305 Theorem

Let $(R, \mathfrak{m})$ be a local noetherian domain of dimension 1. The following are equivalent:

(i) R is a discrete valuation ring

(ii) R is integrally closed

(iii) R is a regular local ring

(iv) $\mathfrak{m}$ is a principal ideal

(v) R is a PID

(vi) R is a UFD

Proof The equivalences (ii)-(iv) have essentially been done by our recent discussion of normality and regularity. Proving (i) is equivalent to these is left as an exercise for the enthusiastic student. (v)-(vi) are also left as exercises; they are mostly consequences of the observation that there exists $t \in R$ such that for all $x \in R$, $x = ut^{v(x)}$ for a unit $u \in R$; here t is a generator for $\mathfrak{m}$.

306 Definition

Suppose $(A, \mathfrak{m})$, $(B, \mathfrak{n})$ are local rings such that $A \subset B$ and $\mathfrak{m} = A \cap \mathfrak{n}$. In this case, we say B dominates A.

307 Remark

Note that only $\mathfrak{m} \supset A \cap \mathfrak{n}$ is guaranteed; e.g. if B is the fraction field of A, then $\mathfrak{n} = 0$. Equivalently, units in A must be units in B, so $A - \mathfrak{m} \subset B - \mathfrak{n}$.

308 Theorem

Suppose K is a field, $R \subset K$ is a local ring. R is a valuation ring of K if and only if R is maximal among the local rings in K which are dominated by R. Furthermore, every local ring in K is dominated by some valuation ring in K.

309 Remark

For instance, if we have two valuation rings, one of which dominates the other, they are equal.

March 7th, 2016: Draft

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310 Lemma

Suppose X is a quasi-projective variety. Suppose $P, Q \in X$. Recall that $\mathcal{O}_{X,P}, \mathcal{O}_{X,Q} \subset K(X)$ in a "canonical" way. Then $\mathcal{O}_{X,P} \supset \mathcal{O}_{X,Q}$ implies $P = Q$.

311 Remark

Let $(A, \mathfrak{m})$, $(B, \mathfrak{n})$ be local rings. We say that B dominates A if $B \supset A$ and $\mathfrak{n} \cap A = \mathfrak{m}$. We do *not* assume $\mathcal{O}_{X,P}$ dominates $\mathcal{O}_{X,Q}$, though it is (trivially) true. In general, $B \supset A$ implies $B - \mathfrak{n} \supset A - \mathfrak{m}$ since units in A are certainly units in B.

PROOF Since the statement is local, we may assume $X \subset \mathbb{P}^n$ is projective (by possibly replacing it with its projective closure). We may assume $H_0 := Z(x_0)$ is a hyperplane such that $P, Q \notin H_0$. Now $P, Q \in X - H_0 \subset \mathbb{P}^n - H_0 \cong \mathbb{A}^n$. Then we may assume X is affine, so the points correspond to maximal ideals $\mathfrak{m} := \mathfrak{m}_P, \mathfrak{n} := \mathfrak{m}_Q$ in $A(X)$. The assumption says

$$A(X)_{\mathfrak{n}} \subset A(X)_{\mathfrak{m}} \subset \operatorname{Frac} A(X)$$

using the "natural" embeddings arising from localization. By locality

$$A(X)_{\mathfrak{n}} - \mathfrak{n}A(X)_{\mathfrak{n}} \subset A(X)_{\mathfrak{m}} - \mathfrak{m}A(X)_{\mathfrak{m}}$$

and

$$(A(X)_{\mathfrak{n}} \cap \mathfrak{m}A(X)_{\mathfrak{m}}) \cap A(X) \subset \mathfrak{n}A(X)_{\mathfrak{n}} \cap A(X)$$

so

$$\mathfrak{m} \subset \mathfrak{n}$$

which by maximality implies $\mathfrak{m} = \mathfrak{n}$.

312 Notation

Let K be a finitely generated extension of k (with $k = \overline{k}$) of transcendence degree 1. We can take this as the definition of the phrase **function field of dimension 1**.

313 Definition

With K as above, define

$$\boxed{C_K} := \{R : R \text{ is a DVR of } K/k\}.$$

Our next task is to give this set the structure of a curve whose function field is K. We may think of elements of C_K as points $P \in C_K$ or as rings $R_P \in C_K$.

To topologize C_K, we must use the cofinite topology. If X is a non-singular curve and $K = K(X)$, then in general we have a map of sets $X \to C_K$ given by $P \mapsto \mathcal{O}_{X,P}$. This is well-defined by the theorem from last time that regular local noetherian domains of dimension 1 are DVR's. The lemma from the start of class says that this is an injective map. In particular, C_K is infinite. (This map of sets is mapping into the power set of $K(X)$, i.e. we are implicitly using the "canonical" embeddings of the local rings $\mathcal{O}_{X,P}$.)

314 Definition

An integrally closed noetherian integral domain of dimension 1 is called a **Dedekind domain**.

315 Corollary

Suppose A is an integral domain. A is a Dedekind domain if and only if $A_{\mathfrak{m}}$ is a DVR for every maximal ideal $\mathfrak{m} \subset A$.

316 Theorem (Krull-Akizuki)

The integral closure of a Dedekind domain in a finite extension field of its fraction field is also a Dedekind domain.

317 Remark

Indeed, the fraction field of the integral closure is the finite extension field, which follows by showing that an algebraic element has a multiple (over the base ring) which is integral.

318 Lemma

For $f \in K$, the set $\{R_P \in C_K : f \notin R_P\}$ is finite.

PROOF Recall that if $x \in K \supset R$ where R is a valuation ring, then $x \notin R \Leftrightarrow 1/x \in \mathfrak{m}_R$. The above set is then $\{R_P \in C_K : 1/f \in \mathfrak{m}_P\}$. (We may assume $f \neq 0$, since if $f = 0$ the set is obviously empty.) Write $y := 1/f$. If $y \in k$, then $v(y) = 0$ and we again get an empty set. Now take $y \notin k$. Consider $k[y] \subset K$. Since $k = \overline{k}$, y is transcendental over k. Then $k \subset k(y) \subset K$ where $k \subset k(y)$ has transcendence degree 1 and $k(y) \subset K$ must be an algebraic extension, hence finite since it's finitely generated over k.

Now take B to be the integral closure of $k[y]$ in K. The Krull-Akizuki theorem says that B is a Dedekind domain. By Noether's theorem, B is a finitely generated k-algebra (indeed, a finite $k[y]$-module). Hence, there exists an affine variety X such that $B \cong A(X)$.

Suppose $R \in C_K$ with $y \in R$. Then $k[y] \subset R$, and R is integrally closed, so $B \subset R$. Set $\mathfrak{n} := B \cap \mathfrak{m}_R$. Since B is a Dedekind domain, $\dim B = 1$, and as usual $\mathfrak{n}$ is prime. Hence $\mathfrak{n} = 0$ or $\mathfrak{n}$ is maximal. In the former case, we would have $\operatorname{Frac} B \subset R$. By the second remark in Krull-Akizuki, $\operatorname{Frac} B = K$ says $K \subset R \subset K$, which is a contradiction.

Hence $\mathfrak{n}$ is a maximal ideal in B. Thus there exists $P \in X$ such that $R \cong \mathcal{O}_{X,P}$. Moreover, X is a non-singular curve. We've now shown that every ring in C_K is a local ring for some such X with $K = K(X)$. We will finish the argument next time.

319 Remark

Attempting to carry out this construction in higher dimensions runs into fundamental problems immediately. For one, the topology is no longer forced. For another, given a variety of dimension at least two, blow it up at a point; the resulting function fields are the same, so they are birational, but the blow up is not isomorphic to the original variety.

March 28th, 2016: Integrality, Finite Morphisms, and Finite Fibers

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320 Remark

Today is the start of the second quarter. Hurray!

321 Notation

For this lecture, let $A \hookrightarrow B$ be an integral extension of rings, i.e. every element of B is the root of a monic polynomial with coefficients in (the image of) A.

322 Lemma

Suppose I is a proper ideal of A. Then $IB \neq B$.

PROOF It clearly suffices to consider $I := \mathfrak{m}$ maximal. Then we certainly want to apply Nakayama's lemma. To do so, first localize at $\mathfrak{m}$, so replace A with $A_\mathfrak{m}$ and B with $B_\mathfrak{m}$, or equivalently assume A is a local ring. Now $A \hookrightarrow B$ need not make B a finitely generated A-module. However, if $\mathfrak{m}B = B$, then $1 = \sum_i a_i b_i$ for $a_i \in \mathfrak{m}$, $b_i \in B$. Set $B' := A[b_1, \ldots, b_r] \subset B$. Since each b_i is integral over A, it follows that B' is a finitely generated A-module. Moreover, $\mathfrak{m}B' = B'$ since $\mathfrak{m}B'$ contains 1. A standard version of Nakayama's lemma says that in this situation $B' = 0$, so $1 = 0$, a contradiction.

323 Theorem (Going Up)

Let $\mathfrak{p}$ be a prime ideal in A and J be an ideal in B such that $J \cap A \subset \mathfrak{p}$. Then there exists a prime $\mathfrak{q}$ in B such that *(i)* $J \subset \mathfrak{q}$

(ii) $\mathfrak{q} \cap A = \mathfrak{p}$

PROOF We have $A/J \cap A \hookrightarrow B/J$, or equivalently we may assume $J = 0$. Now localize at $\mathfrak{p}$ to get $A_\mathfrak{p} \hookrightarrow B_\mathfrak{p}$, so $\mathfrak{p}A_\mathfrak{p}$ is a maximal ideal. By the lemma, $\mathfrak{p}B_\mathfrak{p} \neq B_\mathfrak{p}$ is proper, so it is contained in some maximal ideal $\mathfrak{m}$, so $\mathfrak{m} \cap A_\mathfrak{p} \supset \mathfrak{p}A_\mathfrak{p}$. By locality, $\mathfrak{m} \cap A_\mathfrak{p} = \mathfrak{p}A_\mathfrak{p}$, and the result follows.

324 Lemma
Assume additionally that B is a domain. Let $0 \neq J$ be an ideal in B. Then $J \cap A \neq 0$.

PROOF Pick $0 \neq b \in J$, which by assumption is integral over A. Hence there exists some $a_i \in A$ such that
$$b^m + a_{m-1}b^{m-1} + \cdots + a_0 = 0 \in B.$$
We may divide off enough copies of b to assume $a_0 \neq 0$. But then $a_0 \in J \cap A$ since the rest of the terms are in J.

325 Corollary
Assume additionally that B is a domain. Then A is a field if and only if B is a field.

PROOF Given a non-zero ideal J in B, then $J \cap A \neq 0$. If A is a field, then $J \cap A$ contains a unit in A, hence B, so $J = B$ and B is a field. On the other hand, if B is a field, let $\mathfrak{m}$ be a maximal ideal in A. By the Going Up theorem, there exists a prime ideal $\mathfrak{q}$ in B such that $\mathfrak{q} \cap A = \mathfrak{m}$. Since B is a field, $\mathfrak{q} = 0$, so $\mathfrak{m} = 0$ and A is a field.

326 Corollary
Let $\mathfrak{p}$ be a prime ideal in A and let $\mathfrak{q}$ be a prime ideal in B such that $\mathfrak{p} = \mathfrak{q} \cap A$. Then $\mathfrak{p}$ is maximal if and only if $\mathfrak{q}$ is maximal.

PROOF We have $A/\mathfrak{p} \hookrightarrow B/\mathfrak{q}$, which is an integral extension of domains. Now used the preceding corollary.

327 Corollary
Let $\mathfrak{q} \subset \mathfrak{q}'$ be primes in B such that $\mathfrak{q} \cap A = \mathfrak{q}' \cap A$. Then $\mathfrak{q} = \mathfrak{q}'$.

PROOF Consider $A/\mathfrak{q} \cap A \hookrightarrow B/\mathfrak{q}$, or equivalently assume $\mathfrak{q} = 0$ and $\mathfrak{q}' \cap A = 0$. Non-trivial ideals in $B/\mathfrak{q}$ intersect non-trivially with $A/\mathfrak{q} \cap A$, so $\mathfrak{q}' = 0$, meaning $\mathfrak{q} = \mathfrak{q}'$.

328 Corollary
$\dim A = \dim B$.

PROOF Given a chain of primes in B
$$\mathfrak{q}_0 \subsetneq \cdots \subsetneq \mathfrak{q}_r \subset B$$
we have a chain of primes in A
$$\mathfrak{q}_0 \cap A \subsetneq \cdots \subsetneq \mathfrak{q}_r \cap A \subset A.$$
Hence $\dim A \geq \dim B$. On the other hand, given a chain of primes in A
$$\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_r \subset A,$$
by the Going Up theorem we have $\mathfrak{q}_0$ prime in B such that $\mathfrak{p}_0 = \mathfrak{q}_0 \cap A \subset \mathfrak{p}_1$. Hence there exists $\mathfrak{q}_1$ prime in B such that $\mathfrak{p}_1 = \mathfrak{q}_1 \cap A \subset \mathfrak{p}_2$. It follows that $\dim A \leq \dim B$. (Of course, they could both be infinite dimensional.)

329 Corollary
Assume additionally that A and B are noetherian. Let $\mathfrak{p}$ be prime in A. Then
$$|\{\mathfrak{q} \text{ prime in } B : \mathfrak{q} \cap A = \mathfrak{p}\}| < \infty.$$

PROOF First set $J := \mathfrak{p}B$, so $J \subset \mathfrak{q}$ for any such $\mathfrak{q}$. The Going Up theorem says such a $\mathfrak{q}$ exists, so $\mathfrak{p} \subset J \cap A \subset \mathfrak{q} \cap A = \mathfrak{p}$, and $\mathfrak{p} = J \cap A$. Hence $A/\mathfrak{p} \hookrightarrow B/J$, which may be localized at $\mathfrak{p}$ to give $\mathrm{Frac}(A/\mathfrak{p}) \hookrightarrow (B/J)_{\mathfrak{p}}$. By the preceding corollary, if B/J were a domain, then $(B/J)_{\mathfrak{p}}$ would be a field, so there would be a unique maximal/prime ideal, and the set in question has size 1.

Without assuming B/J is a domain, we still have that $(B/J)_{\mathfrak{p}}$ is a zero dimensional Noetherian ring, which is hence Artinian. Such a ring is a product of finitely many Artin local rings, and it follows that it has only finitely many prime ideals. The result again follows.

330 Remark

What is the geometric meaning of the preceding discussion? First consider the Artinian argument at the end of the preceding proof. A field is Artinian and corresponds to a point. Given two points, there is a field for each, and the coordinate ring is the sum of those fields. To get more complicated Artinian rings one would need to allow nilpotents, e.g. $k[x]/(x^m)$ which is an Artinian ring and as a scheme is a "fuzzy point." Geometrically, the statement that an Artinian ring is a product of finitely many Artin local rings is saying it is a disjoint union of "fuzzy points" with the "fuzz" being caused by nilpotents.

331 Definition

Now consider a morphism of affine varieties $\phi\colon X \to Y$. Equivalently, this is a ring homomorphism $\phi^*\colon A(Y) \to A(X)$. Suppose ϕ^* is an injection, which means that the image of ϕ is dense in Y.

Say that ϕ is a finite morphism if ϕ^* is an injective integral extension.

332 Lemma

If $\phi\colon X \to Y$ is a finite morphism of affine varieties, then for every $Q \in Y$, $\phi^{-1}(Q)$ is finite.

PROOF Consider $X \subset \mathbb{A}^n$ as a closed subset. The coordinates on $\mathbb{A}^n$ have images in $A(X)$, say $\overline{x}_i \in A(X)$. For every i, we have some $b_j \in A(Y)$ such that

$$\overline{x}_i^m + b_{m-1}\overline{x}_i^{m-1} + \cdots + b_0 = 0 \in A(X).$$

Hence

$$(x_i(P))^m + b_{m-1}(Q)(x_i(P))^{m-1} + \cdots + b_0(Q) = 0 \in k.$$

At each coordinate, we thus have finitely many solutions $x_i(P)$ to a fixed polynomial with coefficients in a field, so the preimage of Q is indeed finite.

The preceding proof is the "classical one." We can instead leverage our more abstract reasoning to give a slicker proof. It's actually a more powerful proof in the sense that it works for arbitrary affine schemes, not just those over an algebraically closed field.

First an observation. Suppose $\phi(P) = Q$ with $P \in X$, $Q \in Y$. Then $\phi^*\mathfrak{m}_Q \subset \mathfrak{m}_P$. (Geometrically, this is saying that if we evaluate at Q and get zero, since ϕ^* operates by pre-composing with ϕ, we'll certainly get zero after evaluating at P.) That is, $\mathfrak{m}_Q = \mathfrak{m}_P \cap A(Y)$. We claim $\phi(P) = Q$ if and only if $\mathfrak{m}_Q = \mathfrak{m}_P \cap A(Y)$. We've just done the forward direction. For the backwards direction, let $R := \phi(P)$, so by the forward direction, $\mathfrak{m}_R = \mathfrak{m}_P \cap A(Y)$, so since $\mathfrak{m}_Q = \mathfrak{m}_P \cap A(Y)$, we have $\mathfrak{m}_Q = \mathfrak{m}_R$, so $Q = R$. Hence the finiteness corollary above gives the result.

March 30th, 2016: Draft

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333 Theorem

Let $\phi\colon X \to Y$ be a finite morphism of affine varieties. Then:

(i) $\dim X = \dim Y$

(ii) ϕ is surjective

(iii) ϕ is closed

PROOF By definition, $\phi^*: A(Y) \hookrightarrow A(X)$ is an integral extension. We showed that the dimension of these rings are equal, so the same is true of the corresponding dimensions of varieties. Surjectivity is essentially the going up theorem. That is, for $Q \in Y$, $\phi^{-1}(Q)$ is in one-to-one correspondence with maximal ideals $\mathfrak{m}$ of $A(X)$ such that $\mathfrak{m} \cap A(Y) = \mathfrak{m}_Q$. The Going Up theorem guarantees this set is non-empty, as required.

Finally, suppose $Z \subset X$ is closed, so here Z is affine with corresponding ideal $I_{Z\subset X}$, say. Set $W := \overline{\phi(Z)} \subset Z$, which is also affine with ideal $I_{W\subset Y}$. Now $A(W) = A(Y)/I_{W\subset Y}$ and $A(Z) = A(X)/I_{Z\subset X}$. Now $I_{W\subset Y} = I_{Z\subset X} \cap A(Y)$, which gives us an induced injection $A(W) \hookrightarrow A(Z)$, which remains integral. But then $\phi|_Z: Z \to W$ is finite, hence surjective, as required.

334 Remark

This is roughly the algebraic version of a branched covering. For instance, $\mathbb{A}^1 \to \mathbb{A}^1$ given by $t \mapsto t^n$ is n-to-1 except at the origin.

335 Theorem (Noether normalization)

Let K be a field (not necessarily algebraically closed), and suppose A is a finitely generated K-algebra. Then there exists algebraically independent elements $x_1, \ldots, x_d \in A$ over K and $K[x_1, \ldots, x_d] \subset A$ is an integral extension.

PROOF The existence of $x_1, \ldots, x_d$ is straightforward and very general, where d is the transcendence degree of A. That the resulting extension is integral is the real content, but we do not prove it. (Indeed, this is a standard way to prove that the transcendence degree of a finitely generated K-algebra agrees with its Krull dimension.)

336 Corollary

Let X be an affine variety. Then there exists $\phi: X \to \mathbb{A}^d$ finite.

337 Notation

For the rest of today's lecture, $\phi: X \to Y$ is a morphism of varieties. In particular, we do *not* require X or Y to be affine.

338 Definition

We call ϕ an affine morphism if for all $Q \in Y$, there exists an affine open $Q \in V \subset Y$ such that $\phi^{-1}(V) \subset X$ is an affine open in X.

339 Example

A morphism between affine varieties is affine. The identity morphism is affine for any X, affine or not. More generally, inclusions of affine subvarieties are affine.

Slightly less trivially, $\mathbb{P}^1 \to \mathbb{P}^1$ given by $[u : t] \mapsto u^n : t^n]$ is affine $(n > 1)$. More generally, any non-constant morphism between two curves is affine, essentially since leaving out any single point gives an affine subvariety.

For another example, projection $X \times \mathbb{A}^n \to X$ is an affine morphism, and $\mathbb{A}^n$ can be replaced by any affine variety. In the same vein, structure morphisms for vector bundles are affine.

On the other hand, blow-ups are typically not affine. Roughly this is because the pre-image of the blown up point will contain a positive dimensional projective variety. Indeed, the fibers of an affine morphism are affine, since they are a closed subset of the preimage of the affine near a point, which is affine.

340 Definition

ϕ is finite if for all $Q \in Y$ there exists an open affine neighborhood $V \subset Y$ such that $\phi^{-1}(V) \subset X$ is affine and where $\phi^{-1}(V) \to V$ is a finite morphism of affine varieties.

341 Remark
Our earlier definition of finiteness required the domain and codomain to be affine.

342 Homework
Show the following:

1. Suppose ϕ is an affine morphism. Show that for every open affine subset $V \subset Y$, $\phi^{-1}(V)$ is an open affine in X.

2. If ϕ is finite, then for any open affine $V \subset Y$, the induced map $\phi^{-1}(V) \to V$ is a finite morphism of affine varieties.

343 Theorem
Suppose $\phi: X \to Y$ is a finite morphism. Then $\dim X = \dim Y$ and ϕ is closed and surjective.

Proof All of these conditions are invariant under taking open subsets of Y and covering X with their preimages. The details are left to the reader. Being closed is perhaps slightly subtle; if a set is closed in each element of an open cover, it is literally closed, which is stronger than the notion of being locally closed.

344 Theorem
Let $\phi: X \to Y$ be a dominant morphism (i.e. $\phi(X)$ is dense in Y). Then $\phi(X)$ contains a non-empty open subset of Y.

Proof We will prove this next time.

345 Example
Consider embedding the complement of a punctured line in $\mathbb{A}^2$ into $\mathbb{A}^2$, which intuitively is dominant though the image is not open. However, this complement is not a variety, so we would need some more general constructions to properly handle this example. On the other hand, consider the morphism $\mathbb{A}^2 \to \mathbb{A}^2$ given by $(x, y) \mapsto (x, xy)$. The image of this morphism is the complement of a punctured line.

We will later prove Chevalley's theorem which says $\phi(X)$ is a finite union of locally closed sets.

April 1st, 2016: Draft

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346 Remark
We will begin by proving the theorem from the end of last lecture, namely that the image of a dominant morphism contains a non-empty open set.

Proof We first reduce to the case when X, Y are both affine. Let $V \subset Y$ be a non-empty open affine. Then $\phi^{-1}(V)$ is non-empty and open, so we have some $U \subset \phi^{-1}(V)$ non-empty, open, and affine. By replacing ϕ with $\phi|_U: U \to V$, it suffices to suppose X, Y are affine. Note that $\phi|_U^*: A(V) \hookrightarrow A(U)$ as a consequence of dominance.

Hence, consider $A(Y) \hookrightarrow A(X)$. Localize at $S := A(Y) - \{0\}$ to get $K(Y) \hookrightarrow S^{-1}A(X)$. Note that $A(X)$ is a finitely generated k-algebra, where $k \subset A(Y)$, so $A(X)$ is a finitely generated $A(Y)$-algebra, which is preserved under this localization. By Noether normalization,

$$K(Y)[x_1, \ldots, x_d] \subset S^{-1}A(X)$$

is an integral extension. By clearing denominators, we may assume $x_i \in A(X)$. Now consider

$$A(Y)[x_1, \ldots, x_d] \subset A(X).$$

This is isomorphic to $A(Y \times \mathbb{A}^d)$, which is straightforward to justify either by embedding Y in $\mathbb{A}^n$ or by showing/recalling $A(B \times C) \cong A(B) \otimes_k A(C)$. Hence we have a morphism $\psi: X \to Y \times \mathbb{A}^d$. Set $B := A(Y)[x_1, \ldots, x_d]$ and note that $A(X) = B[f_1, \ldots, f_r]$ for some (not necessarily algebraically independent) f_i. Now for each f_i, by integrality we have

$$f_i^m + a_{i,m-1} f^{m-1} + \cdots + a_{i,0} = 0$$

where $a_{i,j} \in K(Y)[x_1, \ldots, x_d]$. We may clear denominators on these coefficients simultaneously using some $g \in A(Y)$ such that $g a_{ij} \in B$. It follows that $B_g \subset A(X)_g$ is an integral extension. Now set $V := D(g) \subset Y$, so $\varnothing \neq V$ is an (affine) open. Then consider

$$\psi^{-1}(V \times \mathbb{A}^d) \xrightarrow{\psi} V \times \mathbb{A}^d$$

which is finite, hence surjective. Post-composing this with the surjective projection map $Y \times \mathbb{A}^d \to Y$ gives the (restriction of) the original morphism $X \to Y$ which surjects onto V, completing the proof.

347 Remark

The preceding theorem does not say that any dense set in a variety contains an open set; the dense set must be the image of a morphism.

Our next goal is Chevalley's theorem concerning constructible sets.

348 Definition

Let $\boxed{\mathcal{C}(X)}$ denote the set of subsets of a topological space X that satisfy the following conditions:

(i) Every open set is in $\mathcal{C}(X)$

(ii) Every finite intersectino of elements of $\mathcal{C}(X)$ is in $\mathcal{C}(X)$

(iii) $\mathcal{C}(X)$ is closed under complemenets.

A $\boxed{\text{constructible set}}$ of X is an element of $\mathcal{C}(X)$. Note that complements and finite intersections can be used to give finite unions.

349 Homework

(1) Recall the definition of locally closed. (E.g. for any point, there is a neighborhood in which the set is closed.)

(2) Show that any constructible set is the finite union of locally closed subsets.

350 Remark

Recall our morphism $\mathbb{A}^2 \to \mathbb{A}^2$ given by $(x, y) \to (x, xy)$, whose image is the complement of a punctured line. Hence the image of this morphism is not locally closed. One may then ask for a morphism whose image is not even constructible. This is not possible, as the next theorem shows.

351 Theorem (Chevalley)

Suppose $\phi: X \to Y$ is a morphism. Then $\phi(X)$ is constructible.

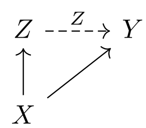
Proof Use induction on $\dim Y$. Take $\dim Y > 0$. We may assume $\overline{\phi(X)} = Y$. Let V be a non-empty open subset of $\phi(X)$ using the theorem above and consider $Z := \overline{\phi(X)} - V$ which is a proper subset of Y. Now set $W_i := \phi^{-1}(Z_i) =: \cup_{j=1}^q W_{ij}$, giving maps $\phi_{ij}: W_{ij} \to Z_i$. By induction, $\phi_{ij}(W_{ij})$ is constructible. Hence

$$\phi(X) = \cup_{i,j} \phi_{ij}(W_{ij}) \cup V$$

is constructible.

352 Lemma

Suppose $\psi\colon Z \dashrightarrow Y$ is a rational map. Then there exists X such that



where $X \to Z$ is birational and projective, and $X \to Y$ is a morphism.

Proof Consider $\Gamma_\psi\colon Z \times Y$. Then $\Gamma_\psi \to Z$ is a birational morphism, and $\Gamma_\psi \to Y$ is a morphism. Using $X \coloneqq \Gamma_\psi$, the diagram above commutes. For projectivity, we may as well assume Y is $\mathbb{P}^n$, from which it follows that $\Gamma_\psi \to Z$ is projective by definition.

353 Lemma

If $\phi\colon X \to Y$ is a projective morphism and $V \subset Y$ is a non-empty open set, then $\phi^{-1}(V) \to V$ is also projective.

Proof By definition, $X \to Y$ factors as $X \to Y \times \mathbb{P}^n \to Y$, so $\phi^{-1}(V) \to V$ factors as $\phi^{-1}(V) \to V \times \mathbb{P}^n \to V$.

(More generally, projective morphisms are invariant under base change.)

354 Remark

Next time we'll again consider "pre-varieties" as defined above, so review them before Monday.

April 4th, 2016: Draft

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355 Lemma

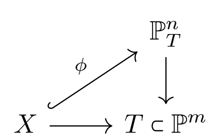
Suppose $X \subset \mathbb{P}^n$ is a projective variety and $\phi\colon X \to T$ is a morphism. Then ϕ is projective.

Proof By definition, this is saying ϕ factors through the structure morphism $\mathbb{P}^n \times T \to T =: \mathbb{P}^n_T$ as a closed embedding. Here the relevant map $X \to \mathbb{P}^n_T$ is just the graph, $x \mapsto (x, \phi(x))$, which is in fact a closed embedding.

356 Lemma

If $\phi\colon X \to T$ is a projective morphism and T is projective, then X is projective.

Proof We have



Now the diagonal arrow is a closed embedding into $\mathbb{P}^n \times \mathbb{P}^m \subset \mathbb{P}^N$, and $T \subset \mathbb{P}^m$ is a closed embedding.

357 Remark

Note that under our definition, X is projective if and only if $X \to *$ is projective, where $*$ is the variety of the coordinate ring of the base field.

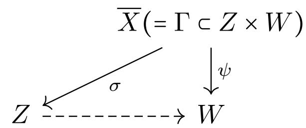
The second lemma could be rephrased (and generalized slightly) as saying the composition of two projective morphisms is projective. The first lemma similarly means that if $\psi \circ \phi$ and ψ are projective, then ϕ is projective.

358 Lemma

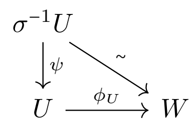
Let $\phi_U : U \to Y$ be a morphism of varieties. Then there exists a projective morphism $\phi : X \to Y$ and an open embedding $U \hookrightarrow X$ such that $\phi|_U = \phi_U$.

(Note: it is not true for schemes that there exists a projective closure, so this lemma is somewhat specific to varieties.)

Proof Set $Z := \overline{U} \subset \mathbb{P}^n$, $W := \overline{Y} \subset \mathbb{P}^m$. From last lecture, the graph yields a diagram



where $\overline{X} \to Z$ is a projective birational morphism. By Lemma 2, $\overline{X}$ is projective. By Lemma 1, $\overline{X} \to W$ is projective. Restricting to the subset U on which the rational map is defined we have



Now $X := \psi^{-1} Y$. Set $\phi := \psi|_X$, so ϕ is projective. Under the embedding $U \cong \sigma^{-1} U \subset X$, we then have $\phi|_U = \phi_U$, as claimed.

359 Lemma

Let $\phi : X \to Y$ is a dominant morphism with finite fibers. Then $\dim X = \dim Y$.

360 Remark

We already showed this works for finite morphisms, but the given assumption is a bit weaker.

Proof We may assume X, Y are affine by taking non-empty open sets and noting the dimension is preserved. Hence we are considering $A(Y) \hookrightarrow A(X)$. Repeating part of the proof of Chevalley's theorem, localizing at $0 \subset A(Y)$ ends up showing the morphism factors through a dominant morphism $\alpha : X \to Y \times \mathbb{A}^d$ where d is the transcendence degree of $K(Y)$. Now by Chevalley's theorem $\alpha(X)$ contains a non-empty open subset of $Y \times \mathbb{A}^d$, which includes some $\mathbb{A}^d$ entirely, but if $d > 0$ then infinitely many points under the composite are sent to the same point of Y, contrary to our assumption. Thus $d = 0$.

We had used Noether normalization to cook up an integral map from a localization of $A(Y)$ to one of $A(X)$, which equivalently says that there is a (dense) open set on which ϕ is finite, so the dimensions indeed agree.

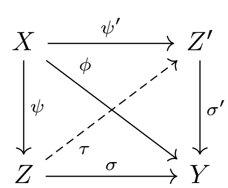
Homework: rewrite and think about this argument.

361 Remark

We'll next define relative normalization, following Grothendieck's philosophy that notions should be relative, i.e. depend on morphisms in a category rather than just objects.

362 Definition

Given $\phi : X \to Y$, the normalization of Y in X is a finite morphism $\sigma : Z \to Y$ with $\psi : X \to Z$ such that $\phi = \sigma \circ \psi$ and for all σ', ψ' as above, there exists a unique $\tau : Z \to Z'$ such that



i.e. $\sigma = \sigma' \circ \tau$, $\psi' = \tau \circ \psi$.

363 Theorem

Let $\phi: X \to Y$ be a morphism of varieties. Then the normalization of Y in X exists and it commutes with locally closed embeddings.

364 Remark

The "commutes" statement means the following. Factoring $X \to Y$ as $X \to Z \overset{\sigma}{\to} Y$ and given a locally closed V in Y where we factor $\phi^{-1}V \to V$ as $\phi^{-1}V \to \sigma^{-1}V \to V$, (something).

Note that if ϕ is finite, we must choose $\sigma = \phi$. Essentially, the normalization is the maximal finite morphism through which the original morphism factors. τ above is the normalization of Z' in X, so in particular it is finite.

We will not prove the theorem at present since it's much easier when the correct machinery has been set up.

365 Example

Consider an "S" projecting down to a curve, which is a finite morphism. Imagine "fattening" the S by adding two extra dimensions to make it 3D and mapping it to a curve by first projecting it onto the unfattened S. The normalization of the fattened S over the curve is the map from the S to the curve.

Given a curve which crosses itself finitely many times, consider a "spread out" version of the curve which does not cross itself. Project each curve to the axis. By the universal property, we have τ from the non-crossing curve to the crossing curve.

April 6th, 2016: Draft

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366 Remark

The terminology in the literature for the precise meaning of "finite" is a bit inconsistent inasmuch as it may or may not include "dominant," and it may mean "dominant over its image" in our terminology.

367 Definition (Stein Factorization)

Let $\phi: X \to Y$ be a dominant projective morphism and let $\sigma: Z \to Y$ be the normalization of Y in X, so we have $X \to Y = X \to Z \to Y$ for some $\psi: X \to Z$. Then ψ is dominant and has connected fibers.

368 Remark

The usual statement is that if ϕ is a projective morphism, then there exists $\sigma: Z \to Y$ finite and $\psi: X \to Z$ where ψ has connected fibers and $\phi = \psi \circ \sigma$.

As mentioned last time, we could have X a line and Y a curve with a single cusp, with $\phi: X \to Y$ naturally. Then we can "factor" this as $X \to X \to Y$ or $X \to Y \to Y$ where $X \to X$ and $Y \to Y$ are identity maps. The normalization is roughly maximal with respect to the finite morphism.

We will not take the time to prove this.

369 Lemma

Let $\psi: X \to Z$ be a bijective projective morphism. Then ψ is finite.

370 Remark

We may prove this later if we have time. One might expect bijections to be isomorphisms, which is a little too strong.

371 Corollary

Let $\phi: X \to Y$ be a dominant projective morphism with finite fibers. Then ϕ is a finite morphism.

Proof Sketch. We have

$$\begin{array}{ccc} X & \xrightarrow{\phi} & Y \\ & \searrow^{\psi} & \uparrow \sigma \\ & & Z \end{array}$$

where σ is finite and ψ is dominant with connected fibers. Since ϕ is projective, it is closed, so since ϕ is dominant, it is surjective. It follows that ψ has finite fibers, so its fibers are points. Hence ψ is bijective (topologically). Hence ψ is bijective and projective, so finite, and ϕ is the composite of two finite morphisms, so is finite.

372 Corollary

If X is a projective variety, then there exists a finite morphism $\sigma: X \to \mathbb{P}^n$ where $n = \dim X$. (Compare with Noether normalization, that the same statement holds for affine morphisms with $\mathbb{P}^n$ replaced by $\mathbb{A}^n$.)

Proof Sketch. Take $X \subset \mathbb{P}^m$ and pick $P \in \mathbb{P}^m - X$. Let $\pi_P: \mathbb{P}^m - \{P\} \to \mathbb{P}^{m-1}$ be projection from P. The restriction $\pi: X \to \mathbb{P}^{m-1}$ is a projective morphism. Fibers of π_P are $\mathbb{A}^1$. The fibers of π are projective varieties in $\mathbb{A}^1$, which forces them to be finite. Hence $X \overset{\pi}{\to} \pi(X)$ is a finite morphism, and $\pi(X)$ is closed. We may now induct. Note that π finite implies X and $\pi(X)$ have the same dimension.

373 Corollary (Grothendieck)

Let $\sigma_U: U \to Y$ be a dominant morphism with finite fibers. Then there exists a finite morphism $\sigma: Z \to Y$ and an open embedding $U \hookrightarrow Z$ such that $\sigma|_U = \sigma_U$.

374 Remark

The intuition is that a dominant morphism with finite fibers that fails to be finite is "missing points" in the domain. For an example, take a curve covering the line with finite fibers and "poke a hole" in the curve, which amounts to localizing on the level of rings, which almost never preserves integrality. Adding the point back in fixes the issue.

We again will not prove this.

375 Theorem (Zariski's main theorem, version 1)

If $\sigma_U: U \to Y$ is a birational morphism with finite fibers and Y is normal, then σ_U is an open embedding.

Proof By the corollary, σ_U extends to $\sigma: Z \to Y$. It follows that $U \to Z \to Y$ gives field embeddings $K(Y) \hookrightarrow K(Z) \hookrightarrow K(U)$. The composite is birational on the varieties, so an isomorphism on function fields. Hence $K(Y) \hookrightarrow K(Z)$ is an isomorphism, so $Z \to Y$ is birational and finite, and also an isomorphism. The result follows.

376 Theorem (Zariski's main theorem, version 2)

If $\phi: X \to Y$ is a birational projective morphism and Y is normal, then ϕ has connected fibers.

Proof As in the previous proof, we have $Z \to Z \to Y$ where $Z \to Y$ is forced to be an isomorphism. Hence $\phi = \psi$, and by Stein factorization, ϕ has connected fibers.

April 8th, 2016: Draft

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377 Notation

Let X be a variety, $\mathcal{O}_X$ its structure sheaf.

378 Definition

We next define $\boxed{\mathcal{O}_X\text{-modules}}$. The philosophy is that you can almost always extend notions from a category to the category of sheaves on that category "locally." Here, if $\mathcal{F}$ is a sheaf (of abelian groups), then for all $U \subset X$ open, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module in a way which is compatible with restriction. Precisely, if $V \subset U$ is an inclusion of open sets, then we require

$$\begin{array}{ccc} \mathcal{O}_X(U) \times \mathcal{F}(U) & \longrightarrow & \mathcal{F}(U) \\ \downarrow & & \downarrow \\ \mathcal{O}_X(V) \times \mathcal{F}(V) & \longrightarrow & \mathcal{F}(V) \end{array}$$

An $\boxed{\mathcal{O}_X\text{-submodule}}$ $\mathcal{G} \subset \mathcal{F}$ is a subsheaf such that $\mathcal{G}(U) \subset \mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module.

379 Remark

To be completely clear, an $\mathcal{O}_X$-module is a sort of collection of modules over a ring. A nice exercise:

380 Homework

$\mathcal{F}/\mathcal{G}$ is an $\mathcal{O}_X$-module. There is a little work required here since one must sheafify for quotients.

381 Definition

Let $\mathcal{F}$ be a sheaf on a topological space X. Define the $\boxed{\text{support}}$ of $\mathcal{F}$ as

$$\boxed{\operatorname{supp} \mathcal{F}} := \{p \in X : \mathcal{F}_p \neq 0\}.$$

382 Example

The following is a good source of counterexamples for things you might naively think are true. In this case, it shows that the support is not necessarily closed. Take $U \subsetneq X$ open and suppose $\mathcal{F}$ is a sheaf on U.

Let $j: U \hookrightarrow X$ be the inclusion map. Define $j_!\mathcal{F}$ (read "j lower shriek") as the sheaf on X associated to the presheaf given by

$$V \mapsto \begin{cases} \mathcal{F}(V) & \text{if } V \subset U \\ 0 & \text{if } V \not\subset U \end{cases}.$$

Note that $\operatorname{supp} j_!\mathcal{F} = \operatorname{supp} \mathcal{F}$ since for all $p \in X$,

$$(j_!\mathcal{F})_p = \begin{cases} \mathcal{F}_p & p \in U \\ 0 & p \notin U \end{cases}$$

In particular, if $\operatorname{supp} \mathcal{F} = U$, we have $\operatorname{supp} \mathcal{F}$ open but not closed. For instance, we could take a constant sheaf; we could take $\mathcal{O}_U$ to be the structure sheaf of a variety on U.

383 Aside

One might ask if we let $k: \overline{U} \hookrightarrow X$ and $\mathcal{F}$ is a sheaf on $\overline{U}$ whether or not

$$j_!(\mathcal{F}|_U) = k|_!(\mathcal{F}).$$

Sandor believes they are indeed equal, though in our context of varieties no proper closed subset contains a non-empty open subset, so they would both be trivially 0.

This construction is in some sense the opposite of the skyscraper sheaf construction: if $p \in X$ is fixed and A is an abelian group, it is defined via

$$\mathcal{A}(U) := \begin{cases} A & \text{if } p \in U \\ 0 & \text{if } P \notin U \end{cases}$$

This is the pushforward $\gamma_* A$ where $\gamma\colon P \hookrightarrow X$. We may contrast this construction with $j_*\mathcal{F}$ where $\mathcal{F}$ is again a sheaf on U. We claim $\operatorname{supp}(j_*\mathcal{F}) = \overline{\operatorname{supp}(\mathcal{F})}$. By definition, $(j_*\mathcal{F})(V) = \mathcal{F}(U \cap V)$. For instance, with $X = \mathbb{A}^2$, $U = \mathbb{A}^2 - \{0\}$, we saw earlier that $j_*\mathcal{O}_U = \mathcal{O}_X$ (which is essentially saying that any regular function on U extends to a regular function on X). This is certainly different from the lower shriek construction's result, which has zero stalk at 0. The $j_!\mathcal{F}$ construction is a good source of non-quasi-coherent $\mathcal{O}_X$-modules, which we next define.

384 Definition

Let X be an affine variety, M an $A(X)$-module. We want to define a sheaf $\widetilde{M}$ on X such that

(1) $\widetilde{M}(D(f)) \cong M_f$

(2) $\widetilde{M}_p \cong M_{\mathfrak{m}_p}$ where $\mathfrak{m}_p$ is the maximal ideal in $A(X)$ corresponding to p.

On $U \subset X$ open, define

$$\widetilde{M}(U) := \{s\colon U \to \cup_{p\in X} M_{\mathfrak{m}_p} \mid *\}$$

where $*$ is the following list of conditions:

(i) For all $p \in U$, $s(p) \in M_{\mathfrak{m}_p}$

(ii) For all $p \in U$, there exists $g \in A(X)$ and $t \in M_g$ such that $p \in D(g) \subset U$ and for any $q \in D(g)$, $t|_Q = s(Q)$ where $t|_Q$ means the following. Recall that $p \in D(g)$ means that $g(p) \neq 0$, i.e. $g \notin \mathfrak{m}_p$, or equivalently $\{1, g, g^2, \ldots\} \cap \mathfrak{m}_p = \varnothing$. Hence we have a natural composite $M \to M_g \to M_{\mathfrak{m}_p}$ where the second map is by definition $t \mapsto t|_p$.

The intuitive version of (ii) is that "locally s looks like some t/g.

385 Homework

Prove that $\widetilde{M}$ is a sheaf, an $\mathcal{O}_X$-module, and that the two conditions above are indeed satisfied.

One should keep in mind that typically when one wants to define a sheaf, it suffices to do so on a basis, and the rest is forced.

386 Example

If $M = A(X)$, then $\widetilde{M} = \mathcal{O}_X$.

387 Definition

Let X be a variety, $\mathcal{F}$ an $\mathcal{O}_X$-module. We say $\mathcal{F}$ is $\boxed{\text{quasi-coherent}}$ if there exists an open affine cover $X = \cup_{i=1}^r U_i$ and $A(U_i)$-modules M_i such that $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$. Hence a quasi-coherent sheaf is one which locally looks like $\widetilde{M}$'s. (Indeed, such M_i are uniquely determined by the fact that $\widetilde{M_i}(U_i) \cong M_i$ since $U_i = D(1)$.)

$\mathcal{F}$ is $\boxed{\text{coherent}}$ if it is quasi-coherent and the M_i are finitely generated $A(U_i)$-modules.

388 Homework

If $\mathcal{F}$ is quasi-coherent, pick any $U \subset X$ open and let $M := \mathcal{F}(U)$. Then $\mathcal{F}|_U \cong \widetilde{M}$.

389 Example

If X is a projective variety, then $\mathcal{O}_X(X) = k$, so $\mathcal{O}_X$ is almost never of the form $\widetilde{\mathcal{O}_{\mathcal{X}}(\mathcal{X})}$ since this would be a constant sheaf. Hence attempting to require "quasi-coherence" on larger than affine sets is not generally advisable.

$\mathcal{O}_X$ is trivially coherent. Finitely many direct sums of $\mathcal{O}_X$ with itself remains coherent, and infinitely many direct sums become quasicoherent. Moreover, if $j\colon U \hookrightarrow X$ with U open, then $j|_!\mathcal{O}_U$ is not quasi-coherent. On the other hand, $j_*\mathcal{O}_U$ is quasi-coherent but usually not coherent. For the punctured affine plane, we had $j_*\mathcal{O}_U = \mathcal{O}_X$, but this was rather special since it expresses the existence of regular function extensions.

April 11th, 2016: Draft

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390 Remark

Suppose X is a variaty, $Z \subset X$ is closed. Then we have an ideal sheaf $\mathcal{I}_Z \subset \mathcal{O}_X$ given as follows. Let $X = \cup_i U_i$ be an open affine cover and consider $Z_i \coloneqq Z \cap U_i \subset U_i$, which corresponds to an ideal $I(Z_i) \subset A(U_i)$, which is an $A(U_i)$-module. Then we have a sheaf $\widetilde{I(Z_i)} \subset \mathcal{O}_{U_i}$. By the remarks from last time, we know what $\widetilde{I(Z_i)}$ is on distinguished ("principal") open sets, from which it is a good exercise to check

$$\widetilde{I(Z_i)}_{U_i \cap U_j} \cong \widetilde{I(Z_j)}_{U_i \cap U_j}.$$

By construction, this is a quasi-coherent sheaf. Since $A(U_i)$ is noetherian, it is in fact coherent.

391 Lemma

Let A be a ring, M a finitely generated A-module, $\operatorname{Ann}(M) \coloneqq \{a \in A : aM = 0\}$ *the annihilator ideal of M, and $\mathfrak{p} \subset A$ a prime ideal. Then* $M_\mathfrak{p} \neq 0 \Leftrightarrow \mathfrak{p} \supset \operatorname{Ann}(M)$.

Proof Suppose $\operatorname{Ann}(M) \not\subseteq \mathfrak{p}$. Then pick $a \in \operatorname{Ann}(M)$ with $a \notin \mathfrak{p}$. Now for each $m/b \in M_\mathfrak{p}$, this is $am/(ab) = 0$ since $am = 0$. Hence $M_\mathfrak{p} = 0$. Here it was important that $a \in A - \mathfrak{p}$ when we muliplied by a/a.

For the other direction, let $M = \langle m_1, \ldots, m_r \rangle$ and note that $\operatorname{Ann}(M) = \cap_i \operatorname{Ann}(m_i)$. Now $\mathfrak{p} \supset \operatorname{Ann}(M) = \operatorname{Ann}(m_i)$ implies that $\mathfrak{p} \supset \operatorname{Ann}(m_i)$ for some i. But then $0 \neq m_i/1 \in M_\mathfrak{p}$ is non-zero.

392 Corollary

If M is a finite $A(X)$-module where X is an affine variety, then $\operatorname{supp} \widetilde{M} = Z(\operatorname{Ann} M)$.

Proof Recall that $\operatorname{supp} \widetilde{M} \coloneqq \{P \in X : (\widetilde{M})_P \neq 0\}$, and that $(\widetilde{M})_P \cong M_{\mathfrak{m}_{X,P}}$ so that $\operatorname{supp} \widetilde{M}$ corresponds to maximal ideals $\mathfrak{m}_{X,P}$ in $A(X)$ such that $\mathfrak{m}_{X,P} \supset \operatorname{Ann}(M)$, which occurs if and only if $f(P) = 0$ for all $f \in \operatorname{Ann}(M)$.

393 Corollary

If $\mathcal{F}$ is a coherent sheaf on a variety, then $\operatorname{supp} \mathcal{F}$ *is closed.*

394 Example

Take a variety X with an open subset U with U non-affine. Now $\mathcal{O}_X(U)$ is an $\mathcal{O}_X(X)$-module, and "typically" the support of this module should be U, which is open, contrary to the preceding corollary.

395 Remark

We will next discuss "almost schemes" (a name Sandor just made up).

396 Definition

Consider local rings $(A, \mathfrak{m})$, $(B, \mathfrak{n})$. A ring homomorphism $\phi\colon A \to B$ is a local homomorphism (of local rings) if $\phi^{-1}(\mathfrak{n}) = \mathfrak{m}$. In general, $\phi^{-1}(\mathfrak{n}) \subset \mathfrak{m}$, since $\phi^{-1}(\mathfrak{n})$ is a prime ideal, which in particular must be proper.

397 Example

Consider $\mathbb{Z} \hookrightarrow \mathbb{Q}$. Localize $\mathbb{Z}$ at any prime and consider the resulting homomorphism of local rings; it is not local since the preimage of the maximal ideal 0 is 0.

398 Definition

A ringed space $(X, \mathcal{F})$ where X is a topological and $\mathcal{F}$ is a sheaf of rings. We may refer to X as the **support** of $(X, \mathcal{F})$. One often thinks of $\mathcal{F}$ as a sheaf of functions, and it will very often be $\mathcal{O}_X$. A **morphism of ringed spaces** $(X, \mathcal{F}) \to (Y, \mathcal{G})$ is a pair $(\phi, \phi^\sharp)$ where $\phi\colon X \to Y$ is a continuous map and $\phi^\sharp\colon \mathcal{G} \to \phi_*\mathcal{F}$ is a morphism of sheaves. The composite of

$$(\phi, \phi^\sharp)\colon (X, \mathcal{F}) \to (Y, \mathcal{G}), \qquad (\psi, \psi^\sharp)\colon (Y, \mathcal{G}) \to (Z, \mathcal{H})$$

is just

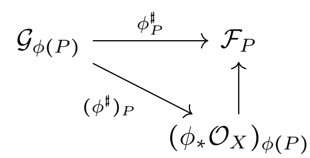
$$(\psi \circ \phi, \psi_*\phi^\sharp \circ \psi^\sharp)\colon (X, \mathcal{F}) \to (Z, \mathcal{H}).$$

399 Remark

Recall that $(\phi_*\mathcal{F})(V) := \mathcal{F}(\phi^{-1}(V))$. Now take an open set $V \subset Y$, which gives a morphism $\mathcal{G}(V) \to \mathcal{F}(\phi^{-1}V)$. Localizing and passing to the limit gives

$$\mathcal{G}_{\phi(P)} = \lim_{V \ni \phi(P)} \mathcal{G}(V) \to \lim_{V \ni \phi(P)} \mathcal{F}(\phi^{-1}V) \to \lim_{U \ni P} \mathcal{F}(U) = \mathcal{F}_P.$$

Hence we have an induced morphism on stalks $\phi^\sharp_P\colon \mathcal{G}_{\phi(P)} \to \mathcal{F}_P$. A slight subtlety: we actually have a triangle



A **locally ringed space** is a ringed space $(X, \mathcal{F})$ such that for all $P \in X$, $\mathcal{F}_P$ is a local ring. One trivial source of non-locally ringed spaces comes from picking a non-local ring and considering its constant sheaf. A **morphism of locally ringed spaces** is a morphism of ringed spaces $(\phi, \phi^\sharp)$ as above such that for every $P \in X$, $\phi^\sharp \mathcal{G}_{\phi(P)} \to \mathcal{F}_P$ is a local homomorphism.

400 Example

Let X be a smooth manifold and let $\mathcal{F}$ be the sheaf of differentiable functions on X. This is locally ringed since the stalks are germs of functions and the unique maximal ideal are those which are zero at the point. Morphisms of differentiable manifolds are morphisms of locally ringed spaces because if a function is zero on Y it must have be zero when pulled back to X.

401 Example

Let X be a topological space with one point P and let Y be a topological space with two points $\{U, Q\}$. Let the open sets of Y be given by $\{\varnothing, \{U\}, \{U, Q\}\}$. Let $R \subset K$ be a DVR included in a field. Define the only interesting restriction map to be $\mathcal{O}_Y(Y) \to \mathcal{O}_Y(\{U\})$ given by $R \to K$. Define $\phi\colon X \to Y$ by $P \mapsto U$. Now $(\phi_*\mathcal{O}_X)(Y) = K$, $(\phi_*\mathcal{O}_X)(U) = K$ with id as the restriction map. Moreover, $\phi^\sharp\colon \mathcal{O}_Y \to \phi_*\mathcal{O}_X$ is given by $K \overset{\text{id}}{\to} K$ on U and $R \hookrightarrow K$ on Y which is compatible with the restriction maps just computed.

As for stalks, we have $\mathcal{O}_{X,P} = K$, $\mathcal{O}_{Y,U} = K$ (the direct limit of $R \to K$), and $\mathcal{O}_{Y,Q} = R$ (the direct limit of $R \overset{\text{id}}{\to} R$). Now $\phi^\sharp_P\colon \mathcal{O}_{Y,\phi(P)} \to \mathcal{O}_{X,P}$ by $K \overset{\text{id}}{\to} K$. This morphism is a morphism of locally ringed spaces.

On the other hand, we can use $\psi\colon X \to Y$ by $P \mapsto Q$. Now $\psi^\sharp\colon \mathcal{O}_Y \to \psi_*\mathcal{O}_X$ is given as follows. Now $(\psi_*\mathcal{O}_X)(U) = \mathcal{O}_X(\varnothing) = 0$ (by convention we use the terminal object here) and

$(\psi_* \mathcal{O}_X)(Y) = K$. Hence $\psi^\sharp$ is given by $K \to 0$ on U and $R \to K$ on Y with restriction maps $R \to K$ and $K \to 0$, which form a commutative square. Now $\psi^\sharp_P : \mathcal{O}_{Y,\phi(P)} \to \mathcal{O}_{X,P}$ is $R \to K$. This is not a local homomorphism since the preimage of 0 is too small to capture the whole maximal ideal in R. Hence ψ is a morphism of ringed spaces but not of locally ringed spaces.

402 Example

Recall that a variety X is non-singular at a point P if $\mathcal{O}_{X,P}$ is a regular local ring. If $X = C$ is a curve, then C is regular at Q if and only if $\mathcal{O}_{C,Q}$ is a DVR. One can roughly model the preceding example at taking U to be the complement of $Q \in C$, so Q is a closed point; we are here imagining "collapsing" the set U to a point. The local ring at U is the function field K and the local ring at Q is R (assume regularity at Q). Now there is a natural way to go from K to functions on subsets of C, namely take a rational function (an element of K) and, if it doesn't have a pole at Q, send it to R, and otherwise send it to U. More next time.

April 13th, 2016: Draft

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403 Remark

The following is a high level discussion of the role of ringed spaces in other geometries. It is imprecise and not entirely true, which is fine.

Roughly, the kind of geometry you do is determined by your choice of functions, what you call "regular." You need some topology, and the choice of functions is essentially a choice of sheaf. In this sense, ringed spaces are about trying to do all geometry at once. Locally ringed spaces arise naturally by considering functions which are zero or non-zero.

Kollar and Mori have a book on birational geometry. They have a section on how their methods apply to analytic spaces. For instance, coherent sheaves make sense in analysis (though the starting definition is different). Proving that $\mathcal{O}_X$ is coherent in the analytic setting is a theorem rather than a trivial observation. An equivalent definition of coherence of a sheaf is the following. Say X is a variety, $\mathcal{F}$ is a sheaf on X. Then $\mathcal{F}$ is coherent if and only if roughly $X = \cup_i U_i$ is a finite open cover such that $\mathcal{F}|_U$ is the cokernel of a morphism of free, sheaves,

$$\oplus \mathcal{O}|_{U_i} \to \oplus \mathcal{O}|_{U_i} \to \mathcal{F}|_{U_i} \to 0.$$

If X is noetherian, if the middle is finite, then the left is finite, since something is finitely generated if and only if it is finitely presented. Sometimes the definition is given in terms of finite presentation.

One could use this language for differentiable manifolds. Consider a surface X over $\mathbb{C}$ mapping to a curve and consider the fiber X_t of a point t. Suppose each X_t is a (smooth) variety of some constant dimension. Now each X_t is in particular an $\mathbb{R}$-manifold. In this context, $X_{t_1} \cong X_{t_2}$ as differentiable manifolds. For instance, say $X_t \subset \mathbb{P}^3$ is a non-singular surface of degree 4. One can consider the space of coefficients of degree 4 polynomials in this context. Then $\mathbb{P}^N \times \mathbb{P}^3 \to \mathbb{P}^N$, and restricting to the open subset of $\mathbb{P}^N$ where the fibers are smooth, the fibers are isomorphic. The cup product gives a $\mathbb{Z}$-lattice structure on $H^2(X, \mathbb{Z})$; for any of the above fibers, this ends up being $E_8^2 \oplus H^3$, where H is the hyperbolic plane defined by $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$. Hence this algebraic rigidity roughly explains some topological rigidity. These X_t are special cases of K3 surfaces. All K3 surfaces have the preceding cohomology.

A very unorthodox definition of K3 surfaces is the following. Take a degree 4 non-singular surface in $\mathbb{P}^3$ together with an arbitrary complex-analytic deformation of that; the result is a K3 surface. This is non-singular, though some people define singular K3 surfaces as surfaces which are birational to one of these smooth ones. Consider $X := \mathbb{C}^2/\mathbb{Z}^4$, which has a $\mathbb{Z}_2$-action given by negating on the level of $\mathbb{C}$.

Then $X/\mathbb{Z}_2$ is a K3 surface. There are 16 singular points, which arise from the fixed points of the group action. (More precisely, they arise from coordinates which are multiples of $1/2$, mod $\mathbb{Z}$. The quotient is a four-dimensional cube with opposite sides identified.) The singularity is not bad; it is essentially the same as the singularity of a cone. If one blows up these points to get a resolution $Y \to X/\mathbb{Z}_2$, one gets 16 $\mathbb{P}^1$'s whose pairwise intersections are related to the Coxeter matrix of E_8. Sandor suggests there are certainly thesis topics related to this discussion; there will be theses written on K3 surfaces for many years to come.

404 Definition

Let $\mathcal{I} \subset \mathcal{O}_{\mathbb{A}^n}$ be a coherent ideal sheaf (i.e. a coherent sub-$\mathcal{O}_{\mathbb{A}^n}$-module). Say $Z := \operatorname{supp} \mathcal{O}_{\mathbb{A}^n}/\mathcal{I} \subset \mathbb{A}^n$. Last time we proved that Z is closed.

405 Homework

If $\mathcal{I}$ is a coherent ideal sheaf, then $\mathcal{O}_X/\mathcal{I}$ is coherent.

Now consider the ringed space $(Z, \mathcal{O}_Z)$ where $\mathcal{O}_Z := \mathcal{O}_{\mathbb{A}^n}/\mathcal{I}$. Call the result an affine scheme. In other words, an affine scheme is a locally ringed space where $Z \subset \mathbb{A}^n$ is a closed subset and there exists $\mathcal{I} \subset \mathcal{O}_{\mathbb{A}^n}$ coherent such that $Z = \operatorname{supp} \mathcal{O}_{\mathbb{A}^n}/\mathcal{I}$ and $\mathcal{O}_Z \cong \mathcal{O}_{\mathbb{A}^n}/\mathcal{I}$. A morphism of affine schemes is just a morphism between affine schemes of locally ringed spaces.

406 Remark

A closed subset of $\mathbb{A}^n$ is an algebraic set, and if Z is irreducible, it is just an affine variety. If $Z \subset \mathbb{A}^n$ is an affine variety, then we have the ideal sheaf $\mathcal{I}_Z := \widetilde{I(Z)}$ where $I(Z) \subset A(\mathbb{A}^n) = k[x_1, \ldots, x_n]$. Hence an affine variety determines an affine scheme $(Z, \mathcal{O}_Z)$ where $\mathcal{O}_Z := \mathcal{O}_{\mathbb{A}^n}/\mathcal{I}_Z$. The definition of affine scheme above is a bit more general. For instance, one could use $(Z, \mathcal{O}_{\mathbb{A}^n}/\mathcal{I}_Z^2)$. There is a morphism $(Z, \mathcal{O}_{\mathbb{A}^n}/\mathcal{I}_Z) \to (Z, \mathcal{O}_{\mathbb{A}^n}/\mathcal{I}_Z^2)$ given by $(\phi, \phi^\sharp)$ where ϕ is the identity and

$$\phi^\sharp : \mathcal{O}_{\mathbb{A}^n/\mathcal{I}_Z^2} \to \mathcal{O}_{\mathbb{A}^n}/\mathcal{I}_Z$$

given by quotienting by $\mathcal{I}_Z/\mathcal{I}_Z^2$. In this example one would expect we cannot go the other way. If $Z \in \{\mathbb{A}^n, \varnothing\}$, then we get an isomorphism trivially. When is there a section of $\phi^\sharp$? Consider the case when Z is a point in $\mathbb{A}^1$. The ideal is then $(t) \subset k[t]$, whose square is (t^2). Then this is $k[t]/(t^2) \to k[t]/(t)$, and we have a section given by including k into $k[t]/(t^2)$.

407 Notation

If $(X, \mathcal{O}_X)$ is a ringed space, we say the ringed space has property P if X does, whenever P is a purely topological notion. For instance, we say $(X, \mathcal{O}_X)$ is connected if X is connected.

408 Remark

Warning: This does not agree with the usual definition of "affine scheme." We have been ignoring non-closed points throughout this course. Our notion is essentially the usual one for schemes of finite type over an algebraically closed field. One convenience with this definition is that our sheaves will be sheaves of functions, which a priori is not true for arbitrary schemes.

April 15th, 2016: Draft

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409 Remark

Last time we defined affine ("almost") schemes. Again, they are almost schemes of finite type over k in a sense we will not make entirely precise.

410 Definition

A locally ringed space $(Z, \mathcal{O}_Z)$ is a scheme if $Z := \cup_i U_i$ is a finite open cover such that $(U_i, \mathcal{O}_Z|_{U_i})$ is an affine scheme in the sense from last time. A morphism of schemes is a morphism of the underlying locally ringed spaces. Note that U_i is an affine algebraic set (not necessarily a variety).

Given a morphism $(\phi, \phi^\sharp) : (Z, \mathcal{O}_Z) \to (W, \mathcal{O}_W)$ where $\phi: Z \to W$ and $\phi^\sharp : \mathcal{O}_W \to \phi_* \mathcal{O}_Z$, the pair $(\phi, \phi^\sharp)$ is an **open immersion** if $\phi: Z \hookrightarrow W$ is an open embedding and $\mathcal{O}|_Z = \mathcal{O}_W|_Z$ (identifying Z with its image). An **open subscheme** is the image of an open immersion. The pair $(\phi, \phi^\sharp)$ is a **closed immersion** if $\phi: Z \hookrightarrow$ is an injection where $\phi(Z) \subset W$ is closed and where $\phi^\sharp : \mathcal{O}_W \to \phi_* \mathcal{O}_Z$ is surjective.

411 Remark

When thinking of closed immersions, we imagine the case of varieties and subvarieties, where the induced map on coordinate rings is a surjection, so the target is a further quotient of the domain.

A **closed subscheme** is an equivalence class of closed immersions $(\phi, \phi^\sharp)$ with $\phi: Z \hookrightarrow U$ where $\phi \sim \phi'$ if there exists a morphism of schemes $\alpha: Z \xrightarrow{\sim} Z$ such that $\phi = \phi' \circ \alpha$.

A **projective scheme** (over k) is a closed subscheme of $\mathbb{P}^n_k$.

412 Notation

We will often drop the sheafy parts of the morphism from the notation. Hence "$\phi: Z \to W$" will really mean a morphism of schemes.

A **quasiprojective scheme** (over k) is an open subscheme of a projective scheme. More generally, a morphism of schemes $\phi: Z \to W$ is a **projective morphism** if it factors through $\mathbb{P}^n_W$ for some n, i.e. we have a closed embedding $j: Z \hookrightarrow \mathbb{P}^n_W$ such that

$$\begin{array}{ccc} & & \mathbb{P}^n_W \\ & \overset{j}{\nearrow} & \downarrow \\ Z & \xrightarrow{\phi} & W \end{array}$$

For instance, if Z is a single point $\{P\}$ where $\mathcal{O}_{\{P\}}$ comes from k, then a morphism $Z \to \{P\}$ is projective when $Z \to \{P\}$ is constant and the map on sheaves is roughly $k = \mathcal{O}_P \to \phi_* \mathcal{O}_Z = \mathcal{O}_Z(Z)$. Now $\mathbb{P}^n_P = \mathbb{P}^n_k$. We may write $\operatorname{Spec} k$ for P, though we won't define it precisely. In any case, the factoring through condition in this case just means that the morphism of rings is actually a k-algebra morphism.1

413 Remark

One may construct non-quasiprojective schemes, though it takes a little bit of effort. This is in contrast to our definitions for varieties, where they're all quasiprojective. That example began by taking a projective variety and constructing a variety which maps onto the projective variety which factors through the map from the projective guy to $\mathbb{P}^n$.

414 Homework

Find a non-quasiprojective scheme.

415 Remark

Let $Z \subset \mathbb{A}^n$ be an affine variety with corresponding ideal sheaf $\mathcal{I}_Z \subset \mathcal{O}_{\mathbb{A}^n}$. Let $\mathcal{O}_Z \coloneqq \mathcal{O}_{\mathbb{A}^n}/\mathcal{I}_Z$, which gives a scheme $(Z, \mathcal{O}_Z)$. (Indeed, the same manipulations work for affine algebraic sets.) In this sense we've (strictly) enlarged our category by going from affine varieties (or algebraic sets) to affine schemes.

Similarly, if Z is an arbitrary quasi-projective algebraic variety (or set), there is a corresponding quasi-projective variety obtained by "gluing together" the schemes coming from standard open affines.

416 Definition

Let $\phi: X \to Y$ be a morphism of varieties. We next describe the corresponding morphism of schemes. That is, we need a map $\phi^\sharp : \mathcal{O}_Y \to \phi_* \mathcal{O}_X$. We may define a map of sheaves locally and we may assume

X and Y live in the same affine space. We showed that a morphism of affine varieties is equivalent to a morphism of their coordinate rings, which is the morphism of the global sections of these sheaves. One must check that the morphisms defined on such open sets glue together. Roughly, one may cover the intersection by principal affine opens to verify this.

417 Homework

Make this argument precise.

418 Definition

Let $(Z, \mathcal{O}_Z)$ be a scheme, $\mathcal{I} \subset \mathcal{O}_Z$ an ideal sheaf. (We defined $\mathcal{O}_X$-modules for varieties, but we can make the same definition word-for-word for a scheme, or generally for a ringed space. An $\mathcal{O}_Z$-submodule of $\mathcal{O}_Z$ is called an ideal sheaf.) Then there exists an ideal sheaf $\sqrt{\mathcal{I}} \subset \mathcal{O}_Z$ called the radical such that

$$(\sqrt{\mathcal{I}})(U) = \sqrt{\mathcal{I}(U)}.$$

419 Homework

Prove that this is indeed an ideal sheaf.

If $(Z, \mathcal{O}_Z)$ is a scheme, let $\mathcal{I} := (0)$ be the zero ideal sheaf, and consider $\sqrt{(0)} \subset \mathcal{O}_Z)$. Define a new scheme $Z_{\text{red}} := (Z, \mathcal{O}_{Z_{\text{red}}})$ where $\mathcal{O}_Z := \mathcal{O}_Z/\sqrt{(0)}$, called the reduced scheme supported on Z. Since $\mathcal{O}_{Z_{\text{red}}}$ is a further quotient of $\mathcal{O}_Z$, Z_{red} is a closed subscheme of Z, i.e. we have $Z_{\text{red}} \hookrightarrow Z$. We have a topological map the other way, but we won't in general have a sheaf morphism to go along with it.

420 Homework

Let $(Z, \mathcal{O}_Z)$ be an irreducible affine scheme. Show that Z_{red} is the affine scheme associated to the affine variety $Z \subset \mathbb{A}^n$.

April 18th, 2016: Draft

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421 Remark

Last time we largely ignored one piece of structure, namely the base field k over which all of our "schemes" are defined. Their local rings are k-algebras and not just rings, so we actually require a morphism of schemes $\phi\colon X \to Y$ to be a morphism of ringed spaces where the ring homomorphisms are all k-algebra homomorphisms, compatible with restriction.

422 Remark

We write $\operatorname{Spec} k = (P, k)$, where P is shorthand for $\{P\}$, and k is shorthand for the sheaf sending $\{P\}$ to k.

423 Remark

Let Z be a scheme. Consider morphisms $\phi\colon \operatorname{Spec} k \to Z$. Topologically, this simply picks a point Q of Z. On the level of sheaves, $\phi^\sharp\colon \mathcal{O}_Z \to \phi_* k$ is a morphism $\mathcal{O}_Z$ to the skyscraper sheaf on Q which is k. Indeed, the "information" in $\phi^\sharp$ is really the local homomorphism $\mathcal{O}_{Z,Q} \to k$. Since the preimage of the maximal ideal 0) is forced to be the kernel, this morphism factors uniquely as

$$\begin{array}{ccc} \mathcal{O}_{Z,Q} & \longrightarrow & k \\ & \searrow & \uparrow \sim \\ & & \mathcal{O}_{Z,Q}/\mathfrak{m}\mathcal{O}_{Z,Q} \end{array}$$

This proves the following:

424 Corollary

The points of Z are in one-to-one correspondence with morphisms $\operatorname{Spec} k \to Z$. *(Warning: This result is more subtle when working with general schemes, namely one must restrict to closed points of Z.)*

425 Remark

Suppose Z is just a ringed space, $U \in Z$, $\overline{\{U\}} \neq \{U\}$, what is $\mathcal{O}_{Z,U}/\mathfrak{m}_{Z,U}$? The function field on U, which often will have transcendence degree ≥ 1, so in particular it will not be k.

426 Remark

Let Z be a scheme and consider morphisms $Z \overset{\psi}{\to} \operatorname{Spec} k$. Topologically, every point must map to P. On the level of sheaves, $\psi^\sharp : k \to \psi_* \mathcal{O}_Z$. This is entirely determined by the induced map on global sections $k \to \mathcal{O}_Z(Z)$, which is to say it is entirely determined by a k-algebra structure on $\mathcal{O}_Z(Z)$. Such a structure restricts via $k \to \mathcal{O}_Z(Z) \to \mathcal{O}_Z(U)$. This motivates the following definition:

427 Definition

Suppose S is a scheme (k-scheme). An $\boxed{S\text{-scheme}}$ is a scheme X and a morphism $\pi : X \to S$. A *morphism of S-schemes* is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\phi} & Y \\ & {\scriptstyle \pi_X} \searrow \quad \swarrow {\scriptstyle \pi_Y} & \\ & S & \end{array}$$

In more generality, there is a scheme $\operatorname{Spec} \mathbb{Z}$, and essentially because every ring is a $\mathbb{Z}$-algebra, every scheme is a $\operatorname{Spec} \mathbb{Z}$-scheme; this is the final object in the category of schemes.

428 Remark

Let $(Z, \mathcal{O}_Z)$ be a scheme. Suppose $f \in \mathcal{O}_Z(Z)$. Define sets

$$\boxed{D(f)} := \{P \in Z : f_P \notin \mathfrak{m}_{Z,P}\}.$$

(Here we denote $\mathcal{O}_Z(Z) \to \mathcal{O}_{Z,P}$ by $f \mapsto f_P$.) If Z is affine, then we write $\boxed{Z_f} := D(f)$.

429 Homework

Prove that if Z is an affine scheme, then $D(f) \subset Z$ is open and $\mathcal{O}_Z(D(f)) \cong \mathcal{O}_Z(Z)_f$.

If $(Z, \mathcal{O}_Z)$ is an affine scheme and M is an $\mathcal{O}_Z(Z)$-module, then we can define a sheaf $\widetilde{M}$ which is an $\mathcal{O}_Z$-module; the construction is identical to our earlier construction which required $(Z, \mathcal{O}_Z)$ to arise from a variety (in particular, it was reduced, but this is not actually important).

430 Homework

Define (quasi-)coherent sheaves on schemes.

431 Definition

Recall that if X, Y are varieties, we defined $X \times Y$ earlier. There was a little subtlety in defining products in the quasi-projective case.

Let X, Y be quasi-projective Z-schemes. That is, we have $\phi : X \to Z$ and $\psi : Y \to Z$. We wish to define $X \times_Z Y$. The definition of $X \times_Z Y$ will be spread out over the rest of this lecture and the start of the next. On the level of topological spaces, consider

$$X \times_Z Y := \{(x, y) \in X \times Y : \phi(x) = \psi(y)\}$$

which has the subspace topological inherited from the topology arising from the products of *varieties*. (It will happen that the topology on $X \times_Z Y$ will not be this in general.)

432 Lemma

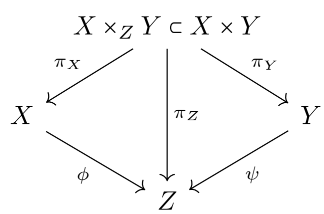
$X \times_Z Y \subset X \times Y$ is closed.

Proof Take $X \times Y \overset{\phi\times\psi}{\to} Z \times Z$ by $(x, y) \mapsto (\phi(x), \psi(y))$, which is a morphism. Warning: $Z \times Z$ does *not* have the product topology, but instead has the topology coming from the product of varieties. Consider the inverse image of the diagonal $\Delta := \{(Z, Z) : z \in Z\}$. One may check that Δ is closed by "gluing" the corresponding result on affine varieties. Then $X \times_Z Y = (\phi \times \psi)^{-1}\Delta$ is closed.

433 Remark

Note that the Zariski topology is not generally Hausdorff. A standard equivalent definition of Hausdorffness is that the diagonal is closed in the product topology. Since varities are almost never Hausdorff, this illustrates how important it is to use to correct topology on $Z \times Z$ above.

Topologically, we have



434 Definition

Let $(X, \mathcal{A})$ be a ringed space, and suppose $\mathcal{F}, \mathcal{G}$ are $\mathcal{A}$-modules. We define a sheaf of $\mathcal{A}$-modules as the sheaf associated to the presheaf

$$\boxed{\mathcal{F} \otimes_{\mathcal{A}} \mathcal{G}} : U \mapsto \mathcal{F}(U) \otimes_{\mathcal{A}(U)} \mathcal{G}(U).$$

435 Remark

Recall that if $\phi: X \to Z$ and $\mathcal{G}$ is a sheaf on Z, we have a sheaf $\phi^{-1}\mathcal{G}$ on X defined by sheafifying

$$\phi^{-1}\mathcal{G}: U \mapsto \lim_{V \supset \phi(V)} \mathcal{G}(V).$$

We define $\mathcal{O}_{X\times_Z Y} := \pi_X^{-1}\mathcal{O}_X \otimes_{\pi_Z^{-1}\mathcal{O}_Z} \pi_Y^{-1}\mathcal{O}_Y)$. We further define $(X \times_Z Y, \mathcal{O}_{X\times_Z Y})$ to be the $\boxed{\text{fibered product}}$ of X and Y over Z.

We will continue our discussion of this next lecture with examples.

April 20th, 2016: Draft

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436 Remark

Last time we talked about tensor products of sheaves. Consider the following exercise:

437 Homework

Let X be an affine scheme, M, N $\mathcal{O}_X(X)$-modules. Then $\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \cong \widetilde{M \otimes_{\mathcal{O}_{\mathcal{X}}(\mathcal{X})} N}$.

Consequently, in this common situation, we do not need to worry about sheafifications when computing the tensor product of sheaves.

438 Homework

Find a tensor product of sheaves and an open set on which the preceding isomorphism can't possibly hold. Find a scheme X (possibly variety) and $\mathcal{O}_X$-modules $\mathcal{F}, \mathcal{G}$ (possibly coherent) such that $\mathcal{F}(X) = \mathcal{G}(X) = 0$ but $(\mathcal{F} \otimes \mathcal{G})(X) \neq 0$.

439 Remark

We recall and expand upon the construction at the end of last lecture. Let $\phi\colon X \to Y$ be a morphism of schemes and let $\mathcal{G}$ be a sheaf on Y. We define $\phi^{-1}\mathcal{G}$ as the sheaf associated to the presheaf

$$U \mapsto \lim_{V \supset \phi(U)} \mathcal{G}(V),$$

called the inverse image sheaf.

If $\mathcal{G}$ is an $\mathcal{O}_Y$-module, the $\mathcal{O}_X$-module inverse image sheaf is defined as follows. The inverse image sheaf $\phi^{-1}\mathcal{G}$ is naturally a $\phi^{-1}\mathcal{O}_Y$-module, essentially because ϕ^{-1} is functorial. Indeed, $\mathcal{O}_X$ is naturally a $\phi^{-1}\mathcal{O}_Y$-module, as follows. It is a fact that ϕ^{-1} is left adjoint to ϕ_*, i.e. given a sheaf $\mathcal{F}$ on X and $\mathcal{G}$ on Y, we have

$$\mathrm{Hom}_X(\phi^{-1}\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_Y(\mathcal{G}, \phi_*\mathcal{F}).$$

Indeed, we have $\phi^{-1}\mathcal{G} \to \mathcal{F}$ and also $\mathcal{G} \to \phi_*\phi^{-1}\mathcal{G} \to \phi_*\mathcal{F}$, and on the other hand we have $\mathcal{G} \to \phi_*\mathcal{F}$ and $\phi^{-1}\mathcal{G} \to \phi^{-1}\phi_*\mathcal{F} \to \mathcal{F}$; it is relatively easy to check that these maps are inverses of each other.

In particular, by this correspondence, the map on sheaves $\phi^\sharp\colon \mathcal{O}_Y \to \phi_*\mathcal{O}_X$ can equivalently be given by a map $\phi^{-1}\mathcal{O}_Y \to \mathcal{O}_X$, so that $\mathcal{O}_X$ is indeed a $\phi^{-1}\mathcal{O}_Y$-module. Now the $\mathcal{O}_X$-module inverse image of $\mathcal{G}$ is

$$\phi^*\mathcal{G} \coloneqq \phi^{-1}\mathcal{G} \otimes_{\phi^{-1}\mathcal{O}_Y} \mathcal{O}_X.$$

We sometimes call $\phi_*\mathcal{F}$ the sheaf direct image; there is a similar notion involving the module structure, as follows. $\mathcal{F}$ is naturally a $\phi_*\mathcal{O}_X$-module, which may be considered as an $\mathcal{O}_Y$-module via the above map. The $\mathcal{O}_Y$-module direct image of $\mathcal{F}$ is then just $\phi_*\mathcal{F}$ considered as an $\mathcal{O}_Y$-module in this way. In particular, the sheaf is the same, in contrast to the inverse image case.

In particular, given a morphism of sheaves $\phi\colon X \to Y$, $\phi^\sharp\colon \mathcal{G} \to \phi_*\mathcal{F}$, we may imagine $\phi^\sharp$ as either just a morphism of sheaves or a morphism of $\mathcal{O}_Y$-modules. Nonetheless, ϕ^* and ϕ_* are also an adjoint pair but now between module categories rather than categories of sheaves on a space,

$$\mathrm{Hom}_{\mathcal{O}_X}(\phi^*\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, \phi_*\mathcal{F}).$$

The right-hand side of this equation is a subset of the right-hand side of the previous one. However, the left-hand sides are rather more different, because $\phi^{-1}\mathcal{G}$ and $\phi^*\mathcal{G}$ are not the same as sheaves.

440 Homework

Let $\phi\colon X \to Y$ be a morphism of affine schemes. Let M be an $\mathcal{O}_X(X)$-module, N an $\mathcal{O}_Y(Y)$-module. We may consider M as an $\mathcal{O}_X(X)$-module or as a $\mathcal{O}_Y(Y)$-module as in the preceding remark. Show that

$$\phi_*\widetilde{M}^X \cong \widetilde{M}^Y,$$

where the superscripts denote which ring we use. Further show that

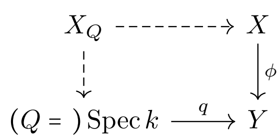
$$\phi^*\widetilde{N} \cong \widetilde{N \otimes_{\mathcal{O}_\mathcal{Y}(\mathcal{Y})} \mathcal{O}_\mathcal{X}}(\mathcal{X}).$$

441 Remark

For affine morphisms and coherent sheaves, the preceding properties determine ϕ_* and ϕ^* entirely. It takes a little more work outside of the affine case.

442 Remark

Let $\phi\colon X \to Y$ be a morphism of scehemes. Recall that points $Q \in Y$ correspond bijectively to morphisms $q\colon \mathrm{Spec}\, k \to Y$. We wish to define the fiber product of ϕ and q,



We define $X_Q := X \times_Y \operatorname{Spec} k$.

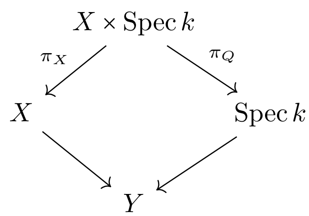
443 Lemma

The topological space of X_Q is $\phi^{-1}(Q)$. However, X_Q as a scheme is not necessarily reduced.

Proof Recall that the underlying set of the product was

$$X \times_Y \operatorname{Spec} k = \{(P, Q) \in X \times \operatorname{Spec} k : \phi(P) = Q\},$$

which is obviously the suggested set. Use the following notation:



By definition, $\mathcal{O}_X|_{X_Q} := \pi_X^{-1}\mathcal{O}_X$. On the other hand, $\pi_Y^{-1}\mathcal{O}_X$ is obtained by computing $\lim_{V \supset \pi_Y(X_Q)} \mathcal{O}_Y(V)$, but $\pi_Y(X_Q) = Q$, so we have a natural isomorphism $\pi_Y^{-1}\mathcal{O}_Y \cong \mathcal{O}_{Y,Q}$, where the right-hand side is interpreted as the constant sheaf (not the constant presheaf; the constant sheaf is locally constant, whereas the constant presheaf is literally constant; to get to the constant sheaf, one must take direct sums for connected components). Further, $\pi_Q^{-1} \operatorname{Spec} k$ is the $\mathcal{O}_{Y,Q}$-module $\mathcal{O}_{Y,Q}/\mathfrak{m}_{Y,Q}$. We had defined

$$\mathcal{O}_{X_Q} := \mathcal{O}_X|_{X_Q} \otimes_{\mathcal{O}_{Y,Q}} \mathcal{O}_{Y,Q}/\mathfrak{m}_{Y,Q} \cong \mathcal{O}_X \otimes_{\mathcal{O}_{Y,Q}} \mathcal{O}_{Y,Q}/\mathfrak{m}_{Y,Q}.$$

Now take $X = \mathbb{A}^1$ with coordinate t, $Y = \mathbb{A}^1$ with coordinate u, and use $\phi\colon X \to Y$ by $t \mapsto t^2 = u$. Then $A(Y) = k[u]$, $A(X) = k[t]$, and the corresponding map $A(Y) \to A(X)$ is $k[u] \hookrightarrow k[t]$ by $u \mapsto t^2$. Consider the fiber at $Q = (u = 0)$. Using the first homework problem from today, we may compute the final expression above on the level of rings via

$$k[t] \otimes_{k[u]_{(u)}} k[u]/(u) = k[t] \otimes_{k[t^2]} k[t^2]/(t^2).$$

(Note: here we are freely using the following fact. Given a maximal ideal $\mathfrak{m}$ in a ring A, there is an isomorphism $A_\mathfrak{m}/\mathfrak{m}A_\mathfrak{m} \cong A/\mathfrak{m}$. This is because localization is an exact functor.) Recall that the last displayed equation is isomorphic to $k[t]/(t^2)$, which is not reduced. Sometimes this is written $k[\epsilon]$ where $\epsilon^2 := 0$.

Geometrically, this example is just projecting a parabola to a line, and we're taking the fiber which contains the vertex. The nilpotent is essentially remembering the tangent vector at the vertex in addition to the point itself.

444 Remark

Why are fibers computed using tensor products? The preceding lemma certainly provides a large amount of justification. Consider a morphism of affine schemes $X \to Y$, so a ring map $A(Y) \to A(X)$. Say $Q \in Y$ has corresponding maximal ideal $\mathfrak{m} \subset A(Y)$. Now X_Q is topologically obvious—the fiber of Q, which is closed. What should the corresponding ideal $I(X_Q) \subset A(X)$ be? The natural choice is $\mathfrak{m}A(X)$. Hence we have

$$0 \longrightarrow \mathfrak{m} \longrightarrow A(Y) \longrightarrow k_Q \longrightarrow 0$$

$$0 \longrightarrow \mathfrak{m}A(X) \longrightarrow A(X) \longrightarrow A(X_Q) \longrightarrow 0$$

where the second sequence can naturally be obtained from the first by applying $A(X) \otimes_{A(Y)} -$ and replacing the left-hand term of the result, $A(X) \otimes_{A(Y)} \mathfrak{m}$ with its image in $A(X)$, namely $\mathfrak{m}A(X)$.

April 22nd, 2016: Draft

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445 Remark

We've discussed fibers briefly. Since we only have so many weeks left, we'll move along even though there's lots more to say. You are invited to try to prove the next theorem.

446 Definition

Let Y be a topological space and let $\chi: Y \to \mathbb{Z}$. The continuous functions to $\mathbb{Z}$ are just the constants, but the upper semicontinuous functions are interesting, namely we require

$$\forall u \in \mathbb{Z}, \{y \in Y : \chi(y) \geq n\} = \chi^{-1}([n, \infty)) \text{ is closed.}$$

The intuition behind upper semicontinuity, say to the reals, is that limits can (weakly) increase your value but not decrease it. Note that $\chi^{-1}([n, \infty))$ forms a nested family of closed subsets.

447 Homework

Let X be a scheme, $\mathcal{F}$ a coherent sheaf (i.e. a coherent $\mathcal{O}_X$-module). Let $\delta(x) := \dim_{\kappa(x)} \mathcal{F} \otimes_{\mathcal{O}_X} \kappa(x)$ where $\kappa(x) := \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$. Prove that $\delta(X)$ is upper semicontinuous.

448 Remark

If $\mathcal{F}$ is a locally free sheaf, then $\mathcal{F} \otimes_{\mathcal{O}_X} \kappa(x)$ can be naturally identified with the (vector space) fiber of the corresponding vector bundle at x. Heuristically, any time we mod out by the maximal ideal of a local ring, we should consider applying Nakayama's lemma in some way, and the above homework is no exception.

449 Definition

Recall we defined dimension of topological spaces above in terms of lengths of maximal chains of irreducible subsets. We use this definition for the dimension of schemes. Note that the dimension before and after quotienting out by all nilpotents ("reducing") is the same.

450 Theorem (Chevalley, say)

Let $\phi: X \to Y$ be a dominant morphism of varieties, $\lambda: Y \to \mathbb{Z}$ given by $\lambda(y) := \dim X_y$. Then λ is upper semicontinuous and $d := \min_{y \in Y} \lambda(y) = \dim(X) - \dim(Y)$.

451 Remark

λ is upper semicontinuous even without the "dominant" adjective. An example of this in action arises from blow-ups at a point, say, where the fiber dimension is 0 except at the blown-up point.

452 Theorem (Chevalley)

In the notation of the preceding theorem, let

$$C_a := \{y \in Y : \dim X_y = a\} = \lambda^{-1}(a).$$

Then

(1) $C_a \subset Y$ is locally closed

(2) $C_a = \varnothing$ *for* $a < d$

(3) C_d *is a dense open set of* Y

PROOF By the theorem, $Z_a := \{y \in Y : \lambda(y) \geq a\} = \lambda^{-1}([a, \infty))$ is closed. Now $C_a = Z_a - Z_{a-1}$ is the difference of closed sets, which is the intersection of a closed set and an open set, giving (i). (ii) is immediate from the preceding theorem. For (iii), $Y = \cup_{a \in \mathbb{Z}} Y$ and $Z_a \supset Z_{a+1} \supset \cdots$, so $C_d = Y - Z_{d+1} \neq \varnothing$ is a non-empty open set.

453 Remark

The next result is not in textbooks, though the following corollary is. The assumptions are relatively mild in that in practice one can usually modify one's situation to assume they hold (e.g. by replacing Z with its image under "ψ".) Most of the argument just uses the underlying topological space; the only place where algebraic geometry really enters is the upper semicontinuity result above.

454 Theorem (Generalized Rigidity Lemma)

Let X, Y, Z *be varieties. Given*

$$\begin{array}{ccc} Z & \xrightarrow{\phi} & Y \\ {\scriptstyle\psi}\downarrow & & \\ Z & & \end{array}$$

where ψ *is surjective. Suppose that the fibers of* ψ *have the same dimension* n*. Then* $\overline{\phi(\psi^{-1}(Z))}$ *all have the same dimension for all* $z \in Z$*.*

PROOF Let $\sigma := \phi \times \psi : X \to Y \times Z$. Let $W := \overline{\mathrm{im}(\sigma)}$. Take $\pi : W \to Z$ be the projection map, so

$$\begin{array}{ccc} X & \longrightarrow & W \subset Y \times Z \\ & {\scriptstyle\psi}\searrow & \downarrow{\scriptstyle\pi} \\ & & Z \end{array}$$

Pick $z \in Z$ and note that

$$\pi^{-1}(z) = \overline{\sigma(\psi^{-1}(z))} \cong \overline{\phi(\psi^{-1}(z))}$$

since

$$\sigma(\psi^{-1}(z)) := \{(\phi(x), \psi(x)) : x \in \psi^{-1}(z)\} = \{(\phi(x), z) : x \in \psi^{-1}(z)\} \leftrightarrow \{\phi(x) : x \in \psi^{-1}(z)\}$$

where the projection $Y \times Z \to Y$ actually yields the required isomorphism of varieties. Hence $\dim \pi^{-1}(z) = \dim \overline{\phi(\psi^{-1}(z))}$.

Now pick $z_0 \in Z$ and set $m := \dim \overline{\phi(\psi^{-1}(z_0))}$. By upper semi-continuity, there exists $z_0 \in U \subset Z$ such that $\dim \pi^{-1}(z) \leq m$ for all $z \in U$ (intuitively, this is because the dimension cannot "jump down" at the closed point z_0). Now pick $w \in \pi^{-1}U \subset W$. Since $\pi^{-1}U$ is an open, non-empty subset, and W is dominated by σ so is irreducible, $\pi^{-1}U$ is dense. By definition, $\mathrm{im}\,\sigma$ is dense and non-empty. We're unfortunately out of time and will finish this next lecture.

455 Corollary (Rigidity lemma)

Take the assumptions of the previous theorem and add the following additional assumptions. Suppose the fibers of ψ *are connected and there exists* $z_0 \in Z$ *such that* $\phi(\psi^{-1}(z_0))$ *is a point. Then for all* $z \in Z$*,* $\phi(\psi^{-1}(z))$ *is a point.*

456 Remark

This is also known as "there are no bowties in algebraic geometry." The corresponding picture is that if ψ is the projection of a square onto a line segment in the x-axis, there is no morphism ϕ which "pinches" the square into a bowtie, since the pinched point would have yield dimension 0 and the others would yield dimension 1.

April 25th, 2016: Draft

457 Remark

We begin by proving the generalized rigidity lemma from last time. Recall the setup:

$$\begin{array}{ccc} X & \xrightarrow{\phi} & Y \\ {\scriptstyle\psi}\downarrow & & \\ Z & & \end{array}$$

where ψ is surjective and the fibers of ψ all have the same dimension (in this case, we call ψ equidimensional). Then for all $z \in Z$, $\phi(\psi^{-1}(z))$ has the same dimension.

Note: we weaken the claim for now to get the "rigidity lemma" from last time; Sandor will think about the more general version later. In addition to the above assumptions, further suppose ψ has connected fibers and that for all $z \in Z$, $\phi(\psi^{-1}(z))$ is a single point in Y (not necessarily the same one as z varies), which in particular holds for some $z_0 \in Z$.

Proof Pick $z_0 \in Z$ and set $m := \dim \phi(\psi^{-1}(z_0))$. Our strategy is to show that the $m \geq \dim \phi(\psi^{-1}(z))$ for all z, which gives the result using the arbitrariness of z_0. Set $\sigma := \phi \times \psi : X \to Y \times Z$ and let $\pi_Z : Y \times Z \to Z$ be the projection morphism. Set $W := \overline{\operatorname{im} \sigma}$, so we may restrict $\pi := \pi_Z|_W : W \to Z$. As we saw last time, for all $z \in Z$,

$$\pi^{-1}(z) \cap \operatorname{im} \sigma = \sigma(\psi^{-1}(z)).$$

We have a one-to-one correspondence between $\sigma(\psi^{-1}(z))$ and $\phi(\psi^{-1}(z))$, since for $x \in \psi^{-1}(x)$ we may correspond $(\phi(x), z)$ and $\phi(x)$. This correspondence is thus induced by projection onto Y, with inverse given by sending $\phi(x)$ to $(\phi(x), z)$.

By Chevalley's theorem, $\operatorname{im} \sigma$ contains an open dense subset. Hence we have some $W_0 \in \operatorname{im} \sigma$ which is dense and open in W. Hence we have $z_1 \in Z$ such that $\pi^{-1}(z_1) \cap W_0 \neq \varnothing$. Thus $\pi^{-1}(z_1) \cap \operatorname{im} \sigma$ is dense in $\pi^{-1}(z_1)$. Sandor will try to finish this later.

Proof We have $\pi^{-1}(z) \cap \operatorname{im} \sigma = \sigma(\psi^{-1}(z))$, which is in correspondence with $\phi(\psi^{-1}(z))$. Again $\operatorname{im} \sigma$ is constructible, so there is some $W_0 \subset \operatorname{im} \sigma$ which is dense and open in W. We will assume the following statement (though be careful with it...).

458 Lemma

Let $\phi : X \to Y$ be a dominant morphism. Suppose $U \subset X$ is a dense open. Then there exists a dense open subset of Y, say V, such that $\overline{X_y \cap U} = X_y$.

Assuming the lemma, we have $z_1 \in Z$ such that $\phi^{-1}(z_1) \cap W_0 \neq \varnothing$.

We will table this until next time.

459 Theorem (Affine dimension theorem)

Let $X, Y \subset \mathbb{A}^n$ be affine varieties. Suppose Z is an irreducible component of $X \cap Y$. Then $\dim Z \geq \dim X + \dim Y - n$.

460 Example

If X and Y are linear subspaces, this reduces to standard linear algebra. Note that $X \cap Y$ may be empty, in which case the result is vacuous; for instance, this occurs when using two parallel codimension 1 hyperplanes.

Proof We consider two cases. In the first, suppose $Y = Z(f)$ for $f \in k[x_1, \ldots, x_n]$ and $\dim Y = n-1$. We must show $\dim Z \geq \dim X - 1$. Consider $\overline{f} \in A(X)$. By assumption, $A(X)$ is a domain, so $\overline{f}$ is not a zero-divisor. Moreover, if it were a unit, then $X \cap Y = \varnothing$, so we may assume it is not a unit. If $\overline{f} = 0$, it follows that $\dim Z = \dim X$, so we may assume not. Hence by Krull's Principal Ideal Theorem, Z corresponds to a prime ideal $\mathfrak{p}$ in $A(X)$ which is a minimal prime containing $(\overline{f})$ with $\operatorname{ht} \mathfrak{p} = 1$. Thus $\dim Z = p - 1$.

In the second case, we identify $X \cap Y$ with $(X \times Y) \cap \Delta$ where $\Delta \subset \mathbb{A}^{2n} = \mathbb{A}^n \times \mathbb{A}^n$ where $X \times Y \subset \mathbb{A}^n \times \mathbb{A}^n$ using coordinates $x_1, \ldots, x_n; y_1, \ldots, y_n$. Now $\Delta = Y_1 \cap \cdots Y_n$ where $Y_i \coloneqq Z(x_i - y_i)$. Now pick an irreducible component of $(X \times Y) \cap Y_1$ which contains Z_1 where Z_1 contains (the image of) Z. By the first case, $\dim Z_1 \supset \dim X + \dim Y - 1$. The next step is $Z_1 \cap Y_2 \supset Z_2 \supset Z$. Continuing inductively will give $\dim Z \geq \dim X + \dim Y - n$.

April 27th, 2016: Draft

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461 Remark

Last time we showed that if $X, Y \subset \mathbb{A}^n$ are affine varieties of dimensions p, q respectively, and if $Z \subset X \cap Y$ is an irreducible component, then $\dim Z \geq p + q - n$.

462 Theorem (Projective dimension theorem)

Let $X, Y \subset \mathbb{P}^n$ be projective varieties of dimensions p, q respectively. Suppose $Z \subset X \cap Y$ is an irreducible component. Then

- $\dim Z \geq p + q - n$
- *Moreover, if $p + q - n \geq 0$, then $X \cap Y \neq \varnothing$*

Proof The dimension estimate follows from the affine dimension theorem. The non-emptiness statement is hence the heart of the matter. Let $\pi \colon \mathbb{A}n + 1 - \{0\} \to \mathbb{P}^n$. let $C(X), C(Y) \subset \mathbb{A}^{n+1}$ be the corresponding affine cones $\overline{\pi^{-1}(X)}, \overline{\pi^{-1}(Y)}$. Recall that $\dim C(X) = \dim X + 1$, etc. If $Z \subset X \cap Y$ is an irreducible component, then $C(Z) \subset C(X) \cap C(Y)$ is as well, so $\dim C(Z) \geq (p+1) + (q+1) - (n+1)$. Since $C(X) \cap C(Y) \neq \varnothing$, it follows from the dimension assumption that $C(X) \cap C(Y)$ must indeed intersect in more than just a point since their intersection is positive-dimensional. Hence $X \cap Y \neq \varnothing$, as required.

463 Remark

We finally prove the rigidity lemma from the last two lectures. We were missing an assumption, as illustrated by the following example:

464 Example

Consider

$$\begin{array}{ccc} \mathbb{A}^2 & \longrightarrow & \mathbb{A}^2 \\ \big\downarrow & & \\ \mathbb{A}^1 & & \end{array}$$

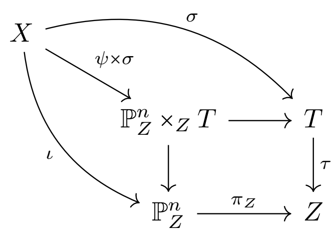
given by $(x, y) \mapsto (x, xy)$ and $(x, y) \mapsto x$. The image of this morphism is a pinched plane, which is exactly what the rigidity lemma disallows.

It turns out we need to assume the vertical arrow is projective. We'll need a generalization of an earlier lemma. Recall that several weeks ago we had shown that the composite of projective morphisms is projective.

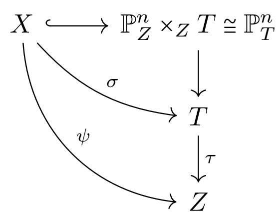
465 Lemma

Let $\psi: X \to Z$ be given by $X \overset{\sigma}{\to} T \overset{\tau}{\to} Z$ by a projective morphism. Then σ is also projective. That is, factoring a projective morphism means the first map is also projective.

PROOF By assumption we have $\iota: X \hookrightarrow \mathbb{P}^n_Z := Z \times \mathbb{P}^n$ where $\psi: X \to Z$ is $X \to \mathbb{P}^n_Z \to Z$. Consider the pullback diagram



where we are using $\mathbb{P}^n_Z \times_Z T = \{(a, b) \mid \forall a \in \mathbb{P}^n_Z, b \in T, \pi_Z(a) = \tau(b)\}$ so $x \mapsto (\iota(x), \sigma(x)) \in \mathbb{P}^n_Z \times T$ since $\pi_Z(\iota(x) = \psi(x) = \tau(\sigma(x)))$. Homework: verify that these are all morphisms of varieties. Now we have

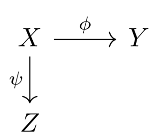


As homework, check all the little details, for instance, verify that $\mathbb{P}^n_Z \times_Z T \cong \mathbb{P}^n_T$.

Warning: the scheme-theoretic version of the preceding lemma strictly speaking requires τ to be "separated," which is automatic for varieties.

466 Lemma (Generalized Rigidity Lemma)

Let



where X, Y, Z are varieties, ψ is surjective, equidimensional, and projective (or proper, though we'll stick with projective here). Then $\dim \phi(\psi^{-1}(z))$ is constant for all $z \in Z$.

PROOF Again consider $\sigma := \phi \times \psi: X \to Y \times Z$. By the lemma, σ is projective, since the composite $X \to Y \times Z \overset{\pi_Z}{\to} Z$ is just ψ, which is projective. Hence σ is a closed morphism. (Properness is really what's needed because the preceding lemma works for properness and it implies closedness. Any reasonable notion satisfying these two properties will satisfy the theorem.) Now set $W := \operatorname{im} \sigma \subset Y \times Z$, which is closed (yay!). One more time, we find

$$\pi^{-1}(z) = \sigma(\psi^{-1}(z)) \leftrightarrow \phi(\psi^{-1}(z))$$

where the $\leftrightarrow$ indicates we have a natural bijective morphism from the left to the right (Sandor doesn't want to worry about non-reducedness; it's fine). Pick $z_0 \in Z$ and set $m := \dim \phi(\psi^{-1}(z_0))$.

The preceding computation shows that $\dim \pi^{-1}(z_0) = m$. By the upper semi-continuity of fibers, there exists an open $z_0 \in U \subset Z$ such that for all $z \in U$, $\dim \pi^{-1}(z) \le m$. Now consider $\pi^{-1}U \subset W$. This is open, non-empty, and W is irreducible (being the image of the irreducible X), so $\pi^{-1}U$ is dense. Pick $w \in \pi^{-1}U$ so $z := \pi(w) \in U \subset Z$. Now $\sigma^{-1}(\pi^{-1}(z)) = \psi^{-1}(z)$ has some dimension n independent of z (for all $z \in Z$, even) since ψ is equidimensional. Since $W = \operatorname{im} \sigma$, σ is surjective onto W, so our earlier dimensionality result using $\sigma^{-1}(\pi^{-1}(z)) \to \pi^{-1}(z)$ says the fibers are of timension at least $n - m$. In particular, $\dim \sigma^{-1}(w) \ge n - m$. By upper semicontinuity, this last estimate holds for any $w \in W$. Now for an arbitrary $w \in W$, set $z := \pi(w)$ and consider $\sigma^{-1}(\pi^{-1}(z)) \to \pi^{-1}(z)$. The dimension of the domain is again n, and the fiber dimension is at least $n - m$, and since the minimal fiber dimension is exactly the difference in dimensions, it follows that the dimension of $\pi^{-1}(z)$ is at most m, now for all $z \in Z$ and not just $z \in U$. Now letting the base point z_0 vary gives the result.

467 Lemma (Rigidity Lemma)

In addition to the assumptions of the previous lemma, further suppose the fibers of ψ are connected, and that there is some z_0 such that $\phi(\psi^{-1}(z_0))$ is a point. The conclusion is then that $\phi(\psi^{-1}(z))$ is a point for all $z \in Z$.

Proof This follows immediately from the generalized rigidity lemma and the fact that a connected zero dimensional fiber is a point.

April 29th, 2016: Draft

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468 Remark

Today we'll discuss Weil divisors.

469 Notation

Let X be a quasi-projective variety with $\dim \operatorname{Sing}(X) \le \dim X - 2$. (Recall that we showed this means X is R_1, meaning if we localize at a height 1 prime, we get a regular local ring.)

470 Definition

Let $Z \subset X$ be a codimension 1 subvariety. We define $\mathcal{O}_{X,Z}$, the local ring of Z in X. Let $U \subset X$ be an open affine, so $U \cap Z \neq \varnothing$. Pick a prime $\mathfrak{p}$ in the domain $A(U)$ with $\mathfrak{p} = I(U \cap Z)$. Then

$$\boxed{\mathcal{O}_{X,Z}} := A(U)_{\mathfrak{p}}.$$

471 Homework

Prove that $\mathcal{O}_{X,Z}$ is independent of $U \subset X$. Hint: first show independence for $U' \subset U$. Restrict to a standard open $D(g) \subset U'$ in U and check $D(g) = D(g|_{U'})$. Check that $A(D(g)) \cong A(U)_g$ and fill in the remaining details.

Since X is R_1 in our context, $\mathcal{O}_{X,Z}$ is a DVR, so we have a valuation $\sigma_Z : K(X)^\times \to \mathbb{Z}$ for each such Z.

472 Definition

A $\boxed{\text{prime divisor}}$ is a codimension 1 closed subvariety. (The name may be connected to unique factorization in Dedekind domains; for instance, over $\mathbb{Z}$ the primes form a multiplicative monoid basis, and over $\mathbb{Q}$ this extends to a group basis.) A $\boxed{\text{Weil divisor}}$ is an element in the free abelian group $\boxed{\operatorname{Div}(X)}$ by the prime divisors. That is, an element $D \in \operatorname{Div}(X)$ is a finite formal sum

$$D = \sum_i a_i D_i$$

where each $a_i \in \mathbb{Z}$ and D_i is a codimension 1 subvariety of X. The support of D is the union of D_i's for which the coefficient a_i is non-zero,

$$\operatorname{supp} D := \cup_{a_i \neq 0} D_i.$$

$\operatorname{Div}(X)$ has a natural ordered group structure $(\operatorname{Div}(X), \geq)$ where the order is defined by

$$D \geq 0 \quad \text{if} \quad \forall i,\ a_i \geq 0$$

so that

$$\sum_i a_i D_i = D \geq D' = \sum_i a_i' D_i' \quad \text{if} \quad \forall i,\ a_i \geq a_i'.$$

A divisor with non-negative coefficients is effective, i.e. when $D \geq 0$.

Given a prime divisor $D \subset X$ with valuation $\sigma_D \colon K(X)^\times \to \mathbb{Z}$, pick $f \in K(X)^\times$. If $\sigma_D(f) > 0$ we say f has a zero along D of order $\sigma_D(f)$. If $\sigma_D < 0$, we say f has a pole along D of order $-\sigma_D(f)$.

473 Lemma

With X as above, for any $f \in K(X)^\times$, consider

$$\{D \text{ prime divisor} : \sigma_D(f) \neq 0\}.$$

This set is finite.

Proof By definition of $K(X)$, we have some $\varnothing \neq U \subset X$ open such that $f|_U$ is regular. Now $X - U$ is a proper closed subset of X, so it has finitely many irreducible components. Since any elements of the set in $X - U$ are top-dimensional in $X - U$, it follows that there are at most finitely of them, so we may assume f is regular on X with X affine. For any $f \in A(X)$, $\sigma_D(f) \geq 0$, and the set in question consists precisely of irreducible components of $Z(f) \subset X$, which is finite.

474 Remark

The intuition behind this argument comes from considering ratios of polynomials in one variable. The zeros of the numerator correspond to connected components of its zero set, etc. (Of course, this can't literally by done in general.)

475 Definition

Given $f \in K(X)^\times$, the divisor of f is

$$\div(f) := \sum_{D \text{ prime div.}} \sigma_D(f) D \in \operatorname{Div}(X).$$

A divisor $D \in \operatorname{Div}(X)$ is called a principal divisor if there exists $f \in K(X)^\times$ such that $D = \div(f)$.

476 Remark

We've been saying just "divisor" and meaning "Weil divisor." So far this has not been confusing, but later we'll get another type of divisor which will make this convention superficially ambiguous. Weil divisors will be the most general type, so this convention actually does no harm.

If $D, D' \in \operatorname{Div}(X)$, we say D and D' are linearly equivalent, written

$$D \sim D'$$

if there exists $f \in K(X)^\times$ such that $D - D' = \operatorname{Div}(f)$. This is an equivalence relation. Indeed, since $v_D(f/g) = v_D(f) - v_D(g)$, it follows that $\div \colon K(X)^\times \to \operatorname{Div}(X)$ is a group homomorphism (multiplicative to additive) with image given by the principal divisors. Linear equivalence is then just equality in the quotient of $\operatorname{Div}(X)$ by this image.

The divisor class group of X is

$$\boxed{\mathrm{Cl}(X)} := \mathrm{Div}(X)/\sim .$$

This turns out to be a very important invariant of X. For instance, it can be used to determine non-isomorphism.

477 Homework

Let X be an affine variety. Prove that $A(X)$ is a UFD if and only if X is normal and $\mathrm{Cl}(X) = 0$. (This can be seen as a generalization of the similar result for Dedekind rings.) As a corollary, $\mathrm{Cl}(\mathbb{A}^n) = 0$.

478 Remark

Functoriality of $\mathrm{Div}(X)$ and $\mathrm{Cl}(X)$ is delicate. In nice circumstances, one can use pullback on divisors, but then one needs linear equivalence to be preserved, which could certainly go wrong. For non-singular varieties, there is no problem, but in general there is at best a non-functorial way to make this work.

479 Example

Let $X \subset \mathbb{P}^n$ with a divisor $D \subset X$. Further suppose $H \subset \mathbb{P}^n$ is a hyperplane and $D = H|_X = H \cap X$. It turns out that any other divisor linearly equivalent to D is of the same form, i.e. it is the intersection of X with some hyperplane of $\mathbb{P}^n$. In some sense this explains the term "linear equivalence."

In this case, if D is locally given by $f = 0$ on X and $D \sim D'$, then D' is locally $f \cdot \frac{g_1}{g_2} = 0$ where $\deg g_1 = \deg g_2$. Using a d-uple embedding $\mathbb{P}^n \hookrightarrow \mathbb{P}^N$ given by

$$[x_0 : \cdots : x_n] \mapsto [\cdots : x^\alpha : \cdots]$$

(where α is a weak composition of size d), then we can essentially view g_1 and g_2 as linear.

For divisors on $\mathbb{P}^n$, it happens that divisors are linearly equivalent if and only if there is an automorphism of $\mathbb{P}^n$ taking one to the other, but this is quite special to $\mathbb{P}^n$ using the fact that $\mathrm{Cl}(\mathbb{P}^n) = \mathbb{Z}$, which is generated by the class of a hyperplane.

If $D \subset \mathbb{P}^n$ is a divisor, then $D = \sum_i a_i D_i$ for prime divisors D_i, where recall a codimension 1 subvariety D_i must be a hypersurface $D_i = V(f_i)$, so we may define $\deg D_i$ to mean the degree of the hypersurface. We may then define $\deg D := \sum_i a_i \deg D_i$. One may show that the degree of a principal divisor is then zero, roughly because the number of zeros and the number of poles must agree, and $\deg\colon \mathrm{Cl}(\mathbb{P}^n) \to \mathbb{Z}$ is in fact a group isomorphism. Indeed, given any $\sum_i a_i D_i$ with $D_i = Z(f_i)$ and $\sum_i a_i \deg D_i = 0$, one may check $\prod_i f_i^{a_i}$ realizes this divisor as a principal divisor. If X is one-dimensional, you can also define the degree by declaring points have degree 1. One may also define degree *up to embedding in* $\mathbb{P}^n$, though different embeddings into different projective spaces could give different notions of degree.

May 2nd, 2016: Draft

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480 Remark

One might ask why we might care about divisors and the divisor class group. At a rough level, algebraic geometry doesn't have a great notion of homology (it does have a great notion of cohomology), and linear equivalence gives a reasonable substitute at least in codimension 1. A generalization without the codimension 1 restriction is given by replacing linear equivalence with rational equivalence, though we won't discuss this.

481 Remark

Whenever we talk about Weil divisors $D \in \mathrm{Div}(X)$, we tacitly assume $\dim \mathrm{Sing}\, X \le \dim X - 2$, as mentioned above, where X is a quasi-projective variety.

482 Notation

Let $\mathcal{F}$ be a sheaf on a topological space X, $U \subset X$ open. We will sometimes use the notation

$$\boxed{\Gamma(U, \mathcal{F})} := \mathcal{F}(U).$$

The advantage of this notation is that we can roughly think of Γ as functorial in each argument, $\Gamma(-, \mathcal{F})$ or $\Gamma(U, -)$.

483 Definition

Let $D \in \mathrm{Div}(X)$. Define the $\boxed{\text{sheaf associated to } D}$, $\boxed{\mathcal{O}_X(D)}$, given as a presheaf by

$$\Gamma(U, \mathcal{O}_X(D)) := \{f \in K(X)^\times : (D + \div(f))|_U \ge 0\} \cup \{0\}.$$

That is, take the rational functions whose poles on U are not worse than D on U. To be clear, for $D \in \mathrm{Div}(X)$ with $D = \sum_i a_i D_i$, we define the $\boxed{\text{restriction of } D \text{ to } U}$ as

$$\boxed{D|_U} := \sum_i a_i (D_i \cap U).$$

(One sometimes defines the divisor of 0 to have an infinite-order zero at every point, in which case we don't need to formally add 0 back in above.)

484 Remark

Sometimes people write $\mathrm{Div}(f) = (f)_0 - (f)_\infty$ where the zero subscript means the part of $\mathrm{Div}(f)$ with non-negative coefficients, and the ∞ subscript means the rest of $\mathrm{Div}(f)$. These are thought of as the zeros and poles, respectively.

485 Remark

This clearly restricts to smaller open sets. The group operation is *addition*, not multiplication. We have

$$\mathrm{Div}(f + g) = \sum_\Delta \sigma_\Delta(f + g) \cdot \Delta$$

where Δ varies over prime divisors of X and σ_Δ is the corresponding discrete valuation. This satisfies $\sigma_\Delta(f + g) \ge \min\{\sigma_\Delta(f), \sigma_\Delta(g)\}$, from which it follows in the above case that $\mathrm{Div}(f + g) \ge -D$. Hence the group is closed under addition, and also under inverses.

One may remember the valuation condition in this context by imagining what happens to the order of vanishing of the sum of two rational functions in one variable.

486 Homework

Prove that the above presheaf is actually a sheaf, and indeed a coherent $\mathcal{O}_X$-module.

From the definition, we see that $\mathcal{O}_X(D)$ is naturally a sub-presheaf of $\mathcal{K}_X$, the constant sheaf on $K(X)$. The "zero" presheaf condition is then trivial.

487 Example

If $D = 0$, the condition above says $\mathrm{Div}(f) \ge 0$ on X, meaning f is regular, so that $\mathcal{O}_X(0) = \mathcal{O}_X$.

488 Remark

One may check effectiveness locally, i.e. D is effective if and only if $D|_U$ is effective for all U in an open cover. To verify this statement, one must note that $D_i \cap U = D_j \cap U \ne \varnothing$ implies $D_i = D_j$ essentially because U is a dense open in X. Note, though, that if we replace U with a *closed* set in X, we can easily find intersections where this fails, e.g. two curves may intersect in a point and we may take U to be a line through that point intersecting each curve only there.

489 Remark

Can one check whether $D \geq 0$ using $\mathcal{O}_X(D)$? Let $D \in \operatorname{Div}(X)$. Then there exists $D' \in \operatorname{Div}(X)$ such that $D \sim D'$ and $D' \geq 0$ if and only if $\Gamma(X, \mathcal{O}_X(D)) \neq 0$. Indeed, given a non-zero global section $f \in K(X)^\times$ such that $D + \operatorname{Div}(f) \geq 0$, use $D' = D + \operatorname{Div}(f)$.

490 Proposition

Let $D, D' \in \operatorname{Div}(X)$. Then $D \sim D' \Leftrightarrow \mathcal{O}_X(D) \cong \mathcal{O}_X(D')$. (This is an abstract $\mathcal{O}_X$-module isomorphism; it is not a $\mathcal{K}_X$-submodule isomorphism.)

491 Homework

Prove the proposition.

492 Homework

As $\mathcal{O}_X$-modules, $\mathcal{O}_X(-D) \cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X(D), \mathcal{O}_X)$, that is, $\mathcal{O}_X(-D)$ is the dual of $\mathcal{O}_X(D)$.

493 Corollary

$\mathcal{O}_X(D)$ is a reflective $\mathcal{O}_X$-module, i.e. its double dual is itself. (More specifically, these are reflective modules of rank 1, which roughly arises from the fact that restricting $\mathcal{O}_X(-D)$ to the complement of D gives $\mathcal{O}_X$, so it is "generically free of rank 1.")

494 Notation

For the rest of the lecture, let X be a non-singular curve.

495 Remark

For curves, we may as well write $D = \sum_{P \in X} a_P P$, since the prime divisors are points.

496 Proposition

Suppose X is a non-singular projective curve and $\phi: X \to Y$ is a non-constant morphism from X to Y, where Y is a curve (not necessarily non-singular or projective). Then

(i) $\phi(X) = Y$

(ii) Y *is projective*

(iii) ϕ *is a finite morphism*

(iv) $\phi^*: K(Y) \hookrightarrow K(X)$ *is injective and induces a finite field extension*

Proof For (i), X is projective, so $\phi(X) \subset Y$ is closed and projective, since Y is quasi-projective. Also, X is irreducible, so Y is irreducible. Hence $\phi(X)$ is a closed irreducible subset of the curve Y which is not a point, and Y's topology forces $\phi(X) = Y$. Hence Y is indeed projective. Since ϕ is surjective, it is dominant, so $\phi^*: K(Y) \hookrightarrow K(X)$ is injective. Since each of these are finitely generated field extensions of the base field k and they both have transcendence degree 1, it follows that ϕ^* must be a finite field extension, giving (iv).

For $X \xrightarrow{\phi} Y$ and V affine open, we want to show the induced map $\phi: \phi^{-1}V \to V$ induces a finite morphism on rings. For this, take $A(V) \subset K(Y) \subset K(X)$ and consider its integral closure $\widetilde{A(V)}$ in $K(X)$. Emmy Noether's theorem says that $\widetilde{A(V)}$ is a finite $A(V)$-module and a finitely generated k-algebra. In particular, $\widetilde{A(V)}$ has dimension 1, and is in fact a Dedekind ring. Now finitely generated k-algebras which are Dedekind rings are coordinate rings of non-singular affine curves, i.e. we have U non-singular affine curve such that $\widetilde{A(V)} \cong A(U)$. Note that $K(X) \cong \operatorname{Frac} \widetilde{A(V)}$, whence U is birational to X. We proved every such U embeds into X, which in fact identifies U with $\phi^{-1}V$. This proves that U is affine and that ϕ is a finite morphism.

May 4th, 2016: Draft

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497 Remark

The proposition from the end of last lecture illustrates a general principle for working with curves: there's "not a lot of room" for things to go horribly wrong. For instance, we used the fact that we can completely describe the Zariski topology on curves in the above. Already for surfaces, things generally get much harder/more varied.

498 Example

Recall the typical example of a non-finite morphism of curves, essentially given by "poking a hole" in the domain curve at a single point. The result is quasi-finite, but the ring extension isn't finite.

499 Definition

Let $\phi\colon X \to Y$ be a (finite) non-constant morphism of curves. Now $\phi^*\colon K(Y) \hookrightarrow K(X)$, and we define the $\boxed{\text{degree}}$ of the morphism ϕ to be $\boxed{\deg \phi} := \deg[K(X) : K(Y)]$.

500 Remark

It is entirely possible that $K(Y)$ does not embed into $K(X)$. If $X = \mathbb{P}^1$ and Y is a non-singular projective cubic, then $K(Y)$ does not embed in $K(X)$. In a similar vein, we have $K(\mathbb{P}^1) \cong k(t^n) \subset k(t) = K(\mathbb{P}^1)$.

On the other hand, suppose $k = \overline{k} \subset K \subset k(t)$ and K/k has transcendence degree > 0; it is a fact that $K \cong k(t)$.

Given ϕ as above where moreover X, Y are non-singular, we may define $\phi^*\colon \mathrm{Div}(Y) \to \mathrm{Div}(X)$ as follows. We think of $Q \in Y$ as being represented by the DVR $\mathcal{O}_{Y,Z}$ with maximal ideal $\mathfrak{m}_{Y,Q} = (t)$; we call such t a $\boxed{\text{local parameter}}$ at Q, meaning it generates the maximal ideal of the local ring. Here we imagine $t \in K(Y)$. We have $\phi^*\colon K(Y) \to K(X)$ which induces morphisms $\mathcal{O}_{Y,Q} \to \mathcal{O}_{X,P}$ whenever $\phi(P) = Q$. In this way, we can define

$$\phi^*(Q) := \sum_{P \in \phi^{-1}Q} v_P(\phi^* t) P$$

where v_P is the valuation corresponding to P. Indeed, we may sum over $P \in X$, since points not in the preimage will have zero evaluation. ϕ^* is extended $\mathbb{Z}$-linearly to all of $\mathrm{Div}(Y)$.

501 Remark

We have $\phi^* \mathrm{Div}(f) = \mathrm{Div}(\phi^* f)$, as follows. We see $v_Q(f) = r$ if $f \in \mathfrak{m}_{Y,Q}^r - \mathfrak{m}_{Y,Q}^{r+1}$. If t generates $\mathfrak{m}_{Y,Q}$, this is equivalent to saying $f = ut^r$ for some unit in $\mathcal{O}_{Y,Q}$. Equivalently, $\phi^* f = \phi^* u \cdot \phi^* t^r$, so that $v_P(\phi^* f) = v_Q(f) v_P(\phi^* t)$. Hence

$$\begin{aligned}\mathrm{Div}(\phi^* f) &= \sum_P v_P(\phi^* f) P = \sum_P v_Q(f) v_P(\phi^* t) \\ &= \phi^*(v_Q(f) Q) = \phi^* \mathrm{Div}(f).\end{aligned}$$

Consequently, $\phi^*\colon \mathrm{Cl}(Y) \to \mathrm{Cl}(X)$ is well-defined.

502 Lemma

With the above notation, $\deg \phi^* D = \deg \phi \cdot \deg D$.

Proof Homework.

503 Lemma

With the above notation and X projective, $f \in K(X)^\times$, then $\deg \mathrm{Div}(f) = 0$.

Proof If $f \in k$, then $\mathrm{Div}(f) = 0$, and this is trivial. If $f \notin k = \overline{k}$, consider $k(f) \subset K(X)$. This must be a transcendence degree 1 extension, so that $k(f) \cong K(\mathbb{P}^1) \hookrightarrow K(X)$. The category of fields with k-morphisms is essentially the same as the category of curves with regular morphisms, so we have a morphism $\phi\colon X \to \mathbb{P}^1$, which is in fact dominant. (More traditionally, you might think of ϕ as literally f, where the base field is thought of as $\mathbb{A}^1$

and wherever f is not defined it's mapping to ∞. This discussion does not require X to be projective; it is a standard trick.)

This morphism is finite by the proposition from last time using projectivity of X. Also, $\mathrm{Div}(f) = \phi^*(0 - \infty)$, where $0 = [0:1]$ and $\infty = [1:0]$, with local parameters x/y and y/x in the usual way. Hence

$$\deg \mathrm{Div}(f) = \deg \phi^*(0 - \infty) = \deg(\deg \phi \cdot \deg(0 - \infty)) = \deg \phi \cdot \deg 0 = 0.$$

504 Remark

Puncturing X destroys finiteness in the preceding argument. This preserves $\deg \phi \cdot \deg D$, while in general it will mutate $\deg \phi^* D$. However, one may show that $\deg \phi^* D \le \deg \phi \cdot \deg D$ if D is effective even without ϕ being finite, essentially because one may pass to the projective closure which does not change birational equivalence classes.

505 Definition

Let X be a non-singular projective curve. We now have

$$\begin{array}{ccc} \deg\colon \mathrm{Div}(X) & \longrightarrow & \mathbb{Z} \\ & \searrow & \uparrow \\ & & \mathrm{Cl}(X) \end{array}$$

where indeed $\deg\colon \mathrm{Cl}(X) \twoheadrightarrow \mathbb{Z}$ is a surjective group homomorphism (since a point maps to 1). Define

$$\boxed{\mathrm{Cl}^0(X)} := \ker \deg\,.$$

506 Example

We have $\mathrm{Cl}(\mathbb{P}^1) \cong \mathbb{Z}$. This is essentially because given any set of points with multiplicities adding up to 0, we can find a rational function with poles and zeros at those points of those multiplicities. It follows that $\mathrm{Cl}^0(\mathbb{P}^1) = 0$.

507 Definition

Call X a rational variety if it is birational to $\mathbb{P}^n$. In particular, X is a rational curve if it is birational to $\mathbb{P}^1$.

508 Corollary

Let X be a non-singular curve. Then X is rational if and only if there exists $P \neq Q \in X$ where P is linearly equivalent to Q.

Proof (Sketch.) ($\Rightarrow$) is easy. For ($\Leftarrow$), we have $f \in K(X)^\times$ with $\mathrm{Div}(f) = P - Q$, and $\phi\colon X \to \mathbb{P}^1$. Now $\phi^*([0,1]) = P$ has degree 1, so $\deg \phi = 1$, so by definition the function fields are isomorphic, so ϕ is birational.

May 6th, 2016: Draft

Missed.

May 9th, 2016: Draft

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509 Remark

Last time we defined the Picard group, which is roughly thought of as the group of invertible sheaves, where the group operation is tensor product. The Picard group exists even for "horrible" schemes. Sometimes Cartier divisors are more useful than Weil divisors, and vice versa.

Recall that by definition a Cartier divisor is a section of a certain quotient sheaf, but we showed there is a natural map from Cartier divisors to Weil divisors (it respects linear equivalence and principality) so we may identify Cartier divisors with their image under this map. Note that not every Weil divisor arises in this fashion in general.

Further recall that diven a divisor D in X, we had defined a subsheaf $\mathcal{O}_X(D) \subset \mathcal{K}_X$.

510 Proposition

Let X be a quasi-projective variety with $\dim \operatorname{Sing} X \leq \dim X - 2$. Let D, D' be Cartier divisors on X. Then:

(i) $\mathcal{O}_X(D)$ is an invertible sheaf;

(ii) the assignment $D \mapsto \mathcal{O}_X(D)$ gives a bijective correspondence between Cartier divisors and invertible subsheaves of $\mathcal{K}_X$;

(iii) $\mathcal{O}_X(D - D') \cong \mathcal{O}_X(D) \otimes \mathcal{O}_X(D')^{-1}$

(iv) $D \sim D'$ if and only if $\mathcal{O}_X(D) \cong \mathcal{O}_X(D')$ as $\mathcal{O}_X$-modules

511 Remark

We could get away without the singular set assumption at the cost of more setup. (ii) and (iv) might seem to be at odds; (ii) compares $\mathcal{O}_X(D)$'s up to equality, whereas (iv) compares them up to abstract isomorphism.

Proof Recall that we had

$$\Gamma(U, \mathcal{O}_X(D)) = \{f \in K(X)^\times : (\operatorname{Div}(f) + D)|_U \geq 0\} \cup \{0\}.$$

Further, a Cartier divisor D is a global section of $\mathcal{K}_X^\times/\mathcal{O}_X^\times$, or equivalently it is a family $\{(U_i, f_i)\}$ where $\{U_i\}$ is an open (affine?) cover of X and $f_i \in K(X)^\times$ such that $f_i/f_j \in \mathcal{O}_X^\times(U_i)$. Let $U \subset U_i$, $D|_U = \operatorname{Div}(f_i)|_U$. Now

$$\Gamma(U, \mathcal{O}_X(D)) = \{f \in K(X)^\times : \operatorname{Div}(f f_i)|_U \geq 0\} \cup \{0\}.$$

Hence $g := f f_i \in \Gamma(U, \mathcal{O}_X)$ for $f \in \Gamma(U, \mathcal{O}_X(D))$. That is, $\Gamma(U, \mathcal{O}_X(D)) = \Gamma(U, \mathcal{O}_X) f_i^{-1}$. (This computation was quite general.) Indeed, as subsheaves of $\mathcal{K}_X|_{U_i}$, we have $\mathcal{O}_X(D)|_{U_i} \cong \mathcal{O}_{U_i} \cdot f_i^{-1} \subset \mathcal{K}_X|_{U_i}$. This is isomorphic to $\mathcal{O}_{U_i}$, though it is not literally equal to $\mathcal{O}_{U_i}$. In particular, $\mathcal{O}_X(D)$ is an invertible sheaf.

From this description, (ii) also follows immediately. Indeed, given an invertible subsheaf $\mathcal{L} \subset \mathcal{K}_X$, we have some open cover $X = \cup U_i$ such that $\mathcal{L}|_{U_i} \cong \mathcal{O}_{U_i}$, where $\mathcal{L}|_{U_i} \subset \mathcal{K}_X|_{U_i}$. Take f_i to be the inverse of the image of $1 \in \mathcal{O}_{U_i}$ in $\mathcal{K}_X$ under these maps. Then one may quickly check that $\{(U_i, f_i)\}$ is the Cartier divisor giving birth to this $\mathcal{L}$.

For (iii), the above computation gives the following. If D is defined by $\{(U_i, f_i)\}$ and D' is defined by $\{(U_i, f_i')\}$, then $\Gamma(U, \mathcal{O}_X(D - D')) = \Gamma(U, \mathcal{O}_X) \cdot (f_i/f_i')^{-1}$. Hence, $\mathcal{O}_X(D - D') = \mathcal{O}_X(D) \cdot \mathcal{O}_X(D')^{-1} \subset \mathcal{K}_X$.

We had a claim in a recent lecture: $\mathcal{H}om(\mathcal{L}, \mathcal{O}_X) \otimes \mathcal{L} \cong \mathcal{O}_X$. If $\mathcal{L}$ is a subsheaf of $\mathcal{K}_X$, then locally $\mathcal{L}$ is $\mathcal{O}_U \cdot f$, and this expression looks like $\operatorname{Hom}(\mathcal{O}_U \cdot f, \mathcal{O}_U) \otimes (\mathcal{O}_U \cdot f)$, and the left-hand side is naturally generated by $\mathcal{O}_U \cdot f^{-1}$. There is a natural map $\mathcal{O}_X(D) \otimes \mathcal{O}_X(D')^{-1} \to \mathcal{O}_X(D) \cdot \mathcal{O}_X(D')^{-1}$. Restricting to any open set on which both sheaves are trivial, (iii) follows. Warning: the tensor product of two ideals is not their product; the preceding computation is special to invertible sheaves.

For (iv), by (iii) it is enough to prove that D is principal if and only if $\mathcal{O}_X(D) \cong \mathcal{O}_X$. Recall that D is principal if and only if D is represented by $\{(X, f)\}$. The preceding computation again holds, but where $\mathcal{O}_X(D)|_X \cong \mathcal{O}_X \cdot f \cong \mathcal{O}_X$.

512 Corollary

We have a natural injective group homomorphism from the Cartier class group to the Picard group,

$$\mathrm{CaCl}(X) \hookrightarrow \mathrm{Pic}\, X.$$

Indeed, this is an isomorphism.

Proof Consider $\mathcal{L} \in \mathrm{Pic}\, X$. Now $\mathcal{L}|_U \cong \mathcal{O}_U$, and $\mathcal{L} \otimes \mathcal{K}_X|_U \cong \mathcal{K}_U$. Note that $\mathcal{L} \otimes \mathcal{K}_X$ is the constant sheaf, since it is a constant sheaf on an open cover, and it is constantly $\mathcal{K}_X$. Since $\mathcal{O}_X \hookrightarrow \mathcal{K}_X$, tensoring with this gives a natural map $\mathcal{L} \to \mathcal{L} \otimes \mathcal{K}_X \cong \mathcal{K}_X$, and one can direct check this is injective.

513 Aside

We remark that $\deg D \geq 0$ and $D \geq 0$ are quite different statements; the former roughly says D has more zeros than poles, whereas the latter roughly says D has no poles ("pointwise"); similar remarks apply to combinations as in $D - D'$. We always have a map from Weil divisors to integers given by $\sum a_i D_i \mapsto \sum a_i$, but this typically doesn't respect linear equivalence so it's not particularly useful. Sometimes we can choose values of $\deg D_i$ for all prime divisors D_i such that the induced group homomorphism respects linear equivalence. This is possible in projective space and some other cases, but not in general.

A simple example of this subtlety is that a circle in $\mathbb{P}^2$ is linearly equivalent to the union of two lines, so the circle needs to have its degree be twice the degree of the lines, say.

May 11, 2016: Draft

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514 Remark

We'll next discuss linear systems, with a goal of discussing why divisors are very, very useful.

515 Notation

Let X be a projective variety (in contrast to the quasi-projective assumption from the last few lectures) and let D be a Cartier divisor.

516 Definition

The complete linear system corresponding to D, written as $|D|$, is

$$\{D' \text{ Cartier divisors } : D' \geq 0, D' \sim D\}.$$

517 Remark

We could have written $D' \in \mathrm{Div}(X)$, which would have been equivalent since being a Cartier divisor is preserved under linear equivalence, since the difference is principal, which is certainly locally principal.

518 Example

If $D = -H$ where $H \subset \mathbb{P}^n$ is a hyperplane, then $|D| = \varnothing$. This is because $\deg\colon \mathrm{Cl}(\mathbb{P}^n) \xrightarrow{\sim} \mathbb{Z}$; the degree of this is -1, while the degree of any effective divisor is non-negative.

519 Theorem

Let X be a projective variety over k, $\mathcal{F}$ a coherent sheaf. (Recall that a coherent sheaf locally corresponds to finitely generated modules.) Then $\Gamma(X, \mathcal{F})$ is a finite dimensional vector space over k.

Proof We will not have time to discuss the proof. This fact is true but not trivial; for instance, it fails without the projective assumption. One may replace k with $\Gamma(X, \mathcal{O}_X)$, which at least works in the most trivial examples using affine space.

520 Proposition

Let X be projective, D a Cartier divisor. Let $\mathcal{L} \cong \mathcal{O}_X(D)$, which is an invertible sheaf. Then we have a surjective map

$$\delta \colon \Gamma(X, \mathcal{L}) - \{0\} \twoheadrightarrow |D|$$

such that for any $s, s' \in \Gamma(X, \mathcal{L}) - \{0\}$, $\delta(s) = \delta(s')$ if and only if there exists $\lambda \in k^\times$ such that $s' = \lambda s$.

Proof The first part works without the "projective" assumption, but it's required for the second part.

First recall the description of $\Gamma(X, \mathcal{L}) = \Gamma(X, \mathcal{O}_X(D)) = \{f \in K(X)^\times : \mathrm{Div}(f) + D \geq 0\} \cup \{0\}$. Removing $\{0\}$, a section s is then really an f, and we may define $\delta(s) := \div(f) + D$, which is in $|D|$ trivially. This is clearly surjective.

For the second statement, let s, s' correspond to f, f', so that $\delta(s) = \delta(s')$ iff $\mathrm{Div}(f) = \mathrm{Div}(f')$, so iff $\mathrm{Div}(f'/f) = 0$, or iff $f'/f \in \Gamma(X, \mathcal{O}_X^\times) \subset \Gamma(X, \mathcal{O}_X) = k$ since X is projective. Note that f'/f is non-zero since neither f nor f' is. This completes the proof.

521 Remark

If we have $D, \overline{D}$ with $D \sim \overline{D}$, then $\mathcal{O}_X(D) \cong \mathcal{O}_X(\overline{D})$, and indeed we have a commutative diagram

$$\begin{array}{ccc} \Gamma(X, \mathcal{O}_X(D)) - \{0\} & \longrightarrow & D \\ & \searrow^{\cong} & \uparrow \\ & & \Gamma(X, \mathcal{O}_X(\overline{D})) - \{0\} \end{array}$$

Indeed, $|D| = |\overline{D}|$ set-theoretically. In this sense, we may view the linear equivalence class of D as represented by some invertible sheaf $\mathcal{L}$, and get a well-defined, otherwise-choice-free map to the corresponding complete linear system.

522 Corollary

$|D| \cong \mathbb{P}(\Gamma(X, \mathcal{O}_X(D))) := (\Gamma(X, \mathcal{O}_X(D)) - \{0\})/k^\times$.

523 Definition

A linear system on X with respect to D is a set $\mathfrak{d} \subset |D|$ such that $\mathfrak{d}$ corresponds to a linear subspace L of $\mathbb{P}(\Gamma(X, \mathcal{L}))$ (where L itself is the projectivization of a vector subspace of $\Gamma(X, \mathcal{L})$.)

524 Example

Let $X = \mathbb{P}^n$, $H \subset \mathbb{P}^n$ a hyperplane. Then

$$\begin{array}{ccc} \mathrm{Cl}(\mathbb{P}^n) & \xrightarrow{\cong} & \mathbb{Z} \\ \cong \uparrow & & \\ \mathrm{CaCl}(\mathbb{P}^n) & & \\ \downarrow \cong & & \\ Z \cong \mathrm{Pic}\,\mathbb{P}^n = \langle \mathcal{O}_{\mathbb{P}^n}(H) \rangle & & \end{array}$$

and in particular $\mathrm{Cl}(\mathbb{P}^n) = \mathbb{Z} \cdot H$. Now set $\mathcal{O}_{\mathbb{P}^n}(d) := \mathcal{O}_{\mathbb{P}^n}(dH)$. We have $\mathcal{O}_{\mathbb{P}^n}(d) \otimes \mathcal{O}_{\mathbb{P}^n}(e) = \mathcal{O}_{\mathbb{P}^n}(d+e)$. Letting $D := dH = \mathrm{Div}(g)$ for some g. Then $|D| = \{\partial\mathrm{iv}(f) : f \text{ deg } d \text{ homog. poly}\}$

525 Definition

Take $X = \mathbb{P}^n$. If f is a degree d homogeneous polynomial, and $\partial\mathrm{iv}(f) := \sum_i a_i Z(f_i)$, where $f = \prod_i f_i^{a_i}$ where the f_i are irreducible polynomials. Now take the ideal sheaf $\mathcal{I}_f$ associated to f, which gives an injection $\mathcal{I}_f \hookrightarrow \mathcal{O}_{\mathbb{P}^n}$. Let Z be the support of the cokernel of this map, and let $\mathcal{O}_Z$ be this cokernel, which in fact gives a ringed space, and a scheme. This scheme is "a little more than" $\partial\mathrm{iv}(f)$. The whole trouble is that we need non-reducedness to accurately track the multiplicities a_i.

Now $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \cong \{\text{deg. } d \text{ homog. poly}\}$. For instance, if $g = x_0^d$, then $\partial\mathrm{iv}(f)$ is $\mathrm{Div}(f/x_0^d) + dH$.

(One may check that $\dim\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = \binom{n+d}{d}$.)

May 13th, 2016: Draft

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526 Definition

Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. We say that $\mathcal{F}$ is generated by global sections $\{s_i\} \subset \Gamma(X, \mathcal{F})$ if $\mathcal{F}_p$ is generated by $\{(s_i)_p\}$ as an $\mathcal{O}_{X,P}$ module. (Sometimes one says "$\mathcal{F}$ is globally generated" instead.)

Warning: the subsheaf generated by $\{s_i\}$ is the smallest subsheaf which contains the restrictions of each s_i to every open U, which is not necessarily saying that these restrictions are generators.

527 Homework

Let X be projective, $\mathcal{L}$ an invertible sheaf. Assume that $\Gamma(X, \mathcal{L}) \neq 0$. Then $\Gamma(X, \mathcal{L}^{-1}) \neq 0$ implies $\mathcal{L} \cong \mathcal{O}_X$.

Idea: pick $s \in \Gamma(X, \mathcal{L}) - \{0\}$ and construct $\mathcal{O}_X \to \mathcal{L}$ by $1 \mapsto s$, and likewise get $\mathcal{O}_X \to \mathcal{L}^{-1}$, which is equivalently $\mathcal{L} \to \mathcal{O}_X$. Hence we have $\mathcal{O}_X \to \mathcal{L} \to \mathcal{O}_X$, and one can check that $\mathcal{O}_X \to \mathcal{L}$ is surjective, which implies an isomorphism in this case.

528 Remark

If $\mathcal{L}$ is an invertible sheaf, then $\mathcal{L}$ is generated by $\{s_i\}$ if and only if for every $P \in X$, there is some i such that $(s_i)_P \notin \mathfrak{m}_{X,P}\mathcal{L}_P$.

529 Aside

That is, the value of some s_i at P is non-zero. We have the map $\mathcal{F}(X) \to \mathcal{F}_P/\mathfrak{m}_{X,P}\mathcal{F}_P$ which can be considered as an evaluation map, i.e. we can plug in $s \in \Gamma(X, \mathcal{F})$ and get a value in a field.

530 Example

We have $\mathcal{O}_{\mathbb{P}^n}(-1) \cong \mathcal{I}_{H \subset X} \subset \mathcal{O}_{\mathbb{P}^n}$, so the global sections of $\mathcal{O}_{\mathbb{P}^n}(-1)$ are constants, and indeed $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-1)) = 0$.

Last time we saw that $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) = \langle x_0, \ldots, x_n \rangle$.

531 Theorem

Let X be a quasi-projective variety. Then:

(i) If $\phi\colon X \to \mathbb{P}^n$ is a morphism and $\mathcal{L} := \phi^*\mathcal{O}_{\mathbb{P}^n}(1)$, then $\mathcal{L}$ is an invertible sheaf which is generated by global the global sections $s_i = \phi^*(x_i)$.

(ii) If $\mathcal{L}$ is an invertible sheaf on X generated by global sections $\{s_0, \ldots, s_n\}$, then there is a unique morphism $\phi\colon X \to \mathbb{P}^n$ such that $\mathcal{L} \cong \phi^*\mathcal{O}_{\mathbb{P}^n}(1)$ such that s_i corresponds to $\phi^*(x_i)$.

Proof For (i), recall that $\phi^*\mathcal{O}_{\mathbb{P}^n} = \mathcal{O}_X$. This is because $\phi^*\mathcal{F} := \phi^{-1}\mathcal{F} \otimes_{\phi^{-1}\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}_X$. It follows that the pullback of an invertible sheaf is invertible, giving the first half. For the second half, it follows very quickly from the local homomorphism aspect of maps of stalks.

For (ii), define $X_i := \{P \in X : (s_i)_P \notin \mathfrak{m}_{X,P}\mathcal{L}_P\}$.

532 Homework

Prove that X_i is open.

By the criterion for generation of invertible sheaves by global sections above, $X = \cup X_i$. Take $U_i := (x_i \neq 0) \subset \mathbb{P}^n$. Note that $U_i \cong \mathbb{A}^n$, where $A(U_i) \cong k[x_0/x_i, \ldots, x_n/x_i]$ (and these isomorphisms are compatible in a natural way). Now consider $(s_i)|_{X_i} \in \Gamma(x_i, \mathcal{L})$. Indeed, $\mathcal{L}|_{X_i} \cong \mathcal{O}_{X_i} \cdot s_i|_{X_i}$. Consider Hence $s_j|_{X_i} = t_j \cdot s_i|_{X_i}$ for some $t_j \in \mathcal{O}_{X_i}(X_i)$, where we are imagining $t_i = s_j/s_i$.

Now define a homomorphism

$$\begin{aligned} A(U_i) &\to \Gamma(X_i, \mathcal{O}_{X_i}) \\ xj/x_i &\mapsto t_j. \end{aligned}$$

Recall that morphisms to affine varieties are determined entirely by the morphism between their global sections, so we have $\phi_i: X_i \to U_i$ where $s_i \leftrightarrow \phi_i^*(x_i)$. It's easy to check that the ϕ_i agree on overlaps:

533 Homework

Check that $\phi_i|_{U_i \cap U_j} = \phi_j|_{U_i \cap U_j}$, so that we have $\phi: X \to \mathbb{P}^n$ with the remaining properties in (ii).

For uniqueness, one checks that the given conditions determine the ϕ_i uniquely, essentially because we are "forced" to send x_j/x_i to t_j.

534 Remark

The above theorem essentially gives an alternative way of considering quasi-projective varieties X paired with morphisms to projective space, namely one can instead look at invertible sheaves X generated by $n+1$ global sections. For instance, we can consider the automorphisms of $\mathbb{P}^n$.

535 Example

Let $M \in \mathrm{GL}_{n+1}(k)$, which corresponds to an $\alpha_M \in \operatorname{Aut} \mathbb{P}^n$. Here we imagine M is mutating $\mathbb{A}^{n+1}$, which descends to an automorphism of $\mathbb{P}^n$. Now given $M, M' \in \mathrm{GL}_{n+1}(k)$, one checks that $\alpha_M = \alpha_{M'}$ if and only if $M' = \lambda M$ for some scalar $\lambda \neq 0$.

536 Definition

Let $\mathrm{PGL}_n(k) := \mathrm{GL}_{n+1}(k)/\sim$ where $M \sim M$" iff $M = \lambda M'$ for some $\lambda \neq 0$. We have a natural map

$$\mathrm{PGL}_n(k) \hookrightarrow \operatorname{Aut} \mathbb{P}^n.$$

Indeed, this is surjective, as follows. Given $\alpha \in \operatorname{Aut} \mathbb{P}^n$, we have $\alpha^*: \operatorname{Pic} \mathbb{P}^n \to \operatorname{Pic} \mathbb{P}^n$. Since $\operatorname{Pic} \mathbb{P}^n = \mathbb{Z} \cdot \mathcal{O}_{\mathbb{P}^n}(1)$, it must be that $\alpha^* \mathcal{O}_{\mathbb{P}^n}(1) \in \{\mathcal{O}_{\mathbb{P}^n}(1), \mathcal{O}_{\mathbb{P}^n}(-1)\}$. Now we had concluded earlier that $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-1)) = 0$ while $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \neq 0$, but $\alpha^* \mathcal{O}_{\mathbb{P}^n}(1)$ is isomorphic to $\mathcal{O}_{\mathbb{P}^n}(1)$, meaning it must be $\mathcal{O}_{\mathbb{P}^n}(1)$. Thus $\alpha^*: \Gamma(\mathbb{P}, \mathcal{O}_{\mathbb{P}^n}(1)) \to \Gamma(\mathbb{P}, \mathcal{O}_{\mathbb{P}^n}(1))$, which is generated (freely) by $x_0, \ldots, x_n$. It follows that α^* can be viewed as an element of GL_{n+1}. By the uniqueness statement in the theorem, it follows that α is the automorphism of $\mathbb{P}^n$ induced by this linear transformation.

537 Aside

There was another lecture yesterday, and Sandor would like to give a few closing remarks relevant to our class. There was a dominant morphism $\phi: \mathbb{P}^1 \to X$ where X was a smooth, projective curve. The conclusion was that $X \cong \mathbb{P}^1$ (the argument used the genus). This corresponded to Lüroth's theorem, namely given a field $k \subsetneq L \subset k(t)$, then $L \cong k(t)$.

What about the higher dimensional analogue? That is, given $k \subsetneq L \subset k(t, u)$, what are the isomorphism types of L? To avoid trivialities, suppose $\operatorname{trdeg}_k L = 2$. In fact, $L \cong k(t, u)$ in this case as well. The analogous geometric question is that if we have a dominant rational map ϕ from $\mathbb{P}^2$ to X where X is a smooth, projective surface, is X birational to $\mathbb{P}^2$? As it turns out, the answer is again yes.

What about the three-dimensional version? In fact, it fails, both algebraically and geometrically. This leads to current research. The condition that X is birational to $\mathbb{P}^n$ goes by the term "X is rational" and the condition that we have a dominant rational map from $\mathbb{P}^n$ to X is known as "unirational." The question is then when unirational implies rational. The above remarks say that the implication holds in dimensions 1 and 2 but not 3. (The simplest counterexample is the function field of a general cubic three-folds, $X \subset \mathbb{P}^4$. This is quite tricky to prove, though. One can also consider quartic three-folds. The general such quartic is rationally connected, but questions about unirationality are unknown in general.)

There is a notion called "rationally connected" where you require that any two points of a variety can be connected by $\mathbb{P}^1$'s. It's easy to see that rational implies unirational implies rationally connected, unirational does not necessarily imply rational. The "most embarassing open problem in algebraic geometry" is that it is unknown whether or not rationally connected implies unirational (at least for smooth varieties). It's likely expected that the implication does not hold in general, though no counterexamples are known.

May 16th, 2016: Draft

Missed.

May 18th, 2016: Draft

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538 Remark

Last time we defined the module of differentials. We briefly recall this. Let B be an A-algebra. If B is a localization of a finitely generated A-algebra, then $\Omega_{B/A}$ is a finite B-module.

Last time there was a little confusion. Note that $\Omega_{B/A}$ is itself a B-module. The elements are not themselves derivations; the derivation is a $d\colon B \to \Omega_{B/A}$, which in some sense is "the only" derivation, since all the actual derivations are induced by this one and homomorphisms. Indeed, $\Omega_{B/A}$ represents the functor of derivations.

We had an exact sequence of the following form. If $C = B/I$, then

$$I/I^2 \to \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to 0$$

We had a proposition which said that $J := \ker B \otimes_A B \to B$ where $b \otimes b' \mapsto bb'$ yields J/J^2 which corresponds to the module of relative differentials.

539 Proposition

Let $(B, \mathfrak{m}, k)$ be a local ring. Suppose $k \subset B$ (that is, B contains a field such that the quotient map by $\mathfrak{m}$ induces an isomorphism to k, i.e. $A := k \subset B \to B/\mathfrak{m} \cong\!=: C$). Then

$$\mathfrak{m}/\mathfrak{m}^2 \to \Omega_{B/k} \otimes_B k$$

is an isomorphism.

Proof The term $\Omega_{C/A} = \Omega_{k/k} = 0$ essentially trivially giving surjectivity from the exact sequence above. One must argue injectivity, which is not too hard, but we'll skip it.

540 Proposition

Let $k \to B \to k$ be an isomorphism. If B is a localization of a finitely generated k-algebra with $k = \overline{k}$ (indeed, k perfect suffices), then $\Omega_{B/k}$ is free of rank $\dim B$ if and only if B is a regular local ring.

Proof We give one direction. The other is somewhat similar, but requires more details like separably generated field extensions.

($\Rightarrow$) Suppose $\Omega_{B/k}$ is a free B-module of rank $\dim B$. It follows that $\Omega_{B/k} \otimes_B B/\mathfrak{m} \cong (B \otimes_B B/\mathfrak{m})^{\dim B} \cong k^{\oplus r}$. By the preceding isomorphism, $\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim B$.

For the converse, this reasoning and Nakayama's lemma say that $\Omega_{B/k}$ is generated by $\dim B$ elements, though one needs to work a bit harder to get $\Omega_{B/k}$ free.

541 Homework

Finish this.

542 Definition

Let $\phi: X \to Y$. The fiber product $X \times_Y X$ comes with a diagonal morphism $\delta: X \to X \times_Y X$ given by $x \mapsto (x, x)$ (i.e. $X \to X$ is the identity). This morphism is locally coming from the maps $B \otimes_A B \to B$ given by $b \otimes b' \mapsto bb'$.

543 Homework

Verify that $\delta: X \overset{\sim}{\to} \operatorname{im} \Delta =: \Delta(X)$.

There is a corresponding ideal sheaf $\mathcal{I} := \mathcal{I}_{\Delta(X)} \subset \mathcal{O}_{X \times_Y X}$, which is locally $\widetilde{J}$ where $J := \ker B \otimes_A B \to B$. Now we have a sheaf $\mathcal{I}/\mathcal{I}^2$ on $\mathcal{O}_{X \times_Y X}$, and we let $\Omega_{X/Y} := \Delta^*(\mathcal{I}/\mathcal{I}^2)$ be the sheaf of relative differentials.

We had many statements about the module of relative differentials last time. These statements translate to statements about the sheaf of relative differentials, which we now give.

If $U \subset X$ is open, then $\Omega_{X/Y}|_U \cong \Omega_{U/Y}$. We can base change as follows. Given

$$\begin{array}{ccc} X' := Y' \times_Y X & \xrightarrow{\psi} & X \\ \downarrow & & \downarrow \phi \\ Y' & \longrightarrow & Y \end{array}$$

we have $\Omega_{X'/Y'} \cong \psi^* \Omega_{X/Y}$.

If additionally $\psi: Y \to Z$ (so $X \overset{\psi}{\to} Y \overset{\psi}{\to} Z$), then we have an exact sequence

$$\phi^* \Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0.$$

Switching notation, let $Z \hookrightarrow X \overset{\phi}{\to} Y$ where $\iota: Z \hookrightarrow X$ is a closed embedding. Then we have an exact sequence

$$\mathcal{I}/\mathcal{I}^2 \to \iota^* \Omega_{X/Y} \to \Omega_{Z/Y} \to 0$$

where $\mathcal{I} := \mathcal{I}_{Z \subset X} \subset \mathcal{O}_X$. (Recall $\iota^* \Omega_{X/Y} \cong \Omega_{X/Y} \otimes \mathcal{O}_Z$.)

544 Remark

What role is the diagonal Δ really playing above? Why would we expect to use it? There is motivation from the normal bundle for smooth manifolds. Suppose we begin with

$$\begin{array}{ccc} X \times_Y X & \xrightarrow{\psi} & X \\ \downarrow & & \downarrow \phi \\ X & \longrightarrow & Y \end{array}$$

The base change theorem says $\Omega_{(X \times_Y X)/X} \cong \psi^* \Omega_{X/Y}$. Now Δ is a closed embedding, i.e. we have $X \hookrightarrow X \times_Y X \to Y$, so that

$$\mathcal{I}/\mathcal{I}^2 \to \Delta^* \Omega_{(X \times_Y X)/X} \to \Omega_{X/Y} \to 0.$$

We also have $X \times_Y X \to X \to Y$ so that

$$\psi^* \Omega_{X/Y} \to \Omega_{(X \times_Y X)/Y} \to \Omega_{(X \times_Y X)/X} \to 0.$$

The base change theorem then says the middle term of this last sequence is sandwiched between isomorphic things. Moreover, one can go through the square at the beginning of this remark by going down then right rather than right then down. When writing out the details, this gives an exact sequence as in the last displayed equation but going left and ending in 0. (The base change is going from ϕ to either the topmost arrow or the leftmost arrow.) It follows that we have a doubly split short exact sequence with $\Omega_{(X \times_Y X)/Y}$ in the middle and $\psi^* \Omega_{X/Y}$ on the left and right.

If we now apply Δ^* to the short exact sequence thus constructed, one ends up getting the exact sequence arising from the closed embedding, i.e. $\Delta^* \psi^* \Omega_{X/Y} \cong \Delta^* \mathcal{I}/\mathcal{I}^2$ and $\Delta^* \psi^* \Omega_{X/Y} \cong \Omega_{X/Y}$. Note that $\Delta^* \psi^* = (\psi \circ \Delta)^*$ and $\psi \circ \Delta$ is the identity. Hence the properties listed above largely force us to choose $\Omega_{X/Y} \cong \Delta^*(\mathcal{I}/\mathcal{I}^2)$.

List of Symbols

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