BIGRASSMANNIANS AND PATTERN AVOIDANCE NOTES

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1. Grassmannians and Bigrassmannians in Type A

Definition 1. A *Grassmannian* $w \in S_n$ is a permutation with ≤ 1 right descent. The only Grassmannian with no descents is the identity.

Proposition 2. *For $w \in S_n$ with one-line notation $w = w_1 \cdots w_n$, the following are equivalent:*

(i) w *is Grassmannian*
(ii) w *is of the form*

$$w_1 < w_2 < \cdots < w_k$$
$$w_{k+1} < w_{k+2} < \cdots < w_n$$

for some $k \in [n-1]$.
(iii) w *is* 321-, 2143-, *and* 3142-*avoiding.*

Proof.

- (i) $\Leftrightarrow$ (ii): Clear.
- (i) $\Rightarrow$ (iii): Contrapositive. If w has a 321, 2143, or 3142 pattern, it evidently has at least 2 descents.
- (iii) $\Rightarrow$ (i): Contrapositive. Suppose w has two descents at positions $k_1 < k_2$. Let $a = w_{k_1}$, $b = w_{k_1+1}$, $c = w_{k_2}$, $d = w_{k_2+1}$, so $a > b$ and $c > d$.
  If $b \geq c$, then $a > b > d$ gives a 321 pattern, so suppose $b < c$, and note that now each of a, b, c, d are distinct.
  If $a > c$, then $a > c > d$ gives a 321 pattern, so suppose $a < c$. If $b > d$, then $a > b > d$ gives a 321 pattern, so suppose $b < d$. If $a > d$, then $c > a > d > b$ gives a 3142 pattern, whereas if $a < d$, then $c > d > a > b$ gives a 2143 pattern.

□

Corollary 3. *There are $2^n - n$ Grassmannians in S_n.*

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Proof. From (ii) of the proposition, to specify a Grassmannian it suffices to choose a set $S = \{w_1, \ldots, w_k\} \subset [n]$, of which there are 2^n. It's easy to see that the resulting permutation will be the identity if and only if S is of the form $\varnothing$ or $\{1, \ldots, k\}$, which is hence counted $n+1$ times. Otherwise, the result has one descent and is counted just once. □

Remark 4. A permutation w contains a pattern v if and only if w^{-1} contains v^{-1}. This is easy to see using the permutation matrix interpretation of pattern containment–striking rows and columns from the matrix of w yields the matrix of v; transposing such a matrix yields its inverse.

Definition 5. A *bigrassmannian* $w \in S_n$ is a permutation where both w and w^{-1} are Grassmannian.

Proposition 6. *For $w \in S_n$ with one-line notation $w = w_1 \cdots w_n$, the following are equivalent:*

(i) w is bigrassmannian
(ii) w is of one of the following forms:
  (a) $w_1 = 1$ and $w_2 \cdots w_n$ forms a bigrassmannian in $S_{[n]-\{1\}}$.
  (b) $w_1 > 1$ and for some k where $a := w_1 \leq k < n$ we have

$$w = a, a+1, \ldots, k, 1, 2, \ldots, a-1, k+1, \ldots, n.$$

(iii) w is 321-, 2143-, 3142-, *and* **2413**-*avoiding.*

Remark 7. The bold permutation is the only one not appearing on the previous list for Grassmannian pattern avoidance.

Proof.

- (i) ⇔ (iii): w is bigrassmannian if and only if w, w^{-1} are Grassmannian, which by the preceding proposition and remark occurs if and only if w is 321-, 2143-, 3142-, 321^{-1}-, 2143^{-1}-, and 3142^{-1}-avoiding. Only $3142^{-1} = 2413$ is new.
- (ii) ⇒ (iii): in case (a) this is clear. In case (b), such a w is clearly Grassmannian with descent at k, so it is 321-, 2143-, and 3142-avoiding. It is evidently also 2413-avoiding.
- (iii) ⇒ (ii): From (ii) and (iii) of the preceding proposition, we have some j such that w is

$$w_1 < w_2 < \cdots < w_j$$
$$w_{j+1} < w_{j+2} < \cdots < w_n.$$

If $w_j < w_{j+1}$ then w must be id, which is of the proper form, so suppose $w_j > w_{j+1}$. Now if $w_1, \ldots, w_j$ are each successively one larger, w is of the form in (b), so suppose $w_1, \ldots, w_j$ "skips" something. That is, there is some i with $i > j$ such that $w_1 < w_i < w_j$. Now, if $w_1 = 1$, then $w_2 \cdots w_n$ continues to avoid the suggested patterns, so we may inductively assume it forms a bigrassmannian in $S_{[n]-\{1\}}$. (The base case $n = 1$ is trivial.) So, take $w_1 > 1$. Since w is a permutation, something must hit 1, and a moment's thought reveals that $w_{j+1} = 1$ is forced. Now

$$w_{j+1} < w_1 < w_i < w_j \qquad 1 < j < j+1 < i$$

yields a 2413 pattern, a contradiction.

□

Corollary 8. *There are $1 + \binom{n+1}{3}$ bigrassmannians in S_n.*

Proof. Let $B(n)$ denote the number of non-identity bigrassmannians in S_n. Condition (ii)(a) of the proposition contributes $B(n-1)$ bigrassmannians to S_n, while condition (ii)(b) contributes $n - a + 1$ for each $2 \leq a \leq n$, i.e.

$$B(n) = B(n-1) + (n-1) + \cdots + 2 + 1 = B(n-1) + \binom{n}{2}.$$

Now $B(2) = 1$ by hand, which is $\binom{2+1}{3}$. Since $\binom{n+1}{3} = \binom{n}{3} + \binom{n}{2}$, we have $B(n) = \binom{n+1}{3}$. □

(The form in (ii) and this computation can be cleaned up significantly.)

2. Type B

Remark 9. As a reflection group, B_n is generated by reflections through e_i and $\pm e_i \pm e_j$ for $i \neq j$ in $\mathbb{R}^n$. A simple system (cf. Humphreys' "Reflection Groups and Coxeter Groups") is given by $e_i - e_{i+1}$ for $1 \leq i < n$ and e_1.

Another construction of B_n is as a subgroup of S_{2n}, namely all bijections $w\colon [\pm n] \to [\pm n]$ with $w(-a) = -w(a)$ (cf. §8.1 of Björner and Brenti's lovely "Combinatorics of Coxeter Groups"). The simple roots/generators S here (in window notation, a modification of one-line notation) are

$$s_i = [1, \ldots, i-1, i+1, i, i+2, \ldots, n] \qquad 1 \leq i < n$$
$$s_0 = [-1, 2, \ldots, n].$$

Björner and Brenti explicitly compute $\mathrm{inv}(w)$ in this notation and also show:

Proposition 10. *Let $v \in B_n$. Then*

$$\mathrm{Des}(v) = \{s_i \in S : v(i) > v(i+1)\} \qquad \text{where } v(0) := 0.$$

Definition 11. For a general Coxeter system (W, S), $u \in W$ is **Grassmannian** if $|\mathrm{Des}(u)| \leq 1$, where $\mathrm{Des}(u) := \{s \in S : \ell(us) < \ell(u)\}$ and where $\ell(v)$ is the minimal length of an expression of the form $v = s_1 \cdots s_m$ for $s_i \in S$. An element $u \in W$ is **bigrassmannian** if both u and u^{-1} are Grassmannian.

Example 12. In B_n, we can check that

$$|\mathrm{Des}([-2\ 3\ 1\ \ -4\ 5])| = 3,$$

which arises from descents at s_0, s_2, s_3.

Proposition 13. *There are $3^n - n$ Grassmannians in type B_n.*

Proof. From the explicit description of descents in B_n, $w \in B_n$ has at most one descent if and only if

(i) $w_1 < 0$: $w_1 \cdots w_n$ forms an increasing sequence and $w_1 < 0$; or
(ii) $w_1 > 0$: $w_1 \cdots w_n$ is made of two increasing sequences, $w_1 < \cdots < w_k$, $w_{k+1} < \cdots < w_n$ with $w_1 > 0$ and $w_k > w_{k+1}$; or
(iii) $w = \mathrm{id}$.

Hence for each such element we can identify a possibly empty increasing sequence of positive entries followed by a possibly empty increasing sequence thereafter. It follows that for each value $1, \ldots, n$ we may specify whether or not that entry is in the positive initial sequence, or whether it is in the second sequence, and if it is in the second sequence, whether it is positive or negative. This yields 3^n choices, though there are some overcounts. In particular, the identity is obtained in $n+1$ ways in exact analogy to the Grassmannians in S_n. The result follows. ☐

Definition 14. The notion of pattern avoidance in B_n is not simply inherited from pattern avoidance in S_{2n}. Instead, we can imagine elements of B_n as signed permutation matrices, namely permutation matrices but with entries ± 1. We say $u \in B_n$ contains $v \in B_m$ if we can strike rows and columns from the matrix of u to obtain the matrix of v.

We again find that u avoids v if and only if u^{-1} avoids v^{-1}.

Proposition 15. *Grassmannians in B_n are precisely those elements avoiding*

$$[-1\ \ -2], [-1\ 3\ 2], [2\ 1\ \ -3], [-2\ 3\ 1],$$
$$[-2\ 3\ \ -1], [3\ 1\ \ -2], [3\ 2\ 1], [3\ 2\ \ -1],$$
$$[-3\ 1\ \ -2], [-3\ 2\ 1], [-3\ 2\ \ -1], [2\ 1\ 4\ 3], [3\ 1\ 4\ 2].$$

The bigrassmannians in B_n are precisely those elements which additionally avoid

$$[\mathbf{2}\ \ \mathbf{-3}\ \mathbf{1}], [\mathbf{2}\ \ \mathbf{-3}\ \ \mathbf{-1}], [\mathbf{3}\ \ \mathbf{-1}\ \mathbf{2}],$$
$$[\mathbf{-3}\ \ \mathbf{-1}\ \mathbf{2}], [\mathbf{2}\ \mathbf{4}\ \mathbf{1}\ \mathbf{3}].$$

Proof. The bigrassmannian computation follows from the Grassmannian computation as before. For the Grassmannian half, each pattern has at least two descents since an initial negative counts as one, so Grassmannians must avoid them. On the other hand, suppose $|\operatorname{Des}(w)| \geq 2$. At most four indexes are involved in such a descent. Striking everything else from the permutation results in an element of B_2, B_3, or B_4 with two descents. Another quick computation shows that the above list contains all such elements with two descents, where for instance elements of B_4 which contain a pattern in B_3 which also has two descents have been omitted.

(My original proof was an awful and very lengthy proof by cases, which I do not want to transcribe.) ☐

Remark 16. Type B_n bigrassmannians should be those of the following form:

(i) $\mathrm{id} \in B_n$; or
(ii) $w \in S_n \subset B_n$ with signed permutation matrix

$$w = \begin{bmatrix} & Q & \\ P & & \\ & & R \end{bmatrix} \qquad \text{or} \qquad \begin{bmatrix} P & & \\ & & R \\ & Q & \end{bmatrix}$$

in block form where P, Q, R are (square) identity matrices with P non-empty and proper; or
(iii) $w \notin S_n \subset B_n$ with signed permutation matrix

$$w = \begin{bmatrix} A & & & \\ & & & D & \\ & & -C & & \\ & B & & & \\ & & & & E \end{bmatrix}$$

in block form where A, B, D, E are identity matrices and C is a non-empty matrix with 1's along the off-diagonal.

Corollary 17. *We have the following counts of bigrassmannians in B_n of the preceding three types:*

(i) 1
(ii) $\binom{n+1}{3}$
(iii) $\binom{n+3}{4}$.

Hence there are $\binom{n+1}{3} + 1$ bigrassmannians in S_n and $\binom{n+3}{4} + \binom{n+1}{3} + 1$ bigrassmannians in B_n. In terms of monomials, these counts are $\frac{1}{6}(n^3 - n + 6)$ for S_n and $\frac{1}{24}(n^4 + 10n^3 + 11n^2 + 2n + 24)$ for B_n.

Conjecture 18. *In type D_n, there are*

$$\binom{n+5}{4} - 9\binom{n+1}{2} - 5 = \frac{1}{24}(n^4 + 14n^3 - 37n^2 + 46n)$$

bigrassmannians.

Remark 19. Proofs of the above classification should be a straightforward proof by cases which I haven't taken the time to write down. The corollary follows by considering "break points" in a sort of stars and bars argument.

The D_n classification seems harder. Sara suggests it's not a pattern-avoidance property, though maybe some simple extra condition would give it. The count is quite similar to the B_n case (conjectured, for now).

3. Type D

Remark 20. "Editorial" note: this section is incomplete.

Definition 21. D_n is defined as the subgroup of B_n consisting of elements with evenly many negatives (in one-line notation).

Proposition 22. *If $w \in D_n$, then*

$$\mathrm{Des}(w) = \{s_i \in S : v(i) > v(i+1)\}$$

where $v(0) := -v(2)$, $0 \leq i < n$, and the s_i are the simple reflections/generators.

Proof. See Björner and Brenti, Proposition 8.2.2. □

Remark 23. The failure of pattern avoidance to classify Grassmannians/bigrassmannians in D_n arises from the "end" condition $v(0) := -v(2)$, which is "context sensitive", whereas pattern avoidance does not depend on where the indexes are located.

Remark 24. We next give a couple of generalizations of pattern avoidance for use in type D, along with a conjectured characterization. I again haven't taken the time to completely verify it. (It should be a quick generalization of the type B_n case, which I only noticed a nice proof for when writing this up.)

Definition 25. If $w = [w_1 \ \cdots \ w_n] \in B_n$, say it contains $u = [u_1 \ \cdots \ u_k : u_{k+1} \ \cdots \ u_m] \in B_m$ if $w_1 \ \cdots \ w_k$ are in the same relative order and of the same sign as $u_1 \ \cdots u_k$, and, after "flattening", $[w_{k+1} \ \cdots \ w_n]$ contains $[u_{k+1} \ \cdots \ u_m]$.

Example 26. $[2 \ -1 \ 3 \ -4]$ contains $[1 : 2 \ -3]$ using 2 followed by $[3 \ -4]$. It avoids $[-1 : 2 \ -3]$ and $[1 : 3 \ -2]$ since the first entry, 2, is non-negative. It avoids $[1 : 3 \ -2]$ since $3 < |-4|$ while $3 > |-2|$.

Example 27. w avoids $[u_1 \ \cdots \ u_m]$ if and only if w avoids $[: u_1 \ \cdots \ u_m]$.

Conjecture 28. *$w \in D_n$ is Grassmannian if and only if it avoids the following 37 patterns, where each $\pm$ is independent:*

$$\begin{aligned}
&[\pm 1 \ -2 :], \\
&[: 3 \ 2 \ \pm 1], [: 3 \ 1 \ -2], [: 2 \ 1 \ -3], [: 2 \ 1 \ 4 \ 3], [: 3 \ 1 \ 4 \ 2] \\
&[-2 \ \pm 1 : -3], [-3 \ \pm 1 : -2], [-3 \ 2 : -1] \\
&[-2 \ \pm 1 : 4 \ \pm 3], [-3 \ \pm 2 : 4 \ \pm 1], [-3 \ \pm 1 : 4 \ \pm 2] \\
&[-4 \ \pm 2 : 3 \ \pm 1], [-4 \ \pm 1 : 3 \ \pm 2], [-4 \ \pm 3 : 2 \ \pm 1].
\end{aligned}$$

Remark 29. How does the inverse's pattern avoidance relate to the original's pattern avoidance?