# Advanced Commutative Algebra Lecture Notes

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**Abstract**

The following notes were taking during a course on Advanced Commutative Algebra at the University of Washington in Fall 2014. Please send any corrections to `jps314@uw.edu`. Thanks!

## Contents

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# September 24th, 2014: Tensor Product Definition, Universal Property

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**1 Remark**

Course content: look at Eisenbud's index and write down what 30 sections of the book we would like covered, ranked in groups of five. Paul will then figure out a course based on our preferences and will make 30 lectures for us. You can also write down topics you don't want discussed.

Put the result in his mailbox as soon as possible (eg. today).

**2 Definition (Tensor Product of Modules)**

Let $R$ be a ring with 1. If $M$ is a right $R$-module and $N$ is a left $R$-module (so $R$ is a ring with 1), their tensor product is the abelian group $M \otimes_R N$ which is the quotient of the free abelian group with basis $\{(m, n) : m \in M, n \in N\}$ by the subgroup generated by the elements

(a) $(m, n + n') - (m, n) - (m, n')$

(b) $(m + m', n) - (m, n) - (m', n)$

(c) $(mr, n) - (m, rn)$

for all $m, m' \in M, n, n' \in N, r \in R$. We have a natural map (of sets) $M \times N \to M \otimes_R N$, where write $m \otimes n$ for the image of $(m, n)$ in $M \otimes_R N$. Hence we have

(a) $m \otimes (n + n') = m \otimes n + m \otimes n'$

(b) $(m + m') \otimes n = m \otimes n + m' \otimes n$

(c) $mr \otimes n = m \otimes rn$.

**3 Example**

$\mathbb{Z}/2 \otimes_{\mathbb{Z}} \mathbb{Z}/3 = 0$ because

$$\begin{aligned} m \otimes n &= m1 \otimes n = m3 \otimes n \\ &= m \otimes 3n = m \otimes 0 \\ &= m \otimes 0 \cdot 0 = m \cdot 0 \otimes 0 \\ &= 0. \end{aligned}$$

More generally,

$$\mathbb{Z}/r \otimes_{\mathbb{Z}} \mathbb{Z}/s \cong \mathbb{Z}/d$$

where $d = \gcd(r, s)$.

**4 Remark**

In general, $M \otimes_R N$ is **not** an $R$-module. However, if $R$ is commutative, it is! In particular, we don't distinguish between left and right modules (eg. we can define $rm := mr$), and we have an action

$$\begin{aligned} r \cdot (m \otimes n) &:= rm \otimes n = mr \otimes n \\ &= m \otimes rn = m \otimes nr \\ &=: (m \otimes n) \cdot r \end{aligned}$$

Elements of $M \otimes_R N$ are finite sums of pure tensors $m \otimes n$. It can be difficult in general to decide whether such a sum is zero in a tensor product.

**5 Proposition (Right Exactness of Tensor Products)**

*If $0 \to A \to B \to C \to 0$ is a short exact sequence of left $R$-modules and $M$ is a right $R$-module, then there is an exact sequence*

$$M \otimes_R A \xrightarrow{f} M \otimes_R B \xrightarrow{g} M \otimes_R C \to 0$$

*given by*

$$m \otimes a \mapsto m \otimes f(a) \mapsto \cdots .$$

Proof Exactness at $M \otimes_R C$ is easy, exactness at $M \otimes_R B$ is more involved and will be proved next time using the Tensor-Hom Adjunction below. Note that there is no $0 \to \cdots$ preceding the first sequence above.

**6 Example**

Consider

$$0 \to \mathbb{Z} \xrightarrow{3} \mathbb{Z} \to \mathbb{Z}/3 \to 0$$

with $M = \mathbb{Z}/3$ over $R = \mathbb{Z}$. This gives

$$\mathbb{Z}/3 \otimes_{\mathbb{Z}} \mathbb{Z} \to \mathbb{Z}/3 \otimes_{\mathbb{Z}} \mathbb{Z} \to \mathbb{Z}/3 \otimes_{\mathbb{Z}} \mathbb{Z}/3 \to 0.$$

Note that $1 \otimes n \mapsto 1 \otimes 3n = 3 \otimes n = 0 \in \mathbb{Z}/3 \otimes_{\mathbb{Z}} \mathbb{Z}$. However, the first induced map is not injective, so we can't just prepend a 0. To see this, we'll use a little lemma:

**7 Lemma**

*If $M$ is a right $R$-module, then*

$$M \otimes_R R \cong M$$

*via*

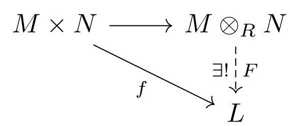
$$\phi\colon m \otimes r \mapsto mr.$$

Proof We need to check that $\phi$ is well-defined, which is a consequence of the next proposition.

We have that $m \otimes r = mr \otimes 1$, so every element in $M \otimes_R R$ is of the form $m \otimes 1$ for some $m \in M$. Hence $\phi(m \otimes 1) = 0$ says $m = 0$, so $\phi$ is injective. It is clearly surjective.

By the lemma, the $\mathbb{Z}/3 \otimes_{\mathbb{Z}} \mathbb{Z}$'s on the left are each $\cong \mathbb{Z}/3$, while the induced map is multiplication by 3 in $\mathbb{Z}/3$, which is the zero map, so not injective.

**8 Proposition (Universal Property of the Tensor Product)**
*Let $R, M, N$ be as above. Let $L$ be an abelian group. If $f\colon M \times N \to L$ is a bilinear map such that $f(mr, n) = f(m, rn)$ for all $m \in M, n \in N, r \in R$, then there exists a unique homomorphism of abelian groups $F\colon M \otimes_R N \to L$ such that the composition*




*(The horizontal map sends $(m, n)$ to $m \otimes n$.)*

Proof Use the universal property of the quotient.

**9 Remark**
We can use this proposition to justify the existence of $\phi$ in the preceding lemma: use $M \times R \to M$ given by $(m, r) \mapsto mr$.

**10 Proposition (Tensor-Hom Adjunction)**
*Let $R$ and $S$ be rings, $N$ a left $R$-module and a right $S$-module such that*

$$(sn)r = s(nr)$$

*for all $s \in S, n \in N, r \in R$. (This is the definition of an $S, R$-bimodule). Further suppose $M$ is a right $R$-module and $L$ is a right $S$-module. Then*

$$\mathrm{Hom}_S(M_R \otimes_R {}_R N_S, L_S) \cong \mathrm{Hom}_R(M_R, \mathrm{Hom}_S(N_S, L_S))$$

*via*

$$\Phi(f)(m)(n) := f(m \otimes n).$$

*(The subscripts are just a memory aid.)*

**11 Remark**
Indeed, this isomorphism is functorial in the rigorous sense of adjoint functors. In particular, $- \otimes_R N\colon R\text{-mod}_{\text{right}} \to S\text{-mod}_{\text{right}}$ is left adjoint to $\mathrm{Hom}_S(N, -)\colon S\text{-mod}_{\text{right}} \to R\text{-mod}_{\text{right}}$.

Proof Next time.

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# September 26th, 2014: Tensor-Hom Adjunction, Right Exactness of Tensor Products, Flatness

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**12 Remark**

Recall the Tensor-Hom adjunction isomorphism $\Phi$ from the end of last class, $\Phi(f)(m)(n) = f(m \otimes n)$. Today we'll show there is a map back the other way. The rest of the proof is straightforward.

Proof Let $\phi \in \mathrm{Hom}_R(M, \mathrm{Hom}_S(N, L))$. The map $M \times N \to L$ given by $(m, n) \mapsto \phi(m)(n)$ has the property that $(mr, n)$ and $(m, rn)$ map to the same thing,

$$\phi(mr)(n) = (\phi(m) \cdot r)(n) = \phi(m)(rn) = \phi(m)(rn).$$

**13 Aside**

How do we remember which way rings act on morphisms? If $A, B, C, D$ are rings, $M$ is an $A, B$-bimodule, and $N$ is a $C, D$-bimodule, then $\mathrm{Hom}_{\mathbb{Z}}({}_AM_B, {}_CN_D)$ is a $B \otimes C, A \otimes D$-bimodule via

$$((b \otimes c)f(a \otimes d))(m) := cf(amb)d.$$

The key is that if we have four rings, there's only one way we can reasonably write down the above. Since $\mathrm{Hom}(-, N)$ is contravariant, the order of $A, B$ "switched"; likewise since $\mathrm{Hom}(M, -)$ is covariant, the order of $C, D$ "stayed the same".

Our map $M \times N \to L$ is bilinear, so there exists a unique homomorphism $f\colon M \otimes_R N \to L$ such that

$$f(m \otimes n) = \phi(m)(n).$$

Now define $\Psi\colon \mathrm{Hom}_R(M, \mathrm{Hom}_S(N, L)) \to \mathrm{Hom}_S(M \otimes_R N, L)$ by

$$\Psi(\phi)(m \otimes n) := \phi(m)(n).$$

The correspondence between $f$ and $\phi$ shows that $\Phi$ and $\Psi$ are mutual inverses to each other.

More general statements are possible: we can add a left $C$ action to $L$ and a left $A$ action to $M$, making $\Phi$ an isomorphism of $C, A$-bimodules.

**14 Lemma**

*A sequence $M \to M' \to M'' \to 0$ of right $R$-modules is exact if and only if the sequence*

$$0 \to \mathrm{Hom}_R(M'', X) \to \mathrm{Hom}_R(M', X) \to \mathrm{Hom}_R(M, X)$$

*is exact for all right $R$-modules $X$.*

Proof Exercise.

**15 Proposition (Right Exactness of Tensor Products)**

*If $M \to M' \to M'' \to 0$ is an exact sequence of right $R$-modules and $N$ is a left $R$-module, then $M \otimes_R N \to M' \otimes_R N \to M'' \otimes_R N \to 0$ is exact.*

Proof We need only show exactness as abelian groups, i.e. $\mathbb{Z}$-modules. From the lemma, this occurs if the following is exact for all abelian groups $L$:

$$0 \to \mathrm{Hom}(M'' \otimes_R N, L) \to \mathrm{Hom}(M' \otimes_R N, L) \to \mathrm{Hom}(M \otimes_R N, L).$$

By the Tensor-Hom adjunction, this sequence is isomorphic to

$$0 \to \mathrm{Hom}_R(M'', \mathrm{Hom}(N, L)) \to \mathrm{Hom}_R(M', \mathrm{Hom}(N, L)) \to \mathrm{Hom}_R(M, \mathrm{Hom}(N, L)).$$

This is exact by the other direction of the lemma, so our original sequence is exact.

**16 Definition**

A left $R$-module $X$ is a $\boxed{\text{flat } R\text{-module}}$ if $- \otimes_R X$ is an exact functor, i.e. a short exact sequence of right $R$-modules

$$0 \to M \to M' \to M'' \to 0$$

implies

$$0 \to M \otimes_R X \to M' \otimes_R X \to M'' \otimes_R X \to 0$$

is exact.

**17 Remark**

Since $- \otimes_R X$ is right exact in general, ${}_R X$ is flat if and only if $M \otimes_R X \to M' \otimes_R X$ is injective, i.e. iff $f \otimes 1 \colon M \otimes_R X \to M' \otimes_R X$ is injective for all $f \in \mathrm{Hom}_R(M, M')$. (Here $(f \otimes 1)(m \otimes x) := f(m) \otimes x$.)

**18 Example**

Flatness in action:

(1) ${}_R R$ is flat because $- \otimes_R R$ is isomorphic to the identity functor on right $R$-modules, which is trivially exact.

(2) **19 Lemma**

*$- \otimes_R X$ commutes with $\oplus$, i.e., the natural map*

$$(\oplus_{i \in I} M_i) \otimes_R X \to \oplus_{i \in I} (M_i \otimes_R X)$$

*is an isomorphism. A direct sum is flat if and only if each summand is flat.*

Proof Exercise.

(3) **20 Proposition**

*Projective $R$-modules are flat.*

Proof Free modules are flat from the previous lemma. Projective modules are direct summands of free modules, hence are also flat from the lemma.

(4) Recall that finitely presented flat modules are projective, giving a partial converse to the above. Moreover, over a Noetherian ring, a finitely generated module is flat if and only if it is projective.

(5) **21 Proposition**

*Let $\mathcal{S}$ be a multiplicatively closed set in a commutative ring $R$. Localization at $S$ is an exact functor. Equivalently, $R\mathcal{S}^{-1}$ is a flat $R$-module.*

Proof Recall the usual construction of $M\mathcal{S}^{-1}$ for an $R$-module $M$ via equivalence classes. One can prove the following directly (though tediously): if $0 \to M \to M' \to M'' \to 0$ is exact, so is $0 \to M\mathcal{S}^{-1} \to M'\mathcal{S}^{-1} \to M''\mathcal{S}^{-1} \to 0$; this gives exactness of localization. On the other hand, one may show $M\mathcal{S}^{-1} \cong M \otimes_R R\mathcal{S}^{-1}$. It follows that $R\mathcal{S}^{-1}$ is flat.

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# September 29th, 2014: Tensors and Quotients, Support of a Module

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**22 Lemma**

*The map $R/I \otimes_R M \to M/IM$ given by $(x + I) \otimes m \mapsto xm + IM$ is an isomorphism for all left $R$-modules $M$ and all right ideals $I$.*

*(In general, this is an isomorphism of abelian groups, though if $R$ is commutative, both sides have $R$-module structures which are preserved.)*

Proof This is well-defined as usual. Apply the right exact functor $-\otimes_R M$ to the short exact sequence $0 \to I \to R \to R/I \to 0$ to get the exact sequence

$$\begin{array}{ccccccc} I \otimes_R M & \longrightarrow & R \otimes_R M & \longrightarrow & R/I \otimes_R M & \longrightarrow & 0 \\ & \searrow^{\phi} & \downarrow \wr & \nearrow & & & \\ & & M & & & & \end{array}$$

By exactness in the middle, $R/I \otimes_R M \cong M/\operatorname{im}(\phi)$. It's very easy to see $\operatorname{im}(\phi) = IM$.

**23 Proposition**

*$R/I \otimes_R R/J \cong R/(I+J)$ if $R$ is commutative.*

Proof By the lemma,

$$R/I \otimes_R R/J \cong \frac{R/J}{I(R/J)} = \frac{R/J}{(I+J)/J} \cong \frac{R}{I+J}.$$

**24 Definition**

The support of a module $M$ over a commutative ring $R$ is the set

$$\boxed{\operatorname{Supp}(M)} := \{\mathfrak{p} \in \operatorname{spec}(R) : M_{\mathfrak{p}} \neq 0.\}.$$

(Recall: $M_{\mathfrak{p}} := M(R - \mathfrak{p})^{-1}$, whereas $M_x := M\{1, x, x^2, \ldots\}^{-1}$.)

**25 Proposition (Atiyah-Macdonald Exercise 19, Page 46)**

*Let $R$ be a commutative ring with $M, L, N$ $R$-modules.*

1. *If $0 \to L \to M \to N \to 0$ is a short exact sequence of $R$-modules, then*

$$\operatorname{Supp}(M) = \operatorname{Supp}(L) \cup \operatorname{Supp}(N).$$

2. $\operatorname{Supp}(R/I) = \{\mathfrak{p} \in \operatorname{spec}(R) : I \subset \mathfrak{p}\} =: V(I)$.
3. $\operatorname{Supp}(M) = \varnothing$ *if and only if* $M = \{0\}$.
4. $\operatorname{Supp}(M \otimes_R N) = \operatorname{Supp}(M) \cap \operatorname{Supp}(N)$ *if $M$ and $N$ are finitely generated.*
5. $\operatorname{Supp}(\sum_i M_i) = \cup_i \operatorname{Supp}(M_i)$ *(where the sum has arbitrary cardinality and is not necessarily direct).*
6. *If $M$ is finitely generated,* $\operatorname{Supp}(M) = V(\operatorname{Ann} M)$.

Proof Two useful results first:

**26 Lemma**

*Let $M$ be an $R$-module. If $I$ is an ideal in $R$ maximal with respect to being the annihilator of a non-zero element in $M$, then $I$ is a prime ideal.*

Proof Suppose $I = \operatorname{Ann}(m)$ for $m \neq 0$. Suppose $ab \in I$. Hence $abm = 0$. If $bm \neq 0$, then $\operatorname{Ann}(bm) \supset I + (a)$. By maximality of $I$, $\operatorname{Ann}(bm) = I$, so $a \in I$. Otherwise, $bm = 0$, so $b \in I$.

**27 Proposition**

*If $M$ is a non-zero finitely generated module over a noetherian ring $R$, then there exists a chain of submodules*

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$$

*such that $M_j/M_{j-1} \cong R/\mathfrak{p}_j$ for some $\mathfrak{p}_j \in \operatorname{spec}(R)$ for all $j = 1, \ldots, n$.*

PROOF Pick $\mathfrak{p}_1$ maximal with respect to being the annihilator of a non-zero element $m_1$ of $M$; it may be the zero ideal. By the lemma, $\mathfrak{p}_1$ is prime. Now $(m_1) \cong R/\operatorname{Ann}(m_1) = R/\mathfrak{p}_1$, so define $M_1 = (m_1)$ and apply the same argument to $M/M_1$, yielding some prime $\mathfrak{p}_2 = \operatorname{Ann}(m_2 + M_1)$. We have

$$R/\mathfrak{p}_2 = \frac{R}{\operatorname{Ann}(m_2 + M_1)} = R(m_2 + M_1) \subset M/M_1.$$

Letting $M_2 = (m_2) + M_1 \subset M$ gives $R(m_2 + M_1) = M_2/M_1$. In general, induct. The noetherian hypothesis ensures we must stop after finitely many steps.

*(1) Since localization is an exact functor,*

$$0 \to L_\mathfrak{p} \to M_\mathfrak{p} \to N_\mathfrak{p} \to 0$$

*is exact, so* $M_\mathfrak{p} = 0$ *iff both* $L_\mathfrak{p} = 0$ *and* $N_\mathfrak{p} = 0$.

*(2) By exactness and flatness,* $(R/I)_\mathfrak{p} \cong R_\mathfrak{p}/I_\mathfrak{p}$, *so*

$$\begin{aligned}\mathfrak{p} \in \operatorname{Supp}(R/I) &\Leftrightarrow R_\mathfrak{p}/I_\mathfrak{p} \neq 0 \\ &\Leftrightarrow 1 \notin I_\mathfrak{p} = I(R - \mathfrak{p})^{-1} \\ &\Leftrightarrow I \cap (R - \mathfrak{p}) = \varnothing \\ &\Leftrightarrow I \subset \mathfrak{p}.\end{aligned}$$

*(3) It suffices to show that* $M \neq 0$ *implies* $\operatorname{Supp}(M) \neq \varnothing$, *the converse being obvious. Suppose* $0 \neq m \in M$. *Then* $Rm \cong R/I$ *for some ideal* $I \neq R$. *Now* $\operatorname{Supp}(R/I) \neq \varnothing$ *by (2) (here we use the fact that* $V(I) \neq \varnothing$ *for* $I$ *proper; for instance, it contains all maximal ideals containing* $I$*). By (1) applied to*

$$0 \to Rm \to M \to M/Rm \to 0,$$

*we have* $\operatorname{Supp}(M) \neq \varnothing$.

*(4) One may check that for any multiplicatively closed set* $\mathcal{S} \subset R$,

$$(M \otimes_R N)\mathcal{S}^{-1} \cong M\mathcal{S}^{-1} \otimes_R N\mathcal{S}^{-1}.$$

*(By the next lemma, we don't need* $\otimes_{R\mathcal{S}^{-1}}$*; this generalizes and is used in the last step of the following computation.)*

**28 Lemma**
$R\mathcal{S}^{-1} \otimes_R R\mathcal{S}^{-1} \cong R\mathcal{S}^{-1}$ *by the multiplication map.*

*In any case, if* $\mathfrak{p} \in \operatorname{Supp}(M \otimes_R N)$, *then*

$$\begin{aligned}0 &\neq (M \otimes_R N)_\mathfrak{p} \\ &\cong M \otimes_R N \otimes_R R_\mathfrak{p} \\ &\cong M \otimes_R N \otimes_R R_\mathfrak{p} \otimes_R R_\mathfrak{p} \\ &\cong (M \otimes_R R_\mathfrak{p}) \otimes_R (N \otimes_R R_\mathfrak{p}) \\ &\cong M_\mathfrak{p} \otimes_R N_\mathfrak{p} \\ &\cong M_\mathfrak{p} \otimes_{R_\mathfrak{p}} N_\mathfrak{p}.\end{aligned}$$

*Hence* $M_\mathfrak{p} \neq 0$ *and* $N_\mathfrak{p} \neq 0$. *In particular,* $\operatorname{Supp}(M \otimes_R N) \subset \operatorname{Supp}(M) \cap \operatorname{Supp}(N)$. *We'll finish next time.*

# October 1st, 2014: Prime Avoidance Lemma, Associated Primes

**Summary** Today we finish the proof from last time, introduce associated primes, and prove some of their basic properties.

PROOF (Continued from last time.)

We proved $\subset$ of (4) last time. For the other direction, suppose $\mathfrak{p} \in \operatorname{Supp}(M) \cap \operatorname{Supp}(N)$, so $M_\mathfrak{p} \neq 0$ and $N_\mathfrak{p} \neq 0$. Note that $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ is a field ($R_\mathfrak{p}$ being a local ring), and

$$R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} \otimes_{R_\mathfrak{p}} M_\mathfrak{p} \cong M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p} \neq 0.$$

Likewise $N_\mathfrak{p}/\mathfrak{p}N_\mathfrak{p} \neq 0$, and these are each $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$-vector spaces. Hence

$$0 \neq M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p} \otimes_{R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}} N_\mathfrak{p}/\mathfrak{p}N\mathfrak{p}.$$

Now $M_\mathfrak{p} \otimes_{R_\mathfrak{p}} N_\mathfrak{p}$ surjects onto the above since $\otimes$ is right exact, so $M_\mathfrak{p} \otimes_{R_\mathfrak{p}} N_\mathfrak{p} \neq 0$. From last time this is $(M \otimes_R N)_\mathfrak{p}$, so $\mathfrak{p} \in \operatorname{Supp}(M \otimes_R N)$, giving $\supset$.

For (5), we first note in general from (1) that if $A \twoheadrightarrow B$ or indeed if $B \hookrightarrow A$, then $\operatorname{Supp}(B) \subset \operatorname{Supp}(A)$. Hence since $\oplus M_i \twoheadrightarrow \sum M_i$, we have $\operatorname{Supp}(\sum M_i) \subset \operatorname{Supp}(\oplus M_i)$ by (1). Since $\otimes$ commutes with arbitrary $\oplus$, it follows that

$$\operatorname{Supp}(\oplus_i M_i) = \cup_i \operatorname{Supp}(M_i).$$

Conversely, $M_0 \hookrightarrow \sum M_i$ for each particular $M_0$, so $\operatorname{Supp}(M_0) \subset \operatorname{Supp}(\sum M_i)$, hence $\cup \operatorname{Supp}(M_i) \subset \operatorname{Supp}(\sum M_i)$.

For (6), let $I = \operatorname{Ann}(M)$. Take generators $m_1, \ldots, m_n$ of $M$ and consider the map

$$R/I \to \oplus_{i=1}^n M$$

given by $x + I \mapsto (xm_1, \ldots, xm_n)$. This is well-defined and evidently injective. Hence

$$V(\operatorname{Ann} M) = V(I) = \operatorname{Supp}(R/I) \subset \operatorname{Supp}(\oplus_{i=1}^n M) = \operatorname{Supp}(M).$$

(This direction fails with infinitely many generators, since we must use the direct product rather than the direct sum.) On the other hand, if $\mathfrak{p} \notin V(\operatorname{Ann} M)$, then there is some $x \in \operatorname{Ann} M$ such that $x \notin \mathfrak{p}$. Hence $x$ is a unit in $R_\mathfrak{p}$, but $xM_\mathfrak{p} = 0$, so $M_\mathfrak{p} = 0$, and $\mathfrak{p} \notin \operatorname{Supp}(M)$.

**29 Lemma (Prime Avoidance Lemma)**

*Let $I$ be an ideal in a commutative ring and let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be prime ideals.*

1. *If $I \subset \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_n$, then $I$ is contained in somce $\mathfrak{p}_i$.*
2. *If $I$ is not contained in any $\mathfrak{p}_i$, then there exists $x \in I$ such that $x \notin \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_n$.*

PROOF These are equivalent (contrapositives), so we'll prove (2). Argue by induction. $n = 1$ is trivial. Write $\mathfrak{q} := \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_{n-1}$. Since $I$ is not contained in $\mathfrak{p}_n$, there is some $a \in I - \mathfrak{p}_n$. If $\mathfrak{q} \subset \mathfrak{p}_n$, then $a \notin \mathfrak{p}_n = \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_n$ is of the required form. So, suppose $\mathfrak{q} \not\subset \mathfrak{p}_n$, giving some $r \in \mathfrak{q} - \mathfrak{p}_n$. Inductively, there is some $x \in I$ with $x \notin \mathfrak{q}$. If $x \notin \mathfrak{p}_n$, we would be done, so suppose $x \in \mathfrak{p}_n$.

Consider $x + ra$. This is in $I$ since $x$ and $a$ are. If $x + ra \in \mathfrak{q}$, then since $r \in \mathfrak{q}$, we would have $x \in \mathfrak{q}$, contrary to our assumption. Hence $x + ra \notin \mathfrak{q}$, and we need only show $x + ra \notin \mathfrak{p}_n$ for it to be of the required form. Now if $x + ra \in \mathfrak{p}_n$, then since $x \in \mathfrak{p}_n$, we have $ra \in \mathfrak{p}_n$. Since $\mathfrak{p}_n$ is prime, either $r \in \mathfrak{p}_n$ or $a \in \mathfrak{p}_n$, a contradiction in either case. Hence $x + ra \notin \mathfrak{p}_n$ is of the required form.

**30 Definition**

The associated primes of an $R$-module $M$ are

$$\boxed{\mathrm{Ass}(M)} := \{\mathfrak{p} \in \operatorname{spec} R : \mathfrak{p} = \mathrm{Ann}(m) \text{ for some } 0 \neq m \in M\}.$$

**31 Lemma**

*Let $N$ be a non-zero submodule of $M$. If $\mathfrak{p}$ is an ideal in $R$ that is maximal subject to being the annihilator of some non-zero element in $N$, then $\mathfrak{p} \in \mathrm{Ass}(M)$.*

Proof By a previous lemma, $\mathfrak{p}$ is prime.

**32 Proposition**

*If $0 \neq N \subset M$ is a submodule of an $R$-module $M$, then $\mathrm{Ass}(N) \subseteq \mathrm{Ass}(M)$.*

Proof Trivial from the definition.

**33 Lemma**

*If $R$ is a noetherian ring and $M$ is a noetherian $R$-module, then:*

*(1)* $\mathrm{Ass}(M) = 0$ *if and only if* $M = 0$

*(2)* $\mathrm{Ass}(M) \subseteq \mathrm{Supp}(M)$

*(3) If* $\mathfrak{p} \in \mathrm{spec}(R)$, *then* $\mathrm{Ass}(R/\mathfrak{p}) = \{\mathfrak{p}\}$

*(4) If $N$ is a submodule of $M$, then*

$$\mathrm{Ass}(N) \subseteq \mathrm{Ass}(M) \subseteq \mathrm{Ass}(N) \cup \mathrm{Ass}(M/N).$$

*(5)* $\mathrm{Ass}(M)$ *is finite.*

Proof (1) is obvious from the lemma: there is an ideal in $R$ that is maximal subject to being the annihilator of a nonzero element in $M$.

(2) Suppose $\mathfrak{p} = \mathrm{Ann}(m)$ is prime for $m \neq 0$. Then

$$Rm \cong R/\mathrm{Ann}(m) = R/\mathfrak{p} \hookrightarrow M,$$

so $\mathfrak{p} \in \mathrm{Supp}(Rm) \subset \mathrm{Supp}(M)$.

(3) Pick $0 \neq m + \mathfrak{p} \in R/\mathfrak{p}$. Since $R/\mathfrak{p}$ is a domain, we see $\mathrm{Ann}_{R/\mathfrak{p}}(m+\mathfrak{p}) = 0$, whence $\mathrm{Ann}_R(m+\mathfrak{p}) = \mathfrak{p}$. Further, such an $m + \mathfrak{p}$ exists by (1).

(4) We must show that if $\mathfrak{p} \in \mathrm{Ass}(M) - \mathrm{Ass}(N)$, then $\mathfrak{p} \in \mathrm{Ass}(M/N)$. By hypothesis there exists $m \in M - N$ such that $\mathfrak{p} = \mathrm{Ann}(m)$. One would naturally hope that $\mathfrak{p} = \mathrm{Ann}(m + N)$ as well. Let $I = \mathrm{Ann}(m + N)$. Now $Rm \cong R/\mathfrak{p}$ and the annihilator of every nonzero element of $Rm$ is $\mathfrak{p}$ as in (3). If $Rm \cap N \neq 0$, by (1) we have $\mathfrak{p} \in \mathrm{Ass}(N)$; hence $Rm \cap N = 0$. That is, $Rm \hookrightarrow M \twoheadrightarrow M/N$ is injective, or equivalently $R/\mathfrak{p}$ embeds in $M/N$. Thus

$$\{\mathfrak{p}\} = \mathrm{Ass}(R/\mathfrak{p}) \subset \mathrm{Ass}(M/N).$$

(5) Using a proposition above, we may find $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$ with $M_i/M_{i-1} \cong R/\mathfrak{p}_i$ with $\mathfrak{p}_i$ prime. By (4) and induction, $\mathrm{Ass}(M) \subset \cup_{i=1}^n \mathrm{Ass}(M/M_{i-1}) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$.

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# October 3rd, 2014: Associated Primes Continued, Minimal Primes, Zero-Divisors, Regular Elements

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**34 Example**

Let $R = k[x,y]$ with $I = (x^2, xy) = (x)(x,y)$. Pictorially, $V(I)$ is the line $x = 0$ with a double point at $(0,0)$ in the $xy$-plane. We show that $\operatorname{Ass}(R/I) = \{(x),(x,y)\}$: we have a filtration $0 \subset kx \subset R/I$ with successive quotients $R/(x,y)$ and $R/(x)$, so by the proof of (5) of the preceding lemma, $\operatorname{Ass}(R/I) \subset \{(x),(x,y)\}$. But $(x) = \operatorname{Ann}(y+I)$ and $(x,y) = \operatorname{Ann}(x+I)$.

Because $(x) \subset (x,y)$ we call $(x,y)$ an **embedded prime**.

**35 Proposition**

*Let $R$ be a commutative noetherian ring, $I$ an arbitrary ideal. Let $\{\mathfrak{p}_i\}_{i\in J}$ be the set of minimal primes over $I$. (Explicitly, $\mathfrak{p} \in \operatorname{spec}(R)$ is minimal over $I$ if $\mathfrak{p} \supset I$ and if $\mathfrak{p}' \in \operatorname{spec} R$ is such that $\mathfrak{p} \supset \mathfrak{p}' \supset I$, then $\mathfrak{p} = \mathfrak{p}'$.)*

*(1) $\{\mathfrak{p}_i\}_{i\in J} = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ is finite and non-empty.*

*(2) $I \supset \mathfrak{p}_1^{i_1} \cdots \mathfrak{p}_n^{i_n}$ for some $i_1, \dots, i_n \geq 1$ and $\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_n = \sqrt{I}$.*

*(3) The irreducible components of $V(I)$ are the $V(\mathfrak{p}_i)$.*

**36 Definition**

Let $M$ be an $R$-module. We call $x \in R$ a **zero-divisor on $M$** if $xm = 0$ for some $0 \neq m \in M$. $x$ is called **$M$-regular** otherwise.

**37 Proposition**

*Let $R$ be a commutative notherian ring, $M$ a noetherian $R$-module. Then*

$$\bigcup_{\mathfrak{p}\in\operatorname{Ass}(M)} \mathfrak{p} = \{\text{zero-divisors on } M\}.$$

Proof If $\mathfrak{p} \in \operatorname{Ass}(M)$, then $\mathfrak{p} = \operatorname{Ann}(m)$ for some $0 \neq m \in M$, so the left-hand side is contained in the right-hand side. On the other hand, let $x$ be a zero-divisor on $M$. Then $x \in \operatorname{Ann}(m)$ for some $0 \neq m \in M$ and, taking a maximal annihilator as usual, there is some $\mathfrak{p} \in \operatorname{Ass}(M)$ containing $\operatorname{Ann}(m)$, so $x \in \mathfrak{p}$.

**38 Definition**

Let $R$ be commutative noetherian, $M$ a noetherian $R$-module. The minimal primes over $\operatorname{Ann}(M)$ are called the **minimal primes of $M$**.

**39 Proposition**

*Let $R$ be commutative noetherian, $M$ a noetherian $R$-module. If $\mathfrak{p}$ is a minimal prime of $M$, then $\mathfrak{p} \in \operatorname{Ass}(M)$.*

Proof Let $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ be the minimal primes of $M$. Since $M$ is noetherian, it is finitely generated, so $\operatorname{Supp}(M) = V(\operatorname{Ann} M)$. In particular, $\mathfrak{p} := \mathfrak{p}_1$ is contained in $\operatorname{Supp} M$, so $M_\mathfrak{p} \neq 0$. For some $i_1, \dots, i_n \geq 1$, $\mathfrak{p}_1^{i_1} \cdots \mathfrak{p}_n^{i_n} \subset \operatorname{Ann}(M)$, i.e. $\mathfrak{p}_1^{i_1} \cdots \mathfrak{p}_n^{i_n} M_\mathfrak{p} = 0$. In $R_\mathfrak{p}$, each $(\mathfrak{p}_i)_\mathfrak{p}$ for $i \neq 1$ contains a unit, so $\mathfrak{p}_\mathfrak{p}^{i_1} M_\mathfrak{p} = 0$. Hence there exists $0 \neq ms^{-1} \in M_\mathfrak{p}$ with $m \in M, s \in R - \mathfrak{p}$ such that $\mathfrak{p}_\mathfrak{p} ms^{-1} = 0$, i.e. $\mathfrak{p}_\mathfrak{p} m = 0$. Since $\mathfrak{p}_\mathfrak{p}$ is the maximal ideal of $R_\mathfrak{p}$, $\operatorname{Ann}_{R_\mathfrak{p}}(m) = \mathfrak{p}_\mathfrak{p}$. Therefore $\operatorname{Ann}_R(m) = \mathfrak{p}_\mathfrak{p} \cap R = \mathfrak{p} \in \operatorname{Ann}(M)$.

**40 Lemma**

*Let $R$ be commutative noetherian, $M$ a noetherian $R$-module. An element $m \in M$ is zero if and only if the image of $m$ under the map $M \to M_\mathfrak{p}$ is zero for all $\mathfrak{p} \in \operatorname{Ass}(M)$.*

Proof $\Rightarrow$ is trivial. For $\Leftarrow$, suppose $m \neq 0$. Then we have some $\operatorname{Ann}(m) \subset \mathfrak{p} \in \operatorname{Ass}(M)$, so $Rm \cong R/\operatorname{Ann}(m)$ and $\operatorname{Ann}(m) \subset \mathfrak{p} \subset R$. Hence we have a short exact sequence

$$0 \to \mathfrak{p}/\operatorname{Ann}(m) \to R/\operatorname{Ann}(m) \to R/\mathfrak{p} \to 0.$$

Localize the sequence at $\mathfrak{p}$. Since $(R/\mathfrak{p})_\mathfrak{p} \neq 0$, $(R/\operatorname{Ann}(m))_\mathfrak{p} \neq 0$, so $(Rm)_\mathfrak{p} \neq 0$. But $(Rm)_\mathfrak{p} = R_\mathfrak{p} m \subset M_\mathfrak{p}$, so the image of $m$ in $M_\mathfrak{p}$ is non-zero.

**41 Lemma**

*Let $R$ be a commutative noetherian ring, $M, N$ noetherian $R$-modules. An $R$-module homomorphism $f: M \to N$ is injective if and only if $f_\mathfrak{p}: M_\mathfrak{p} \to N_\mathfrak{p}$ is injective for all $\mathfrak{p} \in \operatorname{Ass}(M)$.*

Proof For any $\mathfrak{p} \in \operatorname{spec} R$, localizing away from $\mathfrak{p}$ gives the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \ker f & \longrightarrow & M & \xrightarrow{f} & N \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & (\ker f)_\mathfrak{p} & \longrightarrow & M_\mathfrak{p} & \xrightarrow{f_\mathfrak{p}} & N_\mathfrak{p} \end{array}$$

In particular $\ker(f_\mathfrak{p}) = (\ker f)_\mathfrak{p}$, giving $\Rightarrow$. For $\Leftarrow$, suppose $\ker(f_\mathfrak{p}) = 0$ for all $\mathfrak{p} \in \operatorname{Ass}(M)$. If $m \in \ker f$, then running it through the diagram immediately gives $m_\mathfrak{p} = 0 \in \ker(f_\mathfrak{p}) \subset M_\mathfrak{p}$. By the previous lemma, $m = 0$.

**42 Proposition**

*Let $R$ be a commutative noetherian ring, $M$ a noetherian $R$-module, $I$ an ideal in $R$. If every element of $I$ is a zero-divisor on $M$, then $I$ is contained in some $\mathfrak{p} \in \operatorname{Ass}(M)$.*

Proof Since $I$ is contained in the set of zero-divisors on $M$, it is contained in the union of the associated primes by the proposition above. The prime avoidance lemma then says $I$ is contained in some particular associated prime.

**43 Proposition**

*Let $R$ be a commutative noetherian ring, $M$ a noetherian $R$-module. An ideal $I$ in $R$ contains an $M$-regular element if and only if $\operatorname{Hom}_R(R/I, M) = 0$.*

Proof ($\Rightarrow$) If $x \in I$ is $M$-regular, then given any $f \in \operatorname{Hom}_R(R/I, M)$ we have $0 = f(x) = xf(1)$ since $x \in I$. Because $x$ is $M$-regular, $f(1) = 0$, so $f = 0$.

($\Leftarrow$) (Contrapositive.) If $I$ consists of zero-divisors on $M$, by the previous proposition $I \subset \mathfrak{p} \in \operatorname{Ass}(M)$. Therefore

$$R/I \twoheadrightarrow R/\mathfrak{p} = R/\operatorname{Ann}(m) \subset M$$

for some $0 \neq m \in M$. Hence $\operatorname{Hom}_R(R/I, M) \neq 0$.

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# October 6th, 2014: Duality Theories, Injective Envelopes, and Indecomposable Injectives

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**44 Remark**

Here's a rough discussion of where we're headed and what Cohen-Macaulay means.

Cohen-Macaulay-ness is a "duality" theory. The prototypical duality theory comes from linear algebra. If $V$ is a finite dimensional vector space, there is a natural map $V \to \operatorname{Hom}_k(\operatorname{Hom}_k(V, k)) =: V^{**}$ given by $v \mapsto (\alpha \mapsto (v \mapsto \alpha(v)))$. This map is part of a natural transformation from the identity functor to the double-dual functor. Further, it restricts to an isomorphism on the subcategory of finite dimensional vector spaces.

Many more sophisticated dualities rely on this elementary duality, eg. Serre duality, Poincare duality; Fourier transforms are in the same spirit, etc. For instance, we can replace the field $k$ with $\mathbb{Z}$. We generally need a finiteness assumption for these sorts of dualities. For instance, suppose $F$ is a finitely generated projective (equivalently, free) $\mathbb{Z}$-module. The natural map $F \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}(F,\mathbb{Z}),\mathbb{Z})$ is then an isomorphism. If $G$ is a finitely generated torsion $\mathbb{Z}$-module, then the natural map

$$G \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_{\mathbb{Z}}(G,\mathbb{Q}/\mathbb{Z}),\mathbb{Q}/\mathbb{Z})$$

is an isomorphism. Roughly, over $\mathbb{Z}$ we have to use two different dualizing objects, $\mathbb{Z}$ and $\mathbb{Q}/\mathbb{Z}$. In fact, the minimal injective resolution of $\mathbb{Z}$ is

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0.$$

Indeed, this discussion is a special case of Pontryagin duality: if $G$ is a locally compact abelian group, then the natural map

$$G \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_{\mathbb{Z}}(G,S^1),S^1)$$

with $S^1 := \{z \in \mathbb{C} : |z| = 1\}$ is an isomorphism. Note that $\mathbb{Q}/\mathbb{Z} \hookrightarrow S^1$ via $z \mapsto e^{2\pi i z}$ (the roots of unity) and $\mathbb{Z} \hookrightarrow S^1$ via $z \mapsto e^{iz}$.

One of Grothendieck's great inventions was the "derived category" $\mathcal{D}$ of an abelian category $\mathcal{A}$. We won't go into this category very far, though the objects are complexes of objects in $\mathcal{A}$. If $M, N \in \mathcal{D}$, one may naturally define a complex $\mathrm{RHom}(M,N) \in \mathcal{D}$. Roughly, if $R$ is Cohen-Macaulay, there exists an object $\omega_R \in \mathcal{D}$ such that the natural map

$$M \to \mathrm{RHom}(\mathrm{RHom}(M,\omega_R),\omega_R)$$

is an isomorphism for all $M$. Indeed, $H^i(\mathrm{RHom}(M,N)) = \mathrm{Ext}^i(M,N)$. It also happens that $\omega_R$ is concentrated in a single degree, i.e. it is of the form $0 \to \cdots \to 0 \to \omega_R \to 0 \to \cdots$; this turns out to be very restrictive. We'll formulate as much of this result as we can just using Ext groups, without developing the machinery of derived categories.

What about Gorenstein rings? They satisfy the above "duality" isomorphism with the additional condition that $\omega_R$ is 'invertible", i.e. there exists some $\omega_R^{-1}$ such that $\omega_R \otimes \omega_R^{-1} \cong R$.

Here's a duality theorem; we won't define all the terms quite yet; it involves injective envelopes, local cohomology, and Ext groups.

**45 Theorem**

*Let $(R,\mathfrak{m},k)$ be a local noetherian ring and*

$$\omega_R := \mathrm{Hom}(H^d_{\mathfrak{m}}(R), E(R/\mathfrak{m})).$$

*Then the functor*

$$M \mapsto \mathrm{Ext}^p_R(M,\omega_R) =: M^\vee$$

*gives a duality between Cohen-Macaulay modules $M$ of depth $p$ and Cohen-Macaulay modules of depth $d-p$. In particular, the natural map $M \to M^{\vee\vee}$ is an isomorphism.*

**46 Remark**

First some remarks on the classification of indecomposable injective modules over a noetherian ring. This is related to the problem of reconstructing a commutative ring $R$ from the category of $R$-modules. The next theorem for instance allows us to in a way to reconstruct $\operatorname{spec} R$ from $R \mod$.

**47 Definition**

An $R$-module $M$ is indecomposable if $M = M' \oplus M''$ implies either $M' = 0$ or $M'' = 0$. Likewise if $M = M' \oplus M''$, we call $M'$ or $M''$ a direct summand of $M$.

**48 Remark**

In nice categories we can write any object as a sum of indecomposables, though generally not uniquely. Coherent sheaves offer an exception.

**49 Definition**

Let $M$ be an $R$-module. Define the injective envelope of $M$ (also called the injective hull), written $E(M)$, as the "smallest" injective module which contains $M$. To make this precise, we first say that a submodule $P \subset M$ is an essential submodule of $M$ if for all non-zero submodules $Q \subset M$, $P \cap Q \neq \{0\}$. In this situation we call $M$ an essential extension of $P$. $M$ is an essential extension of itself trivially.

The set of essential extensions of a module $M$ has maximal elements by Zorn's lemma. One may check the following:

**50 Proposition (Matsumura, Theorem B4)**

*An $R$-module $M$ is injective if and only if it has no essential extensions other than $M$.*

We define $E(M)$ as such a maximal extension. One may show that any two such maximal extensions are isomorphic in a way which fixes $M$, though not in general uniquely.

Finally, $E(M)$ is a minimal injective extension of $M$. Indeed, suppose $E'$ is a module such that $M \subset E' \subset E(M)$. Any non-zero module $P \subset E(M)$ has $P \cap M \neq 0$, so trivially $P \cap E' \neq 0$, hence $E(M)$ is an essential extension of $E'$. If $E' \neq E(M)$, by the theorem above $E'$ is not injective.

**51 Remark**

One may justify the above definitions in a variety of ways. The above is taken from Matsumura's Appendix B. Another way proceeds by first considering the case $R = \mathbb{Z}$, since injectivity there is simply equivalent to divisibility. In general we have an (exact) forgetful functor $R\text{-mod} \to \mathbb{Z}\text{-mod}$. It has an adjoint which sends injective $\mathbb{Z}$-modules to injective $R$-modules. One can then compute an injective envelope for $M$ as a $\mathbb{Z}$-module and apply the adjoint to get that $M$ embeds as an $R$-module into the corresponding injective object.

**52 Theorem**

*Let $R$ be a commutative ring.*

*(1) If $\mathfrak{p} \in \operatorname{spec} R$, then $E(R/\mathfrak{p})$ is indecomposable.*

*(2) Let $E$ be an injective $R$-module and $\mathfrak{p} \in \operatorname{Ass}(E)$. Then $E(R/\mathfrak{p})$ is a direct summand of $E$. In particular, if $E$ is indecomposable, then $E \cong E(R/\mathfrak{p})$.*

*(3) If $\mathfrak{p}, \mathfrak{q} \in \operatorname{spec} R$, then $E(R/\mathfrak{p}) \cong E(R/\mathfrak{q})$ if and only if $\mathfrak{p} = \mathfrak{q}$.*

Proof (1) Suppose $M, N$ are non-zero submodules of $E(R/\mathfrak{p})$. We will show $M \cap N \neq \{0\}$, which implies $E(R/\mathfrak{p})$ is indecomposable. Because $R/\mathfrak{p}$ is an essential submodule of $E(R/\mathfrak{p})$, $M \cap R/\mathfrak{p} \neq 0$ and $N \cap R/\mathfrak{p} \neq 0$. These intersections correspond to non-zero ideals of $R/\mathfrak{p}$, say $I$ and $J$. Since $\mathfrak{p}$ is prime, $0 \neq IJ \subset I \cap J$; the result follows.

To be continued next lecture.

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# October 8th, 2014: Local Cohomology Defined

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**53 Remark**

We continue proving the theorem from the end of last lecture. A minor note: $E_{R/\mathfrak{p}}(R/\mathfrak{p}) = \operatorname{Frac}(R/\mathfrak{p})$, whereas this is not the case in general for $E_R(R/\mathfrak{p})$. Unless otherwise stated, our $E$'s are $E_R$.

Proof (2) Let $E$ be injective, $\mathfrak{p} \in \operatorname{Ass}(E)$. There is a submodule of $E$ that is isomorphic to $R/\mathfrak{p}$, namely some $(m) \cong R/\operatorname{Ann}(m)$. Hence we have

$$\begin{array}{ccccc} 0 & \longrightarrow & R/\mathfrak{p} & \xrightarrow{\alpha} & E(R/\mathfrak{p}) \\ & & \Big\downarrow{\scriptstyle\beta} & \swarrow{\scriptstyle\gamma} & \\ & & E & & \end{array}$$

$\gamma$ exists since $E$ is injective. Claim: $\gamma$ is injective. Proof: since $R/\mathfrak{p}$ is an essential submodule of $E(R/\mathfrak{p})$, if $\ker\gamma \neq 0$, then $\alpha(R/\mathfrak{p}) \cap \ker\gamma \neq 0$. But $\beta$ is injective, forcing $\ker\gamma = 0$. Hence there is an exact sequence

$$0 \to E(R/\mathfrak{p}) \xrightarrow{\gamma} E.$$

Because $E(R/\mathfrak{p})$ is injective, $\gamma$ splits, whence $E(R/\mathfrak{p})$ is a direct summand of $E$. If $E$ is additionally indecomposable, $E(R/\mathfrak{p})$ is all of $E$.

(3) $\Leftarrow$ is trivial. For $\Rightarrow$, identify $E(R/\mathfrak{p})$ and $E(R/\mathfrak{q})$ and call them $E$. Hence $E$ has essential submodules isomorphic to $R/\mathfrak{p}$ and $R/\mathfrak{q}$. Those submodules have nonzero intersection from the essential condition. That intersection is then of the form $I/\mathfrak{p} \subset R/\mathfrak{p}$ or $J/\mathfrak{q} \subset R/\mathfrak{q}$ for ideals $I \supsetneq \mathfrak{p}, J \supsetneq \mathfrak{q}$. Moreover, their intersection is annihilated by $\mathfrak{p} + \mathfrak{q} \subset R$, so $\mathfrak{q}(I/\mathfrak{p}) = 0$, i.e. $\mathfrak{q}I \subset \mathfrak{p}$, and likewise $\mathfrak{p}J \subset \mathfrak{q}$. Pick $x \in I - \mathfrak{p}$, $q \in \mathfrak{q}$; then $qx \in \mathfrak{p}$, but $x \notin \mathfrak{p}$, so $q \in \mathfrak{p}$. In this way, $\mathfrak{q} \subset \mathfrak{p}$ and likewise $\mathfrak{p} \subset \mathfrak{q}$, so $\mathfrak{p} = \mathfrak{q}$.

**54 Corollary**

*If $R$ is a commutative noetherian ring and $\mathfrak{p} \in \operatorname{spec} R$, then $\operatorname{Ass}(E(R/\mathfrak{p})) = \{\mathfrak{p}\}$ and each element of $E(R/\mathfrak{p})$ is annihilated by some power of $\mathfrak{p}$.*

Proof $R/\mathfrak{p} \subset E(R/\mathfrak{p})$, so $\{\mathfrak{p}\} = \operatorname{Ass}(R/\mathfrak{p}) \subset \operatorname{Ass}(E(R/\mathfrak{p}))$. On the other hand, if $\mathfrak{q} \in \operatorname{Ass}(E(R/\mathfrak{p}))$, from the previous theorem, $E(R/\mathfrak{q}) \cong E(R/\mathfrak{p})$, so $\mathfrak{p} = \mathfrak{q}$.

Let $0 \neq x \in E(R/\mathfrak{p})$. $R/\operatorname{Ann}(x) \cong Rx \hookrightarrow E(R/\mathfrak{p})$, so $\operatorname{Ass}(R/\operatorname{Ann}(x)) = \{\mathfrak{p}\}$. Hence $\operatorname{Ann}(x)$ is $\mathfrak{p}$-primary, i.e. $\operatorname{Rad}\operatorname{Ann}(x) = \mathfrak{p}$ (see Matsumura, Theorem 6.6). Thus any $p \in \mathfrak{p}$ has $p^n \in \operatorname{Ann}(x)$ for $n$ large enough, so $p^n x = 0$. Letting $p$ range over a (necessarily finite) generating set for $\mathfrak{p}$, we see $\mathfrak{p}^N x = 0$ for $N$ large enough.

**55 Theorem**

*A ring $R$ is left noetherian if and only if every direct sum of injective left $R$-modules is injective.*

**56 Exercise**

A product of injective modules is injective.

Proof (Sketch.) One uses Baer's criterion, which says that a left $R$-module $E$ is injective if and only if for all inclusions of ideals $\alpha\colon I \hookrightarrow R$ the dashed arrow always exists given the other arrow $\beta$:

$$\begin{array}{ccccc} 0 & \longrightarrow & I & \overset{\alpha}{\hookrightarrow} & R \\ & & \Big\downarrow{\scriptstyle\beta} & \swarrow{\scriptstyle\exists\gamma} & \\ & & E & & \end{array}$$

(A similar characterization of projectives would require an affirmative answer to Whitehead's conjecture: an abelian group $G$ is free if and only if $\operatorname{Ext}^1_{\mathbb{Z}}(G, \mathbb{Z}) = 0$. Shelah proved that Whitehead's conjecture, among other things, is undecidable in ZFC.)

**57 Theorem**

*If $R$ is commutative noetherian, then every injective $R$-module is isomorphic to a direct sum of $E(R/\mathfrak{p})$'s for $\mathfrak{p} \in \operatorname{spec} R$ with various multiplicities.*

Proof Roughly, apply Zorn's lemma and the fact that a sum of injectives is injective in a noetherian ring.

**58 Definition**
Let $(R, \mathfrak{m}, k)$ be a local noetherian ring. The $\boxed{\text{0th local cohomology group}}$ of an $R$-module $M$ is

$$\boxed{H^0_{\mathfrak{m}}(M)} := \{a \in M : \exists n \text{ s.t. } \mathfrak{m}^n a = 0\}.$$

Indeed, this is an $R$-submodule of $M$.

**59 Proposition**
$H^0_{\mathfrak{m}}(-) \colon R\text{-mod} \to R\text{-mod}$ *is a left exact functor.*

Proof If $\phi \colon M \to N$ is an $R$-module homomorphism and $a \in H^0_{\mathfrak{m}}(M)$ is such that $\mathfrak{m}^n a = 0$, then $0 = \phi(\mathfrak{m}^n a) = \mathfrak{m}^n \phi(a)$, so $\phi(a) \in H^0_{\mathfrak{m}}(N)$. Hence $\phi$ restricts to give $H^0_{\mathfrak{m}}(M) \to H^0_{\mathfrak{m}}(N)$, proving functoriality. For left exactness, if $0 \to L \stackrel{\alpha}{\to} M \stackrel{\beta}{\to} N$ is exact, we get a complex

$$0 \to H^0_{\mathfrak{m}}(L) \stackrel{\alpha}{\to} H^0_{\mathfrak{m}}(M) \stackrel{\beta}{\to} H^0_{\mathfrak{m}}(N).$$

$\alpha$ here is trivially injective. For exactness at $H^0_{\mathfrak{m}}(M)$, we need only show $\ker \beta \subset \operatorname{im} \alpha$. Take $a \in H^0_{\mathfrak{m}}(M)$ and suppose $\beta(a) = 0$. By hypothesis there exists some $b \in L$ such that $\alpha(b) = a$. If $\mathfrak{m}^n a = 0$, then $0 = \mathfrak{m}^n \alpha(b) = \alpha(\mathfrak{m}^n b)$, but $\alpha$ was injective, so $\mathfrak{m}^n b = 0$, so $b \in H^0_{\mathfrak{m}}(L)$.

**60 Example**
Since every $x \in E(R/\mathfrak{m})$ is annihilated by a power of $\mathfrak{m}$, $H^0_{\mathfrak{m}}(E(R/\mathfrak{m})) = E(R/\mathfrak{m})$.

**61 Definition**
Let $(R, \mathfrak{m}, k)$ be a local noetherian ring. The $\boxed{i\text{th local cohomology group}}$ functor, $\boxed{H^i_{\mathfrak{m}}(-)}$, is the $i$th right derived functor of $H^0_{\mathfrak{m}}(-)$.

Explicitly, to compute $H^i_{\mathfrak{m}}(M)$, take an injective resolution of $M$,

$$0 \to M \to I^0 \to I^1 \to \cdots$$

and $H^i_{\mathfrak{m}}(M)$ is the $i$th (co)homology group in the complex

$$0 \to H^0_{\mathfrak{m}}(I^0) \to H^0_{\mathfrak{m}}(I^1) \to H^0_{\mathfrak{m}}(I^2) \to \cdots .$$

As usual, this is independent (up to isomorphism) of the injective resolution. Also as usual, if $0 \to L \to M \to N \to 0$ is an exact sequence of $R$-modules, there is a long exact sequence

$$\begin{aligned} 0 &\to H^0_{\mathfrak{m}}(L) \to H^0_{\mathfrak{m}}(M) \to H^0_{\mathfrak{m}}(N) \\ &\to H^1_{\mathfrak{m}}(L) \to H^1_{\mathfrak{m}}(M) \to H^1_{\mathfrak{m}}(N) \\ &\to \cdots \end{aligned}$$

**62 Definition**
Let $(R, \mathfrak{m}, k)$ be a local noetherian ring. The $\boxed{\text{depth}}$ of an $R$-module $M$ is the smallest integer $d$ such that $H^d_{\mathfrak{m}}(M) \neq 0$.

There are several different ways to define depth. For instance one definition uses the smallest $d$ such that $\operatorname{Ext}^d_R(k, M)$ is nonzero. We'll prove this eventually.

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## October 10th, 2014: Examples of Injectives and Injective Hulls

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**63 Example**

Here are some examples of injective modules. Unlike projectives, these tend to be more "slippery."

(1) If $R$ is a commutative domain, the injective envelope $E(R)$ of $R$ as an $R$-module is the field of fractions $K := \mathrm{Frac}(R)$. A special case of this was mentioned last lecture.

Proof Certainly $R \subset K$. $R$ is an essential submodule of $K$ since if $0 \neq a \in K$, then $a = xy^{-1}$ for some $x, y \in R$, so $x \in Ra \cap I \neq 0$. It suffices to show that $K$ is injective. For that, we use Baer's criterion. Suppose we have an ideal $I$ in $R$ together with a map $f\colon I \to K$. If $a, b \in I - 0$, then $f(ab) = af(b) = bf(a)$, so $a^{-1}f(a) = b^{-1}f(b)$ is well-defined; call this $k$. Define $g\colon R \to K$ by $g(x) = xk$, so for $a \in I - 0$, $g(a) = aa^{-1}f(a) = f(a)$. That is, the following commutes:

$$\begin{array}{ccccc} 0 & \longrightarrow & I & \hookrightarrow & R \\ & & \downarrow f & \swarrow g & \\ & & K & & \end{array}$$

Hence $K$ is injective.

(2) If $R$ is a PID and $K := \mathrm{Frac}(R)$, then $K/R$ is an injective $R$-module. Note that $K/R$ is noetherian, hence is a direct sum of indecomposable $R$-modules, namely

$$K/R \cong \bigoplus_{\mathfrak{m} \in \mathrm{Max}(R)} E(R/\mathfrak{m}) \cong \bigoplus_{x \text{ "irred."}} E(R/(x))$$

where "irred." refers to taking irreducibles only up to associates, i.e. picking only one generator of each $\mathfrak{m} \in \mathrm{Max}(R)$. (Here we use the fact that a PID has Krull dimension 0 or 1.) For instance, $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module and

$$\mathbb{Q}/\mathbb{Z} \cong \bigoplus_{p \text{ prime}} E(\mathbb{Z}/(p)).$$

(It turns out that $E(\mathbb{Z}/(p))$ is a "Prüfer group," namely the direct limit of the groups $\mathbb{Z}/(p^n)$.)

For $x \in R$, we have a sequence of $R$-submodules of $K$: $xR \subset R \subset x^{-1}R \subset x^{-2}R \subset \cdots \subset R[x^{-1}]$. Here

$$\frac{R[x^{-1}]}{R} \cong E\left(\frac{R}{(x)}\right).$$

Note $\frac{x^{-n-1}R}{x^{-n}R} \cong \frac{R}{(x)}$. (For instance, try $R = \mathbb{Z}$, $x = 2$.)

(3) The situation is more complicated for non-PID's. Consider $\mathbb{C}[x, y]$ and the sequence

$$0 \to \mathbb{C}[x, y] \to \mathbb{C}(x, y) \to \mathbb{C}(x, y)/\mathbb{C}[x, y] \to 0.$$

Let $R = \mathbb{C}[x, y]$. It turns out that

$$0 \to \mathbb{C}[x, y] \to \mathbb{C}(x, y) \to \bigoplus_{\mathrm{ht}(\mathfrak{p})=1} E(R/\mathfrak{p}) \to \bigoplus_{\mathfrak{m} \in \mathrm{Max}\, R} E(R/\mathfrak{m}) \to 0,$$

so in particular $\mathrm{Frac}(R)/R$ here is not just $\oplus_{\mathfrak{m}} E(R/\mathfrak{m})$.

(4) Let $(R, \mathfrak{m}, k)$ be a local noetherian commutative ring. Note that $(R_{\mathfrak{m}})/(\mathfrak{m}^n R_{\mathfrak{m}}) \cong (R/\mathfrak{m}^n)_{\mathfrak{m}} \cong R/\mathfrak{m}^n$. For instance, if $R = k[x, y]_{\mathfrak{m}}$ with $\mathfrak{m} = (x, y)$ we have

$$\begin{aligned} k[x, y] &= k \oplus \mathfrak{m} \\ &= k \oplus (kx + ky) \oplus \mathfrak{m}^2 \\ &= k \oplus k[x, y]_1 \oplus k[x, y]_2 \oplus \mathfrak{m}^3 \\ &= \cdots. \end{aligned}$$

Roughly, we "slice" $R$ using powers of $\mathfrak{m}$, which corresponds to "slicing" $E(R/\mathfrak{m})$ using

$$\mathrm{Hom}_R(\mathfrak{m}^i/\mathfrak{m}^{i+1}, R/\mathfrak{m}).$$

We next make this precise.

**64 Remark**

Let $(R, \mathfrak{m}, k)$ be a local noetherian ring. Apply the functor $\mathrm{Hom}_R(-, E)$ to the sequence

$$\cdots \to R/\mathfrak{m}^n \to \cdots R/\mathfrak{m}^2 \to R/\mathfrak{m} \to 0.$$

to get

$$0 \to \mathrm{Hom}_R(R/\mathfrak{m}, E) \to \mathrm{Hom}_R(R/\mathfrak{m}^2, E) \to \cdots .$$

Each of our original maps was surjective, so each of the induced maps are injective (since $\mathrm{Hom}_R(-, E)$ is contravariant and left exact). Take the direct limit of the second sequence, $\varinjlim \mathrm{Hom}_R(R/\mathfrak{m}^n, E)$. The result is sometimes called the "directed union" when the underlying maps are injective. Because $R \to R/\mathfrak{m}^n$ is surjective, there is an injective map $\mathrm{Hom}_R(R/\mathfrak{m}^n, E) \to \mathrm{Hom}_R(R, E) \cong E$. We may then add $E$ to the second diagram above, so by the universal property of direct limits, there is a morphism $\Phi\colon \varinjlim \mathrm{Hom}_R(R/\mathfrak{m}^n, E) \to E$. $\Phi$ is surjective since every element $e$ of $E = E(R/\mathfrak{m})$ is annihilated by a power of $\mathfrak{m}$ as proved last lecture. Hence we have a map $R/\mathfrak{m}^n \to E$ for $n$ large enough with image $e$ in $E$; by commutativity, $e$ is in the image of $\Phi$. On the other hand, since each of the maps to $E$ are injective, $\Phi$ is injective, so $\Phi$ is an isomorphism.

Next consider the sequence

$$0 \to \mathfrak{m}^n/\mathfrak{m}^{n+1} \to R/\mathfrak{m}^{n+1} \to R/\mathfrak{m}^n \to 0.$$

Because $E$ is injective, the sequence

$$0 \to \mathrm{Hom}_R(R/\mathfrak{m}^n, E) \to \mathrm{Hom}_R(R/\mathfrak{m}^{n+1}, E) \to \mathrm{Hom}_R(\mathfrak{m}^n/\mathfrak{m}^{n+1}, E) \to 0$$

is exact. Since $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ is an $R/\mathfrak{m}$-vector space, $\mathrm{Hom}_R(\mathfrak{m}^n/\mathfrak{m}^{n+1}, E) \cong \mathrm{Hom}_R(\mathfrak{m}^n/\mathfrak{m}^{n+1}, R/\mathfrak{m})$.

This $E$ will come up more in future lectures.

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# October 13th, 2014: Local Cohomology and Depth through Ext; Matlis Duality

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**65 Proposition**

*Let $(R, \mathfrak{m}, k)$ be a local noetherian ring. Then*

$$H^d_{\mathfrak{m}}(-) = \varinjlim_n \mathrm{Ext}^d_R(R/\mathfrak{m}^n, -).$$

Proof Recall that

$$\begin{aligned} H^0_{\mathfrak{m}}(M) &= \{a \in M : \mathfrak{m}^n a = 0 \text{ for } n \text{ large}\} \\ &= \bigcup_{n=1}^{\infty} \{a \in M : \mathfrak{m}^n a = 0\} = \bigcup_{n=1}^{\infty} \mathrm{Hom}_R(R/\mathfrak{m}^n, M) \\ &= \varinjlim_n \mathrm{Hom}_R(R/\mathfrak{m}^n, M) \end{aligned}$$

using digram of $R/\mathfrak{m}^n$'s from last time and applying the $\mathrm{Hom}_R(-, M)$ functor to get a directed system in which to compute the direct limit. The above computation is functorial, so $H^0_{\mathfrak{m}}(-) = \varinjlim \mathrm{Hom}_R(R/\mathfrak{m}^n, -)$. Derived functors commute with $\varinjlim$ in this case to give the result.

**66 Proposition**

*Let $(R, \mathfrak{m}, k)$ be a local noetherian ring. Recall the depth of an $R$-module $M$ is $\inf\{i : H^i_{\mathfrak{m}}(M) \neq 0\}$. Equivalently, the depth is $\inf\{i : \mathrm{Ext}^i_R(k, M) \neq 0\}$ where $k := R/\mathfrak{m}$ as an $R$-module.*

Proof Write $d$ for the first infimum and $e$ for the second one; we show $d = e$. If $d < e$, then $\mathrm{Ext}^d_R(k, M) = 0$ and by induction on $n \geq 0$, $\mathrm{Ext}^d_R(R/\mathfrak{m}^n, M) = 0$:

Sketch of induction argument: take a short exact sequence $0 \to L \to X \to N \to 0$ of $R$-modules of finite length, so there exists $t$ such that $\mathfrak{m}^t L = \mathfrak{m}^t X = \mathfrak{m}^t N = 0$. Consider the following part of the associated long exact sequence:

$$\cdots \to \mathrm{Ext}^d_R(N, M) \to \mathrm{Ext}^d(X, M) \to \mathrm{Ext}^d(L, M) \to \cdots .$$

If length is 1, the $L$ and $N$ are $R/\mathfrak{m}$ and inductively the left and right terms vanish, so the middle term (with $X$ of length 2) vanishes; continue in this way.

Hence $\varinjlim_n \mathrm{Ext}^d_R(R/\mathfrak{m}^n, M) = 0$, contradicting the definition of $d$. Therefore $d \geq e$. Because $\mathrm{Ext}^{e-1}_R(k, M) = 0$, $\mathrm{Ext}^{e-1}_R(\mathfrak{m}^n/\mathfrak{m}^{n+1}, M) = 0$ for all $n$: $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ is a $k$-vector space (finite dimensional from the noetherian hypothesis); distribute the finite sum over the Ext. By induction, $\mathrm{Ext}^{e-1}_R(\mathfrak{m}^n/\mathfrak{m}^q, M) = 0$ for all $q \geq n \geq 0$. For instance, consider the sequence $0 \to \mathfrak{m}/\mathfrak{m}^2 \to R/\mathfrak{m}^2 \to R/\mathfrak{m} \to 0$ and the associated long exact sequence at the $e - 1$ part to see how the induction works.

Now, the natural map $\mathrm{Ext}^e_R(R/\mathfrak{m}^n, M) \to \mathrm{Ext}^e_R(R/\mathfrak{m}^q, M)$ is injective for all $n \leq q$: use the short exact sequence $0 \to \mathfrak{m}^n/\mathfrak{m}^q \to R/\mathfrak{m}^q \to R/\mathfrak{m}^n \to 0$ and the corresponding long exact sequence. Hence $H^e_{\mathfrak{m}}(M)$ is the union of its submodules $\mathrm{Ext}^e_R(R/\mathfrak{m}^n, M)$, but $\mathrm{Ext}^e_R(R/\mathfrak{m}, M) \neq 0$, so $H^e_{\mathfrak{m}}(M)$ is non-zero. Therefore $d \leq e$, so $d = e$.

*(There is a minor fiddle: what happens if the* inf*'s above were over empty sets? We may conventionally set them to $-\infty$ in this case, though we will essentially ignore it.)*

**67 Definition**

Let $(R, \mathfrak{m}, k)$ be a local noetherian ring. Write $E := E(R/\mathfrak{m})$. The Matlis dual of an $R$-module $M$ is

$$\boxed{M'} := \mathrm{Hom}_R(M, E).$$

For instance, $R' = E$.

There is a natural map $M \to M''$ given by $a \mapsto \Phi_a$ where $\Phi_a \colon M' \to E$ is given by $\Phi_a(f) := f(a)$.

**68 Theorem**

*Let $(R, \mathfrak{m}, k)$ be a local noetherian ring.*

*(1) The canonical map $M \to M''$ is injective.*

*(2) If $M$ has finite length $\ell(M)$, then $\ell(M') = \ell(M)$ and $M \to M''$ is an isomorphism.*

*(3) If $M$ is artinian, then $M \to M''$ is an isomorphism.*

*(4) $E' = \mathrm{Hom}_R(E, E) \cong \widehat{R}$, the completion of $R$ at $\mathfrak{m}$; this will be defined shortly.*

*(5) If $R$ is complete (i.e. $R = \widehat{R}$), then $\mathrm{Hom}_R(-, E)$ is a duality between the categories of noetherian and artinian $R$-modules.*

*Here a duality between categories $\mathcal{C}$ and $\mathcal{D}$ is a pair of contravariant functors $F \colon \mathcal{C} \to \mathcal{D}$, $G \colon \mathcal{D} \to \mathcal{C}$ such that $FG \cong \mathrm{id}_{\mathcal{D}}$ and $GF \cong \mathrm{id}_{\mathcal{C}}$. Another name for this is an antiequivalence of categories.*

Proof For (1), let $0 \neq a \in M$. Then

$$Ra \cong R/\operatorname{Ann}(a) \twoheadrightarrow R/\mathfrak{m} \hookrightarrow E,$$

and the composite $f\colon Ra \to E$ has $f(a) \neq 0$. Since $E$ is injective, $f$ extends to a homomorphism $f\colon M \to E$ such that $f(a) \neq 0$. The image of $a$ in $M''$ is the map that sends $f$ to $f(a) \neq 0$, so the image of $a$ in $M''$ is non-zero, giving (1).

For (2), we argue by induction on $\ell(M)$. When $\ell(M) = 1$, $M \cong R/\mathfrak{m}$ and $M' \cong \operatorname{Hom}_R(R/\mathfrak{m}, E)$. However, $E$ is the injective envelope of $R/\mathfrak{m}$ so contains a unique submodule isomorphic to $R/\mathfrak{m}$. It follows that $\operatorname{Hom}_R(R/\mathfrak{m}, E) \cong \operatorname{Hom}_R(R/\mathfrak{m}, R/\mathfrak{m}) \cong R/\mathfrak{m}$. Hence $\ell(M') = \ell(M)$ in this case. Take $\ell(M) \geq 2$. Inductively, suppose the result is true for modules of length strictly smaller than $\ell(M)$. Take a short exact sequence

$$0 \to L \to M \to N \to 0$$

with $L \neq 0$ and $N \neq 0$ (any non-zero proper submodule $L$ will do). Since $E$ is injective, $0 \to N' \to M' \to L' \to 0$ is exact, so

$$\ell(M') = \ell(N') + \ell(L') = \ell(N) + \ell(L) = \ell(M).$$

Now, $M \to M''$ is injective and $\ell(M) = \ell(M') = \ell(M'')$, so $M \to M''$ is also surjective, hence an isomorphism.

For (3), first a lemma:

**69 Lemma**

*If $(R, \mathfrak{m}, k)$ is a local noetherian ring with finite length and $E := E(R/\mathfrak{m})$, then $E' \cong R$.*

Proof From (1) we have an injection $R \to R'' = E'$. Since $\ell(R)$ is finite, from (2) this is an isomorphism.

To be continued next time.

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## October 15th, 2014: Matlis Duality Continued

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**70 Remark**

We continue proving the theorem from last time. (It's like "tossing a salad": the more you do it, the better it gets.)

Proof Let $E_n = \{E : \mathfrak{m}^n a = 0\}$. Then $E = \cup_n E_n$, $E_0 \subset E_1 \subset E_2 \subset \cdots$. Since $E_n \cong \operatorname{Hom}_R(R/\mathfrak{m}^n, E) = (R/\mathfrak{m}^n)'$ and $\ell(R/\mathfrak{m}^n) < \infty$, $\ell(E_n) < \infty$, so $E$ is the union of finite length modules. Indeed, $E_n \cong E_{R/\mathfrak{m}^n}(R/\mathfrak{m})$ since $R/\mathfrak{m}$ is essential in $E_n$ and $E_n$ is an $R/\mathfrak{m}$ module; it is injective by a quick application of Baer's criterion.

Now let $f \in E'$. Then $f(E_n) \subset E_n$. Write $f_n := f|_{E_n}$, so $f_n \in \operatorname{Hom}_R(E_n, E_n)$. If $n \geq m$, $f_n|_{E_m} = f_m$. Conversely, given homomorphisms $g_n \in \operatorname{Hom}(E_n, E_n)$ such that $g_n|_{E_m} = g_m$ for all $n \geq m$, we can define $g\colon E \to E$ by $g(a) = g_n(a)$ for $a \in E_n$. Hence

$$\begin{aligned} E' = \operatorname{Hom}_R(E, E) &= \varprojlim_n \operatorname{Hom}_R(E_n, E_n) \\ &= \varprojlim_n \operatorname{Hom}_{R/\mathfrak{m}^n}(E_n, E_n) = \varprojlim (E_n)'_{R/\mathfrak{m}^n} \\ &= \varprojlim (R/\mathfrak{m}^n)''_{R/\mathfrak{m}^n} = \varprojlim (R/\mathfrak{m}^n) \\ &=: \boxed{\widehat{R}}. \end{aligned}$$

This gives (4).

We next consider (5). Every submodule of an artinian module is artinian (from the descending chain condition on submodules), so $Ra \cong R/\operatorname{Ann}(a)$ for $a \in M$ is artinian. Since $R$ is noetherian, $Ra$ is as well, so has finite length. Hence $M$ is the union of finite length modules.

**71 Lemma**

*Let $(R, \mathfrak{m}, k)$ be a noetherian local ring and $E := E(R/\mathfrak{m})$. If $M$ is an artinian $R$-module, then $M$ embeds in $E^n$ for some $n$.*

PROOF Consider all pairs $(f, n)$ where $f\colon M \to E^n$ is a homomorphism. Pick $(f, n)$ such that $\ker f$ is minimal among such kernels. If $\ker f = 0$, we're done; if $\ker f \neq 0$, pick $0 \neq a \in \ker f$. Now there exists a homomorphism $g\colon M \to E$ such that $g(a) \neq 0$ (see proof of (1)). Then $f \oplus g\colon M \to E^n \oplus E$ has smaller kernel, $a \notin \ker(f \oplus g)$, a contradiction.

By the lemma, we have $0 \to M \to E^n$ injective. Apply $\operatorname{Hom}_R(-, E)$ to get $(E^n)' \to M' \to 0$ exact, but $(E^n)' = (E')^n = R^n$ by (4). Hence $M'$ is noetherian.

On the other hand, if $N$ is a noetherian $R$-module, there exists a surjection $R^n \to N$ for some $n$, hence an injective map $N' \to (R^n)' = (R')^n = E^n$. We have seen that $E$ is artinian so $N'$ is artinian as well. Hence $\operatorname{Hom}_R(-, E)$ sends noetherian modules to artinian modules and vice-versa.

(3) is incomplete at present. It is partially proved at the end of the October 17th lecture.

**72 Proposition**

*Let $(R, \mathfrak{m}, k)$ be a local noetherian ring.*

*(1) If $P$ is a projective $R$-module then $P'$ is an injective $R$-module.*

*(2) $\operatorname{Ext}^i_R(M, N') \cong \operatorname{Tor}^R_i(N, M)'$.*

PROOF For (1), let $P$ be a projective $R$-module. Then

$$\operatorname{Hom}_R(M, P') = \operatorname{Hom}_R(M, \operatorname{Hom}_R(P, E)) \cong \operatorname{Hom}_R(M \otimes_R P, E),$$

so $\operatorname{Hom}_R(-, P')$ is the composition of the exact functors $- \otimes_R P$ and $\operatorname{Hom}_R(-, E)$, so is exact. Hence $P'$ is injective. (Indeed, we only needed $P$ flat.)

For (2), when $i = 0$,

$$\operatorname{Hom}_R(M, N') = \operatorname{Hom}_R(M, \operatorname{Hom}_R(N, E)) \cong \operatorname{Hom}_R(M \otimes_R N, E) = (M \otimes_R N)',$$

as required. Now let $P_* \to N$ be a projective resolution. Since $E$ is injective, $\operatorname{Hom}_R(-, E)$ sends exact sequences to exact sequences and therefore commutes with homology, i.e. if $C_*$ is a complex, then $\operatorname{Hom}_R(H^i(C_*), E) = H^i(\operatorname{Hom}_R(C_*, E))$. Therefore $N' \to P'_*$ is an exact sequence, and therefore an injective resolution of $N'$ by (1). Hence

$$\operatorname{Ext}^i_R(M, N') = H^i(\operatorname{Hom}_R(M, P'_*)) = H^i((M \otimes_R P_*)') = (H^i(M \otimes_R P_*))' = \operatorname{Tor}^R_i(M, N)'.$$

**73 Remark**

Last time we showed that $H^i_{\mathfrak{m}}(M) \cong \varinjlim_n \operatorname{Ext}^i(R/\mathfrak{m}^n, M)$. We had commuted $\varinjlim$ and derived functors at one point; here we justify that step.

**74 Lemma**

*Direct limits of exact sequences are exact. More precisely, given exact sequences $X_i \xrightarrow{f_i} Y_i \xrightarrow{g_i} Z_i$, together with morphisms $\alpha_i, \beta_i, \gamma_i$ such that the following commute for all $i \geq 0$*

$$\begin{array}{ccccc} X_i & \xrightarrow{f_i} & Y_i & \xrightarrow{g_i} & Z_i \\ \downarrow{\alpha_i} & & \downarrow{\beta_i} & & \downarrow{\gamma_i} \\ X_{i+1} & \xrightarrow{f_{i+1}} & Y_{i+1} & \xrightarrow{g_{i+1}} & Z_{i+1} \end{array}$$

*then there is an induced exact sequence*

$$\varinjlim X_i \xrightarrow{f} \varinjlim Y_i \xrightarrow{g} \varinjlim Z_i.$$

PROOF The induced sequence always exists; we must only show exactness. Let $y \in \ker g$. We have maps $\overline{\alpha}_i \colon X_i \to \varinjlim X_i$ and if $\alpha_{kj} := \alpha_{k-1} \cdots \alpha_{j+1}\alpha_j$ for $j < k$, $\overline{\alpha}_k \alpha_{kj} = \overline{\alpha}_j$ for all $j < k$. Do the same with the $\beta$'s. Let $y_i \in Y_i$ be such that $y = \overline{\beta}_i(y_i)$. Then $0 = g(y) = g\overline{\beta}_i(y_i) = \overline{\gamma}_i g_i(y_i)$. Therefore for all sufficiently large $k$, $0 = \gamma_{ki} g_i(y_i) = g_k \beta_{ki}(y_i)$. Hence there exists $x_k \in X_k$ such that $f_k(x_k) = \beta_{ki}(y_i)$. Let $x = \overline{\alpha}_k(x_k)$. Then $f(x) = \overline{\beta}_k f_k(x_k) = \overline{\beta}_k \beta_{ki}(y_i) = \overline{\beta}_i(y_i) = y$.

To be continued.

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# October 17th, 2014: Local Cohomology and Injective Hulls Continued

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**75 Remark**

We conclude the remark at the end of last time, namely:

**76 Proposition (Direct limits commute with homology)**

*Suppose* $X_i \xrightarrow{f_i} Y_i \xrightarrow{g_i} Z_i$ *are sequences of* $R$*-modules such that* $g_i f_i = 0$ *for all* $i$*. Further suppose that there are commutative diagrams*

$$\begin{array}{ccccc} X_i & \xrightarrow{f_i} & Y_i & \xrightarrow{g_i} & Z_i \\ \downarrow{\alpha_i} & & \downarrow{\beta_i} & & \downarrow{\gamma_i} \\ X_{i+1} & \xrightarrow{f_{i+1}} & Y_{i+1} & \xrightarrow{g_{i+1}} & Z_{i+1} \end{array}$$

*Then*

$$H(\varinjlim_i X_i \xrightarrow{f} \varinjlim_i Y_i \xrightarrow{g} \varinjlim_i Z_i) \cong \varinjlim_i H(X_i \xrightarrow{f_i} Y_i \xrightarrow{g_i} Z_i).$$

PROOF One may prove this directly by diagram chasing in a manner analogous to the previous lemma. Alternatively, we may use several general facts: the category of $R$-modules is an abelian category; the category of chain complexes of an abelian category is abelian; in this sense, the direct limit is an exact functor between abelian categories; exact functors commute with homology for general abelian categories.

**77 Corollary**

$H^i_{\mathfrak{m}}(M) \cong \varinjlim \operatorname{Ext}^i_R(R/\mathfrak{m}^n, M)$*, as was shown above, modulo the proposition.*

**78 Definition**

Let $M$ be an $R$-module. A minimal injective resolution

$$0 \to M \to I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} \cdots$$

is an injective resolution such that $d^j(I^j)$ is an essential submodule of $I^{j+1}$ for each $j$.

**79 Lemma**

*Let $(R, \mathfrak{m}, k)$ be a noetherian local ring, $M$ an $R$-module. If every element of $M$ is killed by a power of $\mathfrak{m}$, then so is every element of $E(M)$.*

Proof We first show $\operatorname{Ass}(M) = \{\mathfrak{m}\}$. Suppose $\mathfrak{p} \in \operatorname{Ass}(M)$, so we have some $x \in M$ with $\operatorname{Ann}(x) = \mathfrak{p}$. Hence $R/\mathfrak{p} = R/\operatorname{Ann}(x) \cong Rx \subset M$. In particular, $1 + \mathfrak{p}$ is killed by a power of $\mathfrak{m}$, so $\mathfrak{m}^n \subset \mathfrak{p}$ for $n$ large enough. Hence we have a surjection $R/\mathfrak{m}^n \to R/\mathfrak{p}$. Since $R/\mathfrak{m}^n$ is finite length, so is $R/\mathfrak{p}$, which is also noetherian, hence is artinian. We had showed last lecture that an artinian $R$-module embeds into a power of $E := E(R/\mathfrak{m})$, so $R/\mathfrak{p} \hookrightarrow E^m$. But then $\{\mathfrak{p}\} = \operatorname{Ass}(R/\mathfrak{p}) \subset \operatorname{Ass}(E^m) = \{\mathfrak{m}\}$, so $\mathfrak{p} = \mathfrak{m}$.

We next note that if $M \subset N$ is essential, then $\operatorname{Ass}(M) = \operatorname{Ass}(N)$. It suffices to show this for $N = E(M)$, since $\operatorname{Ass}(M) \subset \operatorname{Ass}(N) \subset \operatorname{Ass}(E(M))$. If $\mathfrak{p} \in \operatorname{Ass}(E(M))$, then from the theorem above $E(R/\mathfrak{p})$ is a direct summand of $E(M)$, so $M \cap E(R/\mathfrak{p}) \neq 0$. But we have $\varnothing \neq \operatorname{Ass}(M \cap E(R/\mathfrak{p})) \subset \operatorname{Ass}(E(R/\mathfrak{p})) = \{\mathfrak{p}\}$, so $\{\mathfrak{p}\} = \operatorname{Ass}(M \cap E(R/\mathfrak{p})) \subset \operatorname{Ass}(M)$, giving the reverse inclusion.

Hence $\operatorname{Ass}(E(M)) = \operatorname{Ass}(M) = \{\mathfrak{m}\}$, so $E(M)$ is a sum of copies of $E(R/\mathfrak{m})$. Each element of $E(R/\mathfrak{m})$ is killed by some power of $\mathfrak{m}$, and the same is true of the direct sum, completing the proof.

**80 Proposition**

*Let $(R, \mathfrak{m}, k)$ be a noetherian local ring, $M$ an $R$-module.*

*(1) If $0 \to M \to I^0 \to I^1 \to I^2 \to \cdots$ is an injective resolution of $M$, then*

$$0 \to H^0_{\mathfrak{m}}(I^0) \to H^0_{\mathfrak{m}}(I^1) \to H^0_{\mathfrak{m}}(I^2) \to \cdots$$

*is a subcomplex of the original injective resolution and*

$$H^i_{\mathfrak{m}}(M) \cong H^i(0 \to H^0_{\mathfrak{m}}(I^0) \to H^0_{\mathfrak{m}}(I^1) \to \cdots).$$

*(2) If every element of $M$ is annihilated by a power of $\mathfrak{m}$, then $H^0_{\mathfrak{m}}(M) = M$ and $H^i_{\mathfrak{m}}(M) = 0$ for all $i > 0$.*

Proof We first note that

$$\begin{aligned} H^0_{\mathfrak{m}}(I^j) &= \varinjlim_n \operatorname{Hom}(R/\mathfrak{m}^n, I^j) \\ &= \varinjlim_n \operatorname{Hom}(R/\mathfrak{m}^n, H^0_{\mathfrak{m}}(I^j)) \end{aligned}$$

since each element of the image of $f \colon R/\mathfrak{m}^n \to I^j$ is annihilated by a power of $\mathfrak{m}$. (As far as I can tell, while true, this is irrelevant.) Each $H^0_{\mathfrak{m}}(I^j)$ is a submodule of $I^j$, so the suggested complex is evidently a subcomplex. Further, it is the result of applying the $H^0_{\mathfrak{m}}(-)$ functor to an injective resolution of $M$, so by definition of right-derived functors,

$$H^p_{\mathfrak{m}}(M) = H^p(H^0_{\mathfrak{m}}(I^0) \to H^0_{\mathfrak{m}}(I^1) \to \cdots).$$

For (2), first suppose our injective resolution is minimal. Since every element of $M$ is annihilated by a power of $\mathfrak{m}$, by the lemma, so is every element of $I^0$. Therefore every element in the cokernel of $M \to I^0$ is killed by a power of $\mathfrak{m}$. Hence $I^1$, being the injective envelope of that cokernel, again has the property that every element in it is killed by a power of $\mathfrak{m}$; etc. That is, $H^0_{\mathfrak{m}}(I^j) = I^j$ for each $j$. The complex

$$0 \to H^0_{\mathfrak{m}}(I^0) \to H^{\mathfrak{m}}_0(I^1) \to \cdots$$

is then the same as our original resolution, so is exact, except possibly at $I^0$.

**81 Proposition**

*Let $(R, \mathfrak{m}, k)$ be a local noetherian ring. Suppose $\mathfrak{m} = (a_1, \ldots, a_n)$. Then*

$$\widehat{R} \cong R[[x_1, \ldots, x_n]]/(x_1 - a_1, \ldots, x_n - a_n).$$

*In particular, $\widehat{R}$ is noetherian.*

Proof Taking power series over a noetherian ring yields a noetherian ring, so the second claim follows from the first. For the first, define $S = R[x_1, \ldots, x_n]$, $I = (x_1, \ldots, x_n)$, $J = (x_1 - a_1, \ldots, x_n - a_n)$. The homomorphism $\phi\colon S \to R$ defined by $\phi|_R = \mathrm{id}_R$, $\phi(x_i) = a_i$ is surjective with kernel $J$, so $R \cong S/J$. Hence we can think of $R$ as an $S$-module. Notice that $\phi(I) = \mathfrak{m}$, $\phi(I^2) = \mathfrak{m}^2$, etc. Thus the $\mathfrak{m}$-adic topology on $R$ is the same as the $I$-adic topology on $R$ viewed as an $S$-module.

(Reminder: given an ideal $I$ in a ring $R$, the open sets are of the form $x + I^n$ for all $x, n$. Given an $R$-module $M$, the open sets are of the form $a + I^n M$.)

Hence

$$\widehat{R} \cong \widehat{S/J} \cong \widehat{S}/\widehat{J} = \widehat{S}/J\widehat{S} = R[[x_1, \ldots, x_n]]/(x_1 - a_1, \ldots, x_n - a_n).$$

**82 Proposition**

*Let $(R, \mathfrak{m}, k)$ be a local noetherian ring. Then $E := E(R/\mathfrak{m})$ is artinian.*

Proof Let $E \supset X_1 \supset X_2 \supset \cdots$ be submodules of $E$. Then $\mathrm{Hom}_R(-, E)$ can be considered a functor from $R$-modules to $\widehat{R}$-modules because $\mathrm{End}(E) = \widehat{R}$. Now

$$\mathrm{Hom}_R(E, E) \twoheadrightarrow \mathrm{Hom}_R(X_1, E) \twoheadrightarrow \mathrm{Hom}_R(X_2, E) \twoheadrightarrow \cdots .$$

Since $\widehat{R} = \mathrm{Hom}_R(E, E)$ is noetherian, the ascending chain of kernels of $\widehat{R} \twoheadrightarrow \mathrm{Hom}_R(X, E_i)$ eventually stabilizes. That is, $X_i' \twoheadrightarrow X_{i+1}'$ is eventually an isomorphism. But

$$0 \to X_{i+1} \to X_i \to X_i/X_{i+1} \to 0$$

gives an exact sequence

$$0 \to (X_i/X_{i+1})' \to X_i' \to X_{i+1}' \to 0,$$

so $(X_i/X_{i+1})' = 0$ for $i$ large. We showed that $M' = 0$ if and only if $M = 0$, so $X_i/X_{i+1} = 0$ for $i$ sufficiently large.

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# October 20th, 2014: Auslander-Buchsbaum Formula; Nakayama's Lemma

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**83 Proposition (Neglected Proposition)**

*Let $R$ be a noetherian ring. Every element in $E(R/\mathfrak{p})$ is killed by $\mathfrak{p}^n$ for $n \gg 0$. (Note: an alternative version of this proposition with a different proof was added to the October 8th lecture sometime last week.)*

Proof The only associated prime of $E(R/\mathfrak{p})$ is $\mathfrak{p}$. Let $e \in E(R/\mathfrak{p})$. If $e = 0$, then $\mathfrak{p}e = 0$, so take $e \neq 0$. Then $Re \cong R/I$, where $I = \mathrm{Ann}(e)$. $\mathrm{Ass}(Re)$ contains all minimal primes over $I$, and $\mathrm{Ass}(Re) \subseteq \mathrm{Ass}(E(R/\mathfrak{p})) = \{\mathfrak{p}\}$, so the only minimal prime over $I$ is $\mathfrak{p}$. So $I \supseteq \mathfrak{p}^n$ for $n \gg 0$. Therefore $\mathfrak{p}^n e = 0$.

**84 Example**

Recall that $\mathrm{depth}(M)$ is the smallest $d$ so that $H^d_{\mathfrak{m}}(M) \neq 0$. Let $R = k[x, y]_{(x,y)}/(x^2, xy)$. Then

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Rx & \longrightarrow & R & \longrightarrow & R/(x) & \longrightarrow & 0 \\ & & \updownarrow \cong & & & & \updownarrow \cong & & \\ & & R/\mathfrak{m} & & & & k[y]_{(y)} & & \end{array}$$

These have depths $0, 0, 1$, respectively.

**85 Remark**

Recall that the projective dimension of an $R$-module $M$, pdim$(M)$, is the smallest $n$ such that there exists a projective resolution

$$0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$$

or $\infty$ if no such $n$ exists.

If $\operatorname{pdim}(M) = n$, then $\operatorname{Ext}_R^{n+i}(M, -) \equiv 0$ for all $i > 0$.

**86 Lemma**

*Let $(R, \mathfrak{m}, k)$ be a noetherian local ring and let*

$$0 \to L \to M \to N \to 0$$

*be a short exact sequence of finitely generated $R$-modules. Write $\ell := \operatorname{depth}(L)$, $m := \operatorname{depth}(M)$, $n := \operatorname{depth}(N)$. Then:*

*(a)* $\ell \geq \min\{m, n+1\}$

*(b)* $m \geq \min\{\ell, n\}$

*(c)* $n \geq \min\{\ell - 1, m\}$.

Proof Use the long exact sequence

$$\cdots \to H^i_{\mathfrak{m}}(L) \to H^i_{\mathfrak{m}}(M) \to H^i_{\mathfrak{m}}(N) \to \cdots.$$

**87 Theorem (Auslander-Buchsbaum Formula)**

*Let $(R, \mathfrak{m}, k)$ be a local noetherian ring and $M$ a non-zero finitely generated $R$-module of finite projective dimension. Then*

$$\operatorname{pdim}(M) + \operatorname{depth}(M) = \operatorname{depth}(R).$$

Proof We induct on $\operatorname{pdim}(M) =: n$. If $n = 0$, $M$ is projective, hence free (by the proposition below), so $\operatorname{depth}(M) = \operatorname{depth}(R)$.

For $n = 1$, let $0 \to R^p \xrightarrow{f} R^q \to M \to 0$ be "the" minimal projective resolution of $M$. Then $f$ is right multiplication by a $p \times q$ matrix all of whose entries are in $\mathfrak{m}$. The map $f^*$ in the long exact sequence

$$\cdots \to \operatorname{Ext}^i_R(k, R^p) \xrightarrow{f^*} \operatorname{Ext}^i(k, R^q) \to \operatorname{Ext}^i(k, M) \to \cdots$$

is zero because $k = R/\mathfrak{m}$ is annihilated by $\mathfrak{m}$. So there is a short exact sequence

$$0 \to \operatorname{Ext}^i(k, R^q) \to \operatorname{Ext}^i(k, M) \to \operatorname{Ext}^{i+1}(k, R^p) \to 0.$$

As shown previously, $\operatorname{depth}(R)$ is the smallest $i$ such that $\operatorname{Ext}^i(k, R) \neq 0$. By examining the $\operatorname{depth}(R) - 1$ case of the above sequence, it follows that $\operatorname{depth}(M) = \operatorname{depth}(R) - 1$.

For $n \geq 2$, there is a short exact sequence

$$0 \to M' \to R^p \to M \to 0$$

where $\operatorname{pdim}(M') = \operatorname{pdim}(M) - 1$. By the induction hypothesis, $\operatorname{pdim}(M') = \operatorname{depth}(R) - n + 1 < \operatorname{depth}(R)$. By the lemma, $\operatorname{depth}(M') = \operatorname{depth}(M) + 1 < \operatorname{depth}(R)$. Hence $\operatorname{depth}(M) = \operatorname{depth}(R) - n = \operatorname{depth}(R) - \operatorname{pdim}(M)$.

**88 Lemma (Nakayama)**

*Let $(R, \mathfrak{m}, k)$ be a local noetherian ring, $M$ a finitely generated $R$-module.*

1. *$M = \mathfrak{m}M$ implies $M = 0$.*
2. *If $N$ is a submodule of $M$ such that $M = N + \mathfrak{m}M$, then $N = M$.*
3. *If $a_1, \ldots, a_n \in M$ provide a basis for $M/\mathfrak{m}M$, then $M = Ra_1 + \cdots + Ra_n$.*

Proof Suppose $M \neq 0$. Because $M$ is finitely generated, it has a simple quotient, say $M/M' \cong R/\mathfrak{m}$. Thus $\mathfrak{m}M \subset M'$; this yields (1). For (2), by hypothesis, $\mathfrak{m}(M/N) = (\mathfrak{m}M + N)/N = M/N$, so by (1), $M/N = 0$. For (3), by hypothesis, $M = Ra_1 + \cdots + Ra_n + \mathfrak{m}M$. Use (2).

**89 Proposition**

*Let $(R, \mathfrak{m}, k)$ be a noetherian local ring, $M$ a finitely generated $R$-module. The following are equivalent:*

*(1) $M$ is free;*

*(2) $M$ is projective;*

*(3) $M$ is flat;*

*(4) $\operatorname{Tor}_1^R(M, k) = 0$.*

Proof (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are all immediate. For (4) $\Rightarrow$ (1), pick a basis $a_1, \ldots, a_n \in M$ for $M/\mathfrak{m}M$ as in Nakayama's lemma, part (3). There is a short exact sequence

$$0 \to K \to R^n \to M \to 0.$$

Notice that since $R$ is noetherian, $K$ is finitely generated. Applying $- \otimes_R R/\mathfrak{m}$ and the fact that $R/I \otimes_R M \cong M/IM$ yields the usual long exact sequence

$$\cdots \to \operatorname{Tor}_1^R(M, k) \to K/\mathfrak{m}K \to (R/\mathfrak{m})^n \xrightarrow{\beta} M/\mathfrak{m}M \to 0.$$

By Nakayama (3) $\beta$ is an isomorphism, so $K = \mathfrak{m}K$. By Nakayama (1), $K = 0$, so $M \cong R^n$ is free.

**90 Theorem (Rees)**

*Let $R$ be a ring, $x \in R$ a non-unit, non-zero-divisor. Let $M$ be an $R$-module such that $\operatorname{Ann}_M(x) = 0$. (That is, $x$ is $M$-regular.) Then*

$$\operatorname{Ext}^n_{R/(x)}(L, M/xM) \cong \operatorname{Ext}^{n+1}_R(L, M)$$

*for all $n \geq 0$ and all $R/(x)$-modules $L$.*

*(We may either assume $R$ is a commutative ring or roughly that $xR = Rx$; we take the former approach.)*

Proof Consider the right-derived functors of the left exact functor $\operatorname{Ext}^1_R(-, M) \colon R/(x)\text{-mod} \to Ab$. (Note: we didn't end up using this method.) Continued next lecture.

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# October 22nd, 2014: Dimension Shifting, Rees' Theorem, Depth and Regular Sequences

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**91 Remark**

Paul will be out of town on Friday, so there will be no lecture then.

**92 Proposition (Dimension Shifting)**

*Let $0 \to X \to P \to Y \to 0$ be a short exact sequence of $R$-modules with $P$ projective. If $M$ is an $R$-module,*

$$\operatorname{Ext}^{n+1}(Y, M) \cong \operatorname{Ext}^n(X, M).$$

*for $n \geq 1$.*

Proof There is an exact sequence

$$\cdots \to \operatorname{Ext}^n(P, M) \to \operatorname{Ext}^n(X, M) \to \operatorname{Ext}^{n+1}(Y, M) \to \operatorname{Ext}^{n+1}(P, M) \to \cdots$$

If $n \geq 1$, then since $P$ is projective, the left and right terms vanish.

**93 Remark**

Here we continue proving Rees' theorem from the end of last lecture.

Proof We first show this is true for $n = 0$ and $n = 1$, and then apply dimension shifting for $n > 1$. For $n = 0$, consider the short exact sequence of $R$-modules $0 \to M \xrightarrow{x} M \to M/xM \to 0$. (Injectivity of the first map uses the fact that $\operatorname{Ann}_M(x) = 0$.) Viewing $L$ as an $R$-module, we get a long exact sequence of Ext groups

$$\begin{aligned} 0 &\to \operatorname{Hom}_R(L, M) \xrightarrow{x} \operatorname{Hom}_R(L, M) \to \operatorname{Hom}_R(L, M/xM) \\ &\to \operatorname{Ext}^1_R(L, M) \xrightarrow{x*} \operatorname{Ext}^1_R(L, M) \to \cdots . \end{aligned}$$

$\operatorname{Hom}_R(L, M) = 0$, since if $f\colon L \to M$, then $xf(\ell) = f(x\ell) = f(0) = 0$ for all $\ell$, so $f(\ell) = 0$. On the other hand, the map $x*$ induced by multiplication by $x$ is injective as follows. Let $0 \to M \to I_1 \to I_2 \to \cdots$ be an injective resolution of $M$. We have

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & I_0 & \longrightarrow & I_1 & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle x} & & \\ 0 & \longrightarrow & M & \longrightarrow & I_0 & \longrightarrow & I_1 & \longrightarrow & \cdots \end{array}$$

We compute $\operatorname{Ext}^n_R(L, M)$ by applying $\operatorname{Hom}_R(L, -)$ and computing homology. The induced map $\operatorname{Hom}_R(L, I_1) \xrightarrow{x} \operatorname{Hom}_R(L, I_1)$ is zero since $f\colon L \to I_1 \xrightarrow{x} I_1$ is given by $\ell \mapsto xf(\ell) = f(x\ell) = f(0) = 0$. The induced map on homology is then zero, so $x^*$ is zero. Hence the connected map $\operatorname{Hom}_R(L, M/xM) \to \operatorname{Ext}^1_R(L, M)$ is an isomorphism. But since $L$ and $M/xM$ are in fact $R/xR$-modules, it follows that $\operatorname{Hom}_R(L, M/xM) \cong \operatorname{Hom}_{R/xR}(L, M/xM)$ (as abelian groups, say).

For $n = 1$, since $0 \xrightarrow{x} R \to R/xR \to 0$ is exact (using the fact that $x$ is not a zero-divisor), $\operatorname{pdim}_R(R/xR) \leq 1$. Hence $\operatorname{Ext}^k_R(R/xR, -) = 0$ for $k \geq 2$. Let $P$ be a projective $R/xR$-module and $0 \to K \to P \to L \to 0$ an exact sequence of $R/xR$-modules. Since $P$ is projective, it is a direct summand of a free $R/xR$-module. It follows that $\operatorname{Ext}^k_R(P, -) = 0$ for all $k \geq 2$, since we can pull direct sums out of Ext in exchange for a direct product. Now we have long exact sequences

$$\cdots \to \operatorname{Ext}^1_R(P, M) \to \operatorname{Ext}^1_R(K, M) \to \operatorname{Ext}^2_R(L, M) \to \operatorname{Ext}^2_R(P, M) \to \cdots$$

and

$$\begin{aligned} \cdots &\to \operatorname{Hom}_{R/xR}(P, M/xM) \to \operatorname{Hom}_{R/xR}(K, M/xM) \\ &\to \operatorname{Ext}^1_{R/xR}(L, M/xM) \to \operatorname{Ext}^1_{R/xR}(P, M/xM) \to \cdots . \end{aligned}$$

Since $P$ is projective over $R/xR$, the rightmost terms on each sequence are zero. We've shown the first two terms of each sequence are (naturally) isomorphic in pairs, so it follows that the third terms are isomorphic.

Now suppose $n \geq 2$. Take a projective resolution of $L$ as an $R/xR$-module, $P_* \to L$. Writing out kernels gives

$$\begin{array}{ccccccccccc} \cdots \longrightarrow & P_2 & & \longrightarrow & & P_1 & & \longrightarrow & & P_0 \longrightarrow L \longrightarrow 0 \\ & & \searrow & & \nearrow & & \searrow & & \nearrow & \\ & & & K_1 & & & & K_0 & & \\ & & \nearrow & & \searrow & & \nearrow & & \searrow & \\ & 0 & & & & 0 & & & & 0 \end{array}$$

We have exact sequences

$$0 \to K_i \to P_i \to K_{i-1} \to 0$$

so pieces of long exact sequences

$$0 = \mathrm{Ext}^n_R(P_i, M) \to \mathrm{Ext}^n_R(K_i, M) \to \mathrm{Ext}^{n+1}_R(K_{i-1}, M) \to \mathrm{Ext}^{n+1}_R(P_i, M) = 0.$$

Hence $\mathrm{Ext}^{n+1}_R(K_{i-1}, M) \cong \mathrm{Ext}^n_R(K_i, M)$ for all $n \geq 2$ and $i \geq 0$ (letting $K_{-1} = L$). Hence

$$\mathrm{Ext}^{n+1}_R(L, M) \cong \mathrm{Ext}^n_R(K_0, M) \cong \cdots \cong \mathrm{Ext}^2_R(K_{n-2}, M) \cong \mathrm{Ext}^1_{R/xR}(K_{n-2}, M/xM).$$

By dimension shifting,

$$\mathrm{Ext}^1_{R/xR}(K_{n-2}, M/xM) \cong \mathrm{Ext}^2_{R/xR}(K_{n-3}, M/xM) \cong \cdots \cong \mathrm{Ext}^n_{R/xR}(L, M/xM).$$

The result follows.

**94 Definition**

An element $x \in R$ is $\boxed{M\text{-regular}}$ for an $R$-module $M$ if $xM \neq M$ and $\mathrm{Ann}_M(x) = 0$. (This differs slightly from our previous version of this definition, where we didn't require $xM \neq M$.) An $\boxed{M\text{-regular sequence}}$ is a sequence $x_1, \ldots, x_n \in R$ such that $x_1$ is $M$-regular, $x_2$ is $M/x_1M$-regular, $x_3$ is $M/(x_1M + x_2M)$-regular, etc.

**95 Theorem**

*Let $(R, \mathfrak{m}, k)$ be a local noetherian ring and $M \neq 0$ a noetherian $R$-module. If $x_1, \ldots, x_d$ is an $M$-regular sequence, then*

$$\mathrm{depth}(M) \geq d.$$

*In fact,* $\mathrm{depth}(M)$ *is the maximal length of an $M$-regular sequence. (This was the definition of "depth" before cohomology came around.)*

Proof Since $\mathrm{depth}(M)$ is the smallest $n$ such that $\mathrm{Ext}^n_R(k, M) \neq 0$, we must show that $\mathrm{Ext}^n_R(k, M) = 0$ for all $n < d$. So, let $n < d$. By Rees' Theorem,

$$\mathrm{Ext}^n_R(k, M) \cong \mathrm{Ext}^{n-1}_{R/xR}(k, M/(x_1M)).$$

(Here we use the fact that $k$ is an $R/\mathfrak{m}$-module and hence an $R/x_1R$-module because $x_1 \in \mathfrak{m}$ is a non-unit.) For convenience, write $M_n := M/(x_1, \ldots, x_n)M$. We can iterate Rees' theorem to get

$$\mathrm{Ext}^{n-1}_{R/xR}(k, M_1) \cong \mathrm{Ext}^{n-2}_{R/xR}(k, M_2) \cong \cdots \cong \mathrm{Hom}_{R/xR}(k, M_n).$$

Since $d > n$, $n+1 \leq d$, so $\mathrm{Ann}_{M_n}(x_{n+1}) = 0$. However, if $0 \neq f \in \mathrm{Hom}_R(k, M_n)$, then $f(k) \neq 0$ and $x_{n+1}f(k) = 0$ because $x_{n+1} \in \mathfrak{m}$ is a non-unit, a contradiction. Hence the right-hand side is zero, as we needed.

$\mathrm{depth}(M)$ is then $\geq$ the maximal length of an $M$-regular sequence. For the converse, see next lecture.

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# October 27th, 2014: Krull Dimension of Modules

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**96 Remark**

We continue proving the theorem from the end of last time.

PROOF Now suppose $x_1, \ldots, x_n$ is an $M$-regular sequence with $n$ as large as possible (which is bounded above by the previous inequality). For convenience, write $R_n := R/(x_1, \ldots, x_n)$, $M_n := R_n \otimes_R M = M/(x_1 M + \cdots + x_n M)$. Suppose $n < d$; then $\mathrm{Ext}^n_R(k, M) = 0$. The dimension shifting argument for the $d \geq n$ implication last time gives $\mathrm{Ext}^n_R(k, M) \cong \mathrm{Hom}_{R_n}(k, M_n)$, so $\mathrm{Hom}_{R_n}(k, M_n) = 0$. That is, $\mathfrak{m} \notin \mathrm{Ass}(M_n)$. Since $M_n \neq 0$, there exists some $\mathfrak{p} \in \mathrm{Ass}(M_n)$. The set of non-zero-divisors of $M_n$ is $R - \cup_{\mathfrak{q} \in \mathrm{Ass}(M_n)} \mathfrak{q}$. Since $\mathfrak{m}$ is not contained in any $\mathfrak{q} \in \mathrm{Ass}(M_n)$, by the prime avoidance lemma, there is some $y \in \mathfrak{m}$ such that $y \notin \cup_{\mathfrak{q} \in \mathrm{Ass}(M_n)} \mathfrak{q}$, so $y$ is not a zero-divisor on $M$. Also, $M_n \neq \mathfrak{m} M_n$ by Nakayama, so $y M_n \neq M_n$. Hence $\{y\}$ is an $M_n$-regular sequence, so that $x_1, \ldots, x_n, y$ is an $M$-regular sequence, a contradiction. Hence equality holds.

**97 Remark**

$M$ is Cohen-Macaulay of depth $d$ if $H^i_{\mathfrak{m}}(M) = \delta_{i,d} C$ for some $C$, i.e. there is a unique non-zero $H^i_{\mathfrak{m}}$. Our next goal is to show that $M$ is Cohen-Macaulay if and only if $\mathrm{depth}(M) = \mathrm{Kdim}(M)$. The classical definition is as follows: $M$ is a Cohen-Macaulay if the maximal length of an $M$-regular sequence is the Krull dimension of $M$.

**98 Definition**

If $R$ is a ring, the Krull dimension of $R$, $\mathrm{Kdim}(R)$, is the supremum of the number of strict inclusions of any chain of prime ideals. If $M$ is an $R$-module, the Krull dimension of $M$ is defined as

$$\boxed{\mathrm{Kdim}(M)} := \mathrm{Kdim}(R/\operatorname{Ann} M).$$

Note: if $M = R/I$, then $\mathrm{Kdim}(R/I) = \mathrm{Kdim}(R/\operatorname{Ann}(R/I)) = \mathrm{Kdim}(R/I)$, where $R/I$ on the left is an $R$-module and $R/I$ on the right is a ring.

**99 Proposition**

*Let $R$ be a noetherian ring, $M$ a noetherian $R$-module.*

(1) *If $I \subset J$ are ideals, then* $\mathrm{Kdim}(R/J) \leq \mathrm{Kdim}(R/I)$.

(2) $\mathrm{Kdim}(R/I)$ *is the maximum of* $\{\mathrm{Kdim}(R/\mathfrak{p})\}_{\mathfrak{p}}$ *where $\mathfrak{p}$ ranges over the minimal primes of $R$ over $I$.*

(3) $\mathrm{Kdim}(R/I) = \mathrm{Kdim}(R/I^n)$ *for all $n$ (since $I$ and $I^n$ have the same minimal primes).*

(4) *If $I^n \subset J \subset I$, then* $\mathrm{Kdim}(R/I) = \mathrm{Kdim}(R/J)$.

(5) $\mathrm{Kdim}(R_1 \oplus R_2) = \max\{\mathrm{Kdim}(R_1), \mathrm{Kdim}(R_2)\}$.

(6) $\mathrm{Kdim}(R/I_1 \cap I_2) = \max\{\mathrm{Kdim}(R/I_1), \mathrm{Kdim}(R/I_2)\}$.

(7) $\mathrm{Kdim}(R/I_1 \cap I_2) = \mathrm{Kdim}(R/I_1 I_2)$.

(8) *If $0 \to L_1 \to M \to L_2 \to 0$ is exact, then* $\mathrm{Kdim}(M) = \max\{\mathrm{Kdim}(L_j)\}$.

(9) *If $\mathrm{Kdim}(R) < \infty$, and if $x \in R$ is a regular element, then* $\mathrm{Kdim}(R/xR) < \mathrm{Kdim}(R)$

(10) *If $\mathrm{Kdim}(M) < \infty$, $x \in R$, and $\mathrm{Ann}_M(x) = 0$, then* $\mathrm{Kdim}(M/xM) < \mathrm{Kdim}(M)$.

*(The final two were added during the next lecture.)*

PROOF (1)-(4) are trivial or immediate. For (5), since $(R_1 \oplus 0)(0 \oplus R_2) = R_1 R_2 = 0$, if $\mathfrak{p} \subset R_1 \oplus R_2$ is prime, then $\mathfrak{p} \supset R_1$ or $\mathfrak{p} \supset R_2$. In any chain of prime ideals $\mathfrak{p}_1 \supset \mathfrak{p}_2$, they must both contain the same $R_i$, since otherwise the larger contains both $R_1$ and $R_2$, hence is not proper. The result follows.

For (6), $R/I_1 \cap I_2$ embeds in $R/I_1 \oplus R/I_2$ induced by the natural map $R \to R/I_1 \oplus R/I_2$. Hence $I_1 \cap I_2 = \mathrm{Ann}_R(R/I_1 \oplus R/I_2)$, so $\max\{\mathrm{Kdim}(R/I_i)\} = \mathrm{Kdim}(R/I_1 \oplus R/I_2) = \mathrm{Kdim}(R/I_1 \cap I_2)$.

For (7), use the fact that $(I_1 \cap I_2)^2 \subset I_1 I_2 \subset I_1 \cap I_2$. For (8), let $I_j := \mathrm{Ann}(L_j)$. Then $I_2 M \subset L_1$, so $I_1 I_2 M = 0$. That is, $\mathrm{Ann}(M) \supset I_1 I_2$, so

$$\mathrm{Kdim}(M) \leq \mathrm{Kdim}(R/I_1 I_2) = \mathrm{Kdim}(R/I_1 \cap I_2) = \max\{\mathrm{Kdim}(R/I_j)\} = \max\{\mathrm{Kdim}(L_j)\}.$$

On the other hand, $\mathrm{Ann}(M) \subset I_j$ for $j = 1, 2$, so $\mathrm{Kdim}(M) \geq \mathrm{Kdim}(R/I_j) = \mathrm{Kdim}(L_j)$, which gives the reverse inequality.

For (9), let $n := \mathrm{Kdim}(R/xR)$ and $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n$ be a chain of prime ideals containing $xR$. Since 0 contains a product of minimal primes, $x$ is not in any minimal prime. Hence $\mathfrak{p}_0$ is not a minimal prime. But every prime ideal contains a minimal prime, so there exists a prime $\mathfrak{q} \subsetneq \mathfrak{p}_0$, so $\mathrm{Kdim}(R) \geq n + 1$.

For (10), let $I := \mathrm{Ann}(M)$. Since $x$ is $M$-regular, $x \notin I$. In fact, the image of $x$ in $R/I$ is regular since if $xy \in I$, then $xyM = 0$, so $yM = 0$, so $y \in I$. By (9), since $(R/I)/(xR/I) \cong R/(xR + I)$, $\mathrm{Kdim}(R/(xR + I) < \mathrm{Kdim}(R/I) = \mathrm{Kdim}(M)$. Since $\mathrm{Ann}(M/xM \supset xR + I$, $\mathrm{Kdim}(M/xM) \leq \mathrm{Kdim}(R/(xR + I)) < \mathrm{Kdim}(M)$.

**100 Proposition**

*If $x \in R$ is not a zero-divisor on $M$, then $\mathrm{Kdim}(M/xM) < \mathrm{Kdim}(M)$. (In the non-commutative case, this follows from the definition; in the commutative case, it requires a bit more work.)*

Proof We discuss the non-commutative case briefly. The statement is trivial if $xM = M$, so suppose $xM \neq M$. Since $x$ is not a zero-divisor on $M$, $xM \cong M$ as $R$-modules, giving a strictly descending chain of isomorphic submodules

$$M \supsetneq xM \supsetneq x^2 M \supsetneq \cdots$$

with successive quotients $M/xM$. One defines Krull dimension in the non-commutative case in terms of "slices" of this form needing to be finite, yet we have an infinite sequence, which ends up giving the suggested inequality immediately.

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# October 29th, 2014: Local Cohomology over Varying Rings; Examples: Rings of Invariants

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**101 Remark**

(The final two statements were added to the large proposition on Krull dimension of modules from last lecture.)

**102 Theorem**

*Let $(R, \mathfrak{m}, k)$ be a local noetherian ring, $M$ a noetherian $R$-module. Then $\mathrm{depth}(M) \leq \mathrm{Kdim}(M)$.*

Proof Let $x_1, \ldots, x_n$ be an $M$-regular sequence. Write $M_j := M/(x_1, \ldots, x_j)M$. By (10) of the above proposition, $\mathrm{Kdim}\, M > \mathrm{Kdim}(M_1) > \mathrm{Kdim}(M_2) > \cdots > \mathrm{Kdim}(M_n) \geq 0$, so $\mathrm{Kdim}(M) \geq n$, so $\mathrm{Kdim}\, M$ is $\geq$ the length of the largest $M$-regular sequence, which is $\mathrm{depth}(M)$ from before.

**103 Example**

A good source of Cohen-Macaulay rings is the following: if $G$ is a finite group, $k$ is a field, and $S := k[x_1, \ldots, x_n]$, then if $\mathrm{char}\, k \nmid |G|$, $S^G$ is a Cohen-Macaulay ring. By the characteristic condition, $S$ is a semisimple $kG$-module, so decomposes as some $\oplus_{i=1}^n S_i$ where $V_1, \ldots, V_n$ are the simple $kG$-modules and $S_i$ is the sum of all $kG$-submodules of $S$ isomorphic to $V_i$. Taking $V_1$ as the trivial representation, $S_1 = S^G$. Since $V_1 \otimes V_i \cong V_i$, every $S_i$ is an $S^G$-module.

**104 Theorem**

*Let $(R, \mathfrak{m})$ and $(S, \mathfrak{n})$ be local noetherian rings and $f\colon R \to S$ a ring homomorphism such that $S$ is a finitely generated $R$-module. If $M$ is an $S$-module, then $H^i_{\mathfrak{n}}(M) = H^i_{\mathfrak{m}}(M)$ for all $i$, i.e. local cohomology of $M$ is the same whether $M$ is treated as an $R$-module or an $S$-module.*

PROOF Next time.

**105 Remark**

The preceding theorem has a graded analogue. We next discuss it in the context of the previous example.

**106 Remark**

A graded $k$-algebra $A =: A_0 \oplus A_1 \oplus \cdots$ is connected if $A_0 = k$. (The terminology comes from topology, where the 0th homotopy group counts the number of connected components.) There is a good theory of local cohomology for noetherian graded modules over a noetherian connected graded $k$-algebra where one uses $\mathfrak{m} := A_1 \oplus A_2 \oplus \cdots$. One defines $H^0_{\mathfrak{m}}(M)$ exactly as before; the $H^i_{\mathfrak{m}}(-)$ are again the right-derived functors of $H^0_{\mathfrak{m}}(-)$, though now the category has changed slightly; one defines the depth exactly as before; there is a theorem that $\mathrm{depth}(M)$ is the minimum $d$ such that $\mathrm{Ext}^d_A(k, M) \neq 0$; there is an analogue of the above theorem; etc.

**107 Example**

Continuing the notation of the previous example, suppose $G \subset \mathrm{GL}(n)$, i.e. $G$ acts on the degree 1 part of $S$ by linear automorphisms. We now view $S$ as a graded $k$-algebra with $\deg x_i = 1$. (We really just need the action of $G$ to preserve degree.) Each $S_i$ is a graded vector space since $G$ preserves degree and by Hilbert $S^G$ is noetherian and $S$ is a finitely generated $S^G$-module. By the graded analogue of the preceding theorem, if $M$ is a graded $S$-module, then its local cohomology $H^i_{\mathfrak{m}}(M)$ (where $\mathfrak{m}$ consists of the positive degree portion of $S$) is $H^i_{\mathfrak{n}}(M)$, where the right-hand side is the local cohomology computed with respect to $(S^G, \mathfrak{n} := \mathfrak{m} \cap S^G)$.

We will show (someday) that $S$ is a Cohen-Macaulay ring, i.e. Cohen-Macaulay as a module over itself, i.e. $\mathrm{depth}(S) = n$, i.e. $H^i_{\mathfrak{m}}(S)$ is non-zero if and only if $i = n$. By the theorem, $H^i_{\mathfrak{n}}(S)$ is non-zero if and only if $i = n$. Hence $S$ is Cohen-Macaulay as an $S^G$-module. Since $S = \oplus S_i$ as $S^G$-modules, and since one may check $M_1 \oplus M_2$ is Cohen-Macaulay if and only if $M_1$ and $M_2$ are Cohen-Macaulay of the same depth, each $S_i$ is Cohen-Macaulay as an $S_G$-module.

**108 Remark**

We say $M$ is a maximal Cohen-Macaulay module if $M$ is an $R$-module and $R$ and $M$ are Cohen-Macaulay with $\mathrm{depth}(M) = \mathrm{depth}(R)$. In the previous example, $S$ is a maximal Cohen-Macaulay $S^G$-module. If $R$ has only finitely many maximal Cohen-Macaulay modules, $R$ is said to have finite maximal Cohen-Macaulay type.

**109 Example**

For example, take a finite non-trivial subgroup of $\mathrm{SL}_2(\mathbb{C})$ acting on $\mathbb{C}[x, y]$. Then $\mathbb{C}[x, y]^G$ is not regular, $\mathbb{C}^2/G = \mathrm{spec}(\mathbb{C}[x, y]^G)$ is called a Kleinian singularity. (Klein classified finite subgroups of $\mathrm{SL}_2(\mathbb{C})$.) They are indexed by the Dynkin diagrams of types $A_n, D_n, E_6, E_7, E_8$.

For instance, $A_n$ corresponds to the subgroup consisting of the diagonal matrices with the $n+1$st roots of unit in the upper left. One checks $\mathbb{C}[x, y]^G \cong \mathbb{C}[x^{n+1}, xy, y^{n+1}]$, which is $\mathbb{C}[u, v, w]/(uw - v^{n+1})$. $\mathbb{C}[x, y]^G$ is Cohen-Macaulay, and the maximal Cohen-Macaulay modules for it are the $S_i$ (in the notation above). The minimal resolution of $\mathbb{C}^2/G$, written $\widetilde{\mathbb{C}^2/G} \xrightarrow{\pi} \mathbb{C}^2/G$, has exceptional locus $\pi^{-1}(0)$ which is a union of $\mathbb{P}^1$'s. The dual graph of $\pi^{-1}(0)$, given by putting a vertex for each $\mathbb{P}^1$ in $\pi^{-1}(0)$ and connecting them by an edge when their intersection is non-empty, happens to be the Dynkin diagram above.

Indeed, the bounded derived category of $\mathrm{End}_{\mathbb{C}[x,y]^G}(\mathbb{C}[x, y])$-modules is equivalent to the bounded derived category of $\mathrm{coh}(\widetilde{\mathbb{C}^2/G})$. In some sense, all of this stems from the above theorem.

# October 31st, 2014: Local Cohomology and Ring Changes; Krull's Principal Ideal Theorem

**110 Remark**

Happy Halloween! We'll restate and prove the second theorem from last time.

**111 Theorem**

*Let $(R, \mathfrak{m})$ and $(S, \mathfrak{n})$ be local noetherian rings. Let $f\colon R \to S$ be a ring homomorphism which makes $S$ a finitely generated $R$-module. If $M$ is a finitely generated $S$-module, then for all $i$*

$$H^i_{\mathfrak{m}}(M) = H^i_{\mathfrak{n}}(M).$$

*Note: We suppose $S\mathfrak{m} \subset \mathfrak{n}$. We may need $f$ to be a morphism of local rings to justify this, or it may follow from the finiteness condition, Paul was unsure.*

Proof Because $S$ is a finitely generated $R$-module, $S/S\mathfrak{m}$ is a finitely generated $R/\mathfrak{m}$-module, hence of finite length as an $S$-module. Thus $\mathfrak{n}^t(S/S\mathfrak{m}) = 0$ for $t \gg 0$. That is, $\mathfrak{n}^t \subset S\mathfrak{m} \subset \mathfrak{n}$. Hence

$$\begin{aligned} H^0_{\mathfrak{m}}(M) &= \{a \in M : \mathfrak{m}^k a = 0 \text{ for } k \gg 0\} \\ &= \{a \in M : \mathfrak{n}^t a = 0 \text{ for } k \gg 0\} \\ &= H^0_{\mathfrak{n}}(M). \end{aligned}$$

While $f$ induces a functor $S$-mod $\to R$-mod, and under this functor we have $H^0_{\mathfrak{m}} = H^0_{\mathfrak{n}}$, the right-derived functors of $H^0_{\mathfrak{n}}$ are computed in $S$-mod, not $R$-mod, so we can't immediately conclude the remaining local cohomology functors are equal.

**112 Aside**

Let $F$ be a left exact functor on $S$-modules. To compute the right derived functors of $F$ we use an injective resolution typically. However, we can more generally compute $R^iF$ by using an acyclic resolution.

**113 Definition**

Given a functor $F$, an $S$-module $J$ is acyclic for $F$ if $R^nF(J) = 0$ for all $n \geq 0$. An acyclic resolution of $M$ is an exact sequence

$$0 \to M \to J_0 \to J_1 \to \cdots$$

where each $J_i$ is $F$-acyclic.

Indeed, the usual proof that derived functors are, up to isomorphism, independent of which injective resolution you chose can be easily generalized to cover this case.

Since $H^*_{\mathfrak{n}}(M)$ is computed by taking a resolution of $M$ by injective $S$-modules, it suffices to show that injective $S$-modules are acyclic for $H^*_m(-)$. For this, it suffices to show that every indecomposable injective $S$-module $I$ has the property that $H^i_{\mathfrak{m}}(I) = 0$ for all $i > 0$. We had classified the indecomposable injectives in this case above. If $I$ is the injective envelope of $S/\mathfrak{n}$ (as an $S$-module), then every element of $I$ is annihilated by a power of $\mathfrak{n}$, and therefore by a power of $\mathfrak{m}$. We showed that $H^i_{\mathfrak{m}}(N) = 0$ for all $i > 0$ if every element of $N$ is annihilated by a power of $\mathfrak{m}$, so the $E(S/\mathfrak{n})$ case works.

Now suppose $I \not\cong E(S/\mathfrak{n})$ is an indecomposable injective $S$-module. Let $P_* \to R/\mathfrak{m}^k$ be a projective resolution of $R$-modules for some $k$. Then

$$\begin{aligned}
\operatorname{Hom}_S(\operatorname{Tor}_n^R(S, R/\mathfrak{m}^k), I) &\cong \operatorname{Hom}_S(H^n(S \otimes P_*), I) \\
&\cong H^n(\operatorname{Hom}_S(S \otimes P_*), I)) \\
&\cong H^n(\operatorname{Hom}_R(P_*, \operatorname{Hom}_S(S, I))) \\
&\cong H^n(\operatorname{Hom}_R(P_*, I)) \\
&\cong \operatorname{Ext}_R^n(R/\mathfrak{m}^k, I).
\end{aligned}$$

To compute $\operatorname{Tor}_n^R(S, R/\mathfrak{m}^k)$, we take a projective resolution of $S$ as an $R$-module. Because $S$ is finitely generated as an $R$-module and $R$ is noetherian, we can assume that all the projective $R$-modules in the projective resolution of $S$ are finitely generated. Applying $- \otimes_R R/\mathfrak{m}^k$ to this projective resolution will give a complex of finitely generated $R/\mathfrak{m}^k$-modules. Hence $\operatorname{Tor}_n^R(S, R/\mathfrak{m}^k)$ is a finitely generated $R/\mathfrak{m}^k$-module.

However, $I \not\cong E(S/\mathfrak{n})$, from which it follows that no elements in $I$ are annihilated by a power of $\mathfrak{n}$, hence no element of $I$ is annihilated by a power of $\mathfrak{m}$. Hence $\operatorname{Hom}_S(\operatorname{Tor}_n^R(S, R/\mathfrak{m}^k), I) = 0$, i.e. $\operatorname{Ext}_R^n(R/\mathfrak{m}^k, I) = 0$ for $n \geq 0$. In particular, $H^0_\mathfrak{m}(I) = \varinjlim_k \operatorname{Ext}_R^n(R/\mathfrak{m}^k, I) = 0$ for all $n$. Hence $I$ is acyclic for $H^0_\mathfrak{m}(-)$. The result follows.

**114 Theorem (Krull's Principal Ideal Theorem)**
*Let $R$ be a noetherian ring, $x \in R$, and $\mathfrak{p}$ a minimal prime over $Rx$. Then* $\operatorname{ht}(\mathfrak{p}) \leq 1$.

**115 Definition**
Recall the height of a prime ideal $\mathfrak{p} \subset R$ is the biggest $n$ such that there exists a chain of prime ideals $\mathfrak{p} = \mathfrak{p}_n \supsetneq \mathfrak{p}_{n-1} \supsetneq \cdots \supsetneq \mathfrak{p}_1 \supsetneq \mathfrak{p}_0$.

Proof Suppose not. That is, suppose $\mathfrak{p}$ is minimal over $xR$ and $\mathfrak{q}, \mathfrak{q}'$ are distinct primes different from $\mathfrak{p}$ with $xR \subset \mathfrak{p} \subset R$ and $\mathfrak{q}' \subset \mathfrak{q} \subset \mathfrak{p}$. We can localize at $\mathfrak{p}$ and pass to $(xR)_\mathfrak{p} \subset \mathfrak{p}R_\mathfrak{p} \subset R_\mathfrak{p}$ and $\mathfrak{q}'R_\mathfrak{p} \subset \mathfrak{q}R_\mathfrak{p} \subset \mathfrak{p}R_\mathfrak{p}$. One must show the primes remain distinct here; use lying over/going up. In any case, we can thus assume without loss of generality that $R$ is local with maximal ideal $\mathfrak{p}$. We may then quotient by $\mathfrak{q}'$ without loss of generality, so suppose $\mathfrak{q}' = 0$.

Let $y \in \mathfrak{p} - \mathfrak{q}$ and define $I_k := \{r \in R : rx^k \in yR\}$. Then $I_1 \subset I_2 \subset \cdots$ stabilizes at some $I_t = I_{t+1} = \cdots$. Without loss of generality, we can replace $x$ by $x^t$ and take $t = 1$. Hence if $rx^2 \in yR$, then $rx \in yR$. Since $R/xR$ has only one prime ideal, it has finite length. (Recall that every ideal contains a product of minimal prime ideals, and only $\mathfrak{p}$ is minimal over $xR$, so $xR$ is annihilated by some power of $\mathfrak{p}$, which gives a composition series for $R/xR$ by $R/\mathfrak{p}R$'s, which are fields of finite dimension.) Likewise $R/x^2R$ has finite length. That is,

$$\frac{Rx + Ry}{Rx^2} \quad \text{and} \quad \frac{Rx^2 + Ry}{Rx^2}$$

have finite length.

**116 Lemma**
*Let $x$ be non-zero element in a domain $R$. Then*

*(a) If $y \in R$, $\frac{Rx+Ry}{Rx} \cong \frac{Rx^2+Ry}{Rx^2+Rxy}$*

*(b) If $y \in R - \{0\}$ has the property that $bx^2 \in yR$ implies $bx \in yR$, then*

$$\frac{Rx^2 + Ry}{Rx^2 + Rxy} \cong \frac{R}{Rx}.$$

Proof Exercise. (Note: the statement of this lemma was modified in several ways during the November 3rd lecture. The original proof has been removed.)

To be continued.

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# November 3rd, 2014: Generalized Principal Ideal Theorem

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**117 Remark**

We begin by finishing off the proof of Krull's intersection theorem from last time.

Proof Using parts (1) and (2) of the lemma from last time,

$$\frac{R}{Rx} \cong \frac{Rx + Ry}{Rx}$$

and both have (the same) finite length. However, this is absurd since $Rx + Ry \neq R$, so the right-hand side is a proper submodule, forcing the left-hand side to have strictly larger length than the right hand side, a contradiction. The theorem follows.

**118 Corollary**

*Let $\mathfrak{p} \subseteq \mathfrak{q}$ be prime ideals in a noetherian ring $R$. If there exists a single prime ideal strictly between $\mathfrak{p}$ and $\mathfrak{q}$, then there exists infinitely many.*

Proof By passing to $R/\mathfrak{p}$, we can assume $\mathfrak{p} = 0$. By localizing at $\mathfrak{q}$, all primes are contained in $\mathfrak{q}$. Suppose $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ are all the prime ideals strictly between 0 and $\mathfrak{q}$. By the prime avoidance lemma applied to $\mathfrak{q}$, there exists an element $x \in \mathfrak{q}$ such that $x \notin \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_n$. By Krull's theorem, the height of $\mathfrak{q}$ is $\leq 1$, a contradiction since $n \geq 1$ by assumption. Therefore there must be infinitely many primes strictly between 0 and $\mathfrak{q}$.

**119 Theorem (Generalized Principal Ideal Theorem, GPIT)**

*Let $R$ be a noetherian ring and $x_1, \ldots, x_n \in R$. If $\mathfrak{p}$ is a minimal prime over $(x_1, \ldots, x_n)$, then $\mathrm{ht}(\mathfrak{p}) \leq n$.*

Proof By localizing at $\mathfrak{p}$, we reduce to the case where $R$ is local and $\mathfrak{p}$ is maximal. Pick $\mathfrak{p} \supsetneq \mathfrak{p}_1$ as large as possible (using the noetherian condition). Since $\mathfrak{p}$ is minimal over $(x_1, \ldots, x_n)$, $(x_1, \ldots, x_n) \not\subset \mathfrak{p}_1$. Hence without loss of generality we may take $x_1 \notin \mathfrak{p}_1$. By maximality of $\mathfrak{p}_1$, $\mathfrak{p}$ is the unique minimal prime over $(x_1, \mathfrak{p}_1)$. Hence $\mathfrak{p}^k \subset (x_1, \mathfrak{p}_1)$ for $k \gg 0$. Thus we can write $x_i^k = x_1 a_i + b_i$ for $a_i \in R, b_i \in \mathfrak{p}_1$, $i = 2, \ldots, n$. Now

$$\mathfrak{p} = \sqrt{(x_1, \ldots, x_n)} = \sqrt{(x_1^k, \ldots, x_n^k)} \subseteq \sqrt{(x_1, b_2, \ldots, b_n)} \subset \mathfrak{p}$$

where the final $\subset$ follows since $\mathfrak{p}$ is maximal. Hence $\mathfrak{p}$ is the unique minimal prime over $(x_1, b_2, \ldots, b_n)$. Therefore $\mathfrak{p}/(b_2, \ldots, b_n)$ is the unique minimal prime in $R/(b_2, \ldots, b_n)$ that contains the image of $x$. Krull's Principal Ideal Theorem implies the height of $\mathfrak{p}/(b_2, \ldots, b_n)$ is at most 1. Thus $\mathfrak{p}_1$ is minimal over $(b_2, \ldots, b_n)$. By induction, $\mathfrak{p}_1$ has height $\leq n - 1$, and this reasoning holds for all primes strictly contained in $\mathfrak{p}_i$, so $\mathfrak{p}$ has height $\leq n$.

**120 Corollary**

*We have:*

*(1) In a noetherian ring, every prime ideal has finite height.*

*(2) If $(R, \mathfrak{m})$ is a local noetherian ring, then $\mathrm{Kdim}(R) < \infty$. Indeed, $\mathrm{Kdim}(R) = \mathrm{ht}(\mathfrak{m})$.*

*(3) If $(R, \mathfrak{m})$ is local noetherian, then every finitely generated non-zero $R$-module has finite depth.*

Proof (1) If $\mathfrak{p} \in \mathrm{spec}(R)$, then $\mathfrak{p}$ is finitely generated, so is a minimal prime over itself whose height is bounded above by the number of generators of $\mathfrak{p}$. (2) follows from (1) applied to $\mathfrak{m}$ since any chain of prime ideals can have $\mathfrak{m}$ added at the top. (3) $\mathrm{depth}(M) \leq \mathrm{Kdim}(M) \leq \mathrm{Kdim}(R)$ since $R^n \twoheadrightarrow M$ for some $n$.

**121 Corollary**

*If $k = \overline{k}$ is an algebraically closed field, then* $\operatorname{Kdim} k[x_1, \ldots, x_n] = n$.

PROOF $0 \subsetneq (x_1) \subsetneq (x_1, x_2) \subsetneq \cdots \subsetneq (x_1, \ldots, x_n)$ is a chain of prime ideals, so $\operatorname{ht}(x_1, \ldots, x_n) \geq n$. On the other hand, $\operatorname{ht}(x_1, \ldots, x_n) \leq n$ by the theorem. Hence $\operatorname{Kdim} R_{(x_1,\ldots,x_n)} = n$. Every maximal ideal is, up to a linear change of variables, of this form, and the result follows.

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# November 5th, 2014: Krull Dimension of Polynomial Rings; Systems of Parameters

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**122 Remark**

Paul was unconvinced by a key step in our proof of the generalized principal ideal theorem from last time, so he presented another argument. The original argument has been replaced by this one. Similarly, the next corollary's proof was replaced/completed later. An alternate argument and statement appears in the November 19th lecture. The example at the end of this lecture was also added later.

**123 Corollary (Converse to GPIT)**

*If $R$ is noetherian, $\mathfrak{p} \in \operatorname{spec}(R)$, and $n := \operatorname{ht}(\mathfrak{p})$, then $\mathfrak{p}$ is minimal over some ideal generated by $n$ elements.*

PROOF We construct a sequence of elements $x_1, x_2, \ldots, x_n$ such that each $x_k \in \mathfrak{p}$ is contained in no minimal primes containing $(x_1, \ldots, x_{k-1})$. Indeed, having chosen $x_1, \ldots, x_k$, $(x_1, \ldots, x_k)$ has finitely many minimal primes over it. If one of these is $\mathfrak{p}$, we are done. Otherwise, by the prime avoidance lemma, there is an element $x_{k+1} \in \mathfrak{p}$ not contained in any of these minimal primes. Next we claim that $\mathfrak{p}$ is minimal over $(x_1, x_2, \ldots, x_n)$. Choose a prime $\mathfrak{q}_n \subset \mathfrak{p}$ minimal over $(x_1, \ldots, x_n)$. We construct a sequence of primes $\mathfrak{q}_0 \subsetneq \cdots \subsetneq \mathfrak{q}_n$ such that $\mathfrak{q}_k$ is minimal over $(x_1, \ldots, x_k)$. Indeed, having chosen $\mathfrak{q}_k \subsetneq \cdots \subsetneq \mathfrak{q}_n$, $\mathfrak{q}_k$ contains $(x_1, \ldots, x_{k-1})$, so it contains some minimal prime $\mathfrak{q}_{k-1}$ over $(x_1, \ldots, x_{k-1})$. Since $x_k \in \mathfrak{q}_k$ is not contained in any minimal prime over $(x_1, \ldots, x_{k-1})$, we have $\mathfrak{q}_{k-1} \subsetneq \mathfrak{q}_k$. Now $\mathfrak{q}_0 \subsetneq \cdots \subsetneq \mathfrak{q}_n \subset \mathfrak{p}$ is of length $n = \operatorname{ht} \mathfrak{p}$, so $\mathfrak{q}_n = \mathfrak{p}$, completing the result.

**124 Theorem**

*If $R$ is noetherian, then* $\operatorname{Kdim} R[x_1, \ldots, x_n] = n + \operatorname{Kdim} R$.

PROOF It suffices to treat the $n = 1$ case. Write $x := x_1$. If $\mathfrak{p} \in \operatorname{spec}(R)$, then $R[x]/\mathfrak{p}R[x] \cong (R/\mathfrak{p})[x]$, which is a domain, so $\mathfrak{p}R[x] \in \operatorname{spec} R[x]$. If $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_m$ is a chain of primes in $R$, then $\mathfrak{p}_0 R[x] \subsetneq \mathfrak{p}_1 R[x] \subsetneq \cdots \subsetneq \mathfrak{p}_m R[x] \subsetneq \mathfrak{p}_m R[x] + xR[x]$ is a chain of primes in $R[x]$, so $\operatorname{Kdim}(R[x]) \geq \operatorname{Kdim}(R) + 1$.

On the other hand, if $\operatorname{Kdim}(R) = \infty$, $\operatorname{Kdim}(R[x]) = \infty$, so take $\operatorname{Kdim}(R) < \infty$. Let $\mathfrak{q} \in \operatorname{spec}(R[x])$ and let $\mathfrak{p} := R \cap \mathfrak{q}$. It suffices to show that $\operatorname{ht}(\mathfrak{q}) \leq \operatorname{Kdim}(R) + 1$. For this, it suffices to show that $\operatorname{ht}(\mathfrak{q}R_{\mathfrak{p}}[x]) \leq \operatorname{Kdim}(R) + 1$. Hence we can assume $R$ is local with maximal ideal $\mathfrak{p}$. We will show that $\mathfrak{q}$ is minimal over an ideal generated by $\operatorname{Kdim}(R) + 1$ elements and use the generalized principal ideal theorem to get $\operatorname{ht}(\mathfrak{q}) \leq \operatorname{Kdim}(R) + 1$.

Since $\operatorname{ht}(\mathfrak{p}) \leq \operatorname{Kdim}(R)$, by the previous corollary $\mathfrak{p}$ is minimal over an ideal $I := (x_1, \ldots, x_m)$ where $m := \operatorname{Kdim}(R)$. Since $(R, \mathfrak{p})$ is local, $\mathfrak{p}$ is the unique minimal prime over $(x_1, \ldots, x_m)$, so some power of $\mathfrak{p}$ is contained in it, i.e. $\mathfrak{p}^k \subset I$ for $k \gg 0$. Now the image of $\mathfrak{p}$ in $(R/I)[x]$ is nilpotent. We claim that if $N$ is a nilpotent ideal in a ring $T$, then $\operatorname{Kdim}(T/N) = \operatorname{Kdim}(T)$. This is just because $N$ is contained in every prime ideal, so any chain of primes in $T$ contains $N$ already. Hence $\operatorname{Kdim}((R/I)[x]) = \operatorname{Kdim}((R/\mathfrak{p})[x])$. Since $\mathfrak{p}$ is maximal, $(R/\mathfrak{p})[x]$ is a polynomial ring over a field, hence has Krull dimension 1.

Since $R[x]_\mathfrak{q}/IR[x]_\mathfrak{q}$ is a localization of $(R/I)[x]$, $\mathrm{Kdim}(R[x]_\mathfrak{q}/IR[x]_\mathfrak{q}) \leq 1$. Hence $\mathrm{ht}((\mathfrak{q}R[x]_\mathfrak{q} + IR[x]_\mathfrak{q})/IR[x]_\mathfrak{q}) \leq 1$, so $(\mathfrak{q}R[x]_\mathfrak{q} + IR[x]_\mathfrak{q})/IR[x]_\mathfrak{q}$ is minimal over a principal ideal generated by the image of some $z$. Thus $\mathfrak{q}R[x]_\mathfrak{q}$ is minimal over $(x_1, \ldots, x_m, z)$. Therefore $\mathrm{ht}(qR[x]_\mathfrak{q}) \leq m + 1 = \mathrm{Kdim}(R) + 1$.

*There is a paper in the American Mathematical Monthly (2005) by T. Coquand and H. Lombardi which gives this (at least when $R = k$) in $< 2$ pages. It's elementary and short, though complicated.*

**125 Definition**

A system of parameters in a local noetherian ring $(R, \mathfrak{m}, k)$ of Krull dimension $n$ is a sequence $x_1, \ldots, x_n \in \mathfrak{m}$ such that $\mathfrak{m} = \sqrt{(x_1, \ldots, x_n)}$. Equivalently, the length of $R/(x_1, \ldots, x_n) < \infty$.

Every local noetherian ring has a system of parameters because $\mathfrak{m}$ has some finite height $n$, so $\mathfrak{m}$ will be a minimal prime over an ideal generated by $n$ elements.

**126 Example**

Let $R = k[[x, y, z]]/(x) \cap (y, z)$. Note in general that if $\mathfrak{p}, \mathfrak{p}' \in \operatorname{spec} R$ and $\mathfrak{p} \subsetneq \mathfrak{p}'$, $\mathfrak{p}' \subsetneq \mathfrak{p}$, then $\mathfrak{p}$ and $\mathfrak{p}'$ are the only minimal primes over $\mathfrak{p} \cap \mathfrak{p}'$. Here we take $\mathfrak{p} = (x), \mathfrak{p}' = (y, z)$. Now $R/(x) \cong k[[y, z]]$ and $R/(y, z) \cong k[[x]]$. Since $\mathrm{Kdim}(R)$ is the same as the Krull dimension of the quotient of $R$ by *some* (not any) minimal prime, in this case we have $\mathrm{Kdim}(R) = 2$.

Claim: $y, x + z$ is a system of parameters for $R$ i.e. $\mathfrak{m}^n \subset (y, x + z)$ for $n \gg 0$. For proof, consider $x^2 = x(x+z) \in (y, x+z)$ and $z^2 = z(x+z) \in (y, x+z)$ (in the quotient), whence $(x, y, z)^2 \subset (y, x+z)$. Notice, however, that $xy = 0$, so $y$ is not a regular element in $R$. Hence a minimal system of parameters need not be a regular sequence.

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# November 10th, 2014: Generalized Generalized Principal Ideal Theorem; Regular Rings

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**127 Theorem (Generalized Generalized Principal Ideal Theorem)**

*Let $I := (x_1, \ldots, x_n)$ be an ideal in a noetherian ring $R$. Suppose $\mathfrak{p}$ is a prime ideal containing $I$. Then*

$$\mathrm{ht}(\mathfrak{p}) \leq \mathrm{ht}_{R/I}(\mathfrak{p}/I) + n.$$

Proof We argue by induction on the number $k := \mathrm{ht}_{R/I}(\mathfrak{p}/I)$. The case $k = 0$ is exactly the generalized principal ideal theorem. Suppose $k \geq 1$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ be the minimal primes over $I$. Since $k \geq 1$, $\mathfrak{p}$ is not in this list. By the prime avoidance lemma, $\mathfrak{p} \not\subset \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_n$, that is, there exists $x \in \mathfrak{p}$ such that $x \notin \mathfrak{p}_i$ for all $i$. Hence every chain of prime ideals between $\mathfrak{p}$ and $I + xR$ does not contain any of the $\mathfrak{p}_i$'s. Indeed, any such chain can be extended by a longer chain since the smallest element in the chain is a non-minimal prime over $I$ so contains one of the $\mathfrak{p}_i$ properly. That is,

$$\mathrm{ht}(\mathfrak{p}/(I + xR)) < \mathrm{ht}(\mathfrak{p}/I) = k.$$

That is, $\mathrm{ht}(\mathfrak{p}/(I + xR)) \leq k - 1$. Applying the induction hypothesis to $I + xR$ gives

$$\mathrm{ht}(\mathfrak{p}) \leq (k - 1) + (n + 1) = n + k = n + \mathrm{ht}(\mathfrak{p}/I),$$

completing the proof.

**128 Corollary**

*Let $\mathfrak{p}$ be a prime ideal of height $k$ in a noetherian ring $R$ and $x \in \mathfrak{p}$. Then $\mathrm{ht}_{R/xR}(\mathfrak{p}/xR)$ is*

*(1) either $k$ or $k - 1$ and*

*(2) is $k-1$ if $x$ is not contained in any minimal prime.*

Proof Take $n = 1$ in the GGPIT, so that $k = \mathrm{ht}(\mathfrak{p}) \leq \mathrm{ht}_{R/xR}(\mathfrak{p}/xR) + 1$, i.e. $\mathrm{ht}(\mathfrak{p}/xR) \geq k-1$. However, $\mathrm{ht}(\mathfrak{p}/xR) \leq \mathrm{ht}(\mathfrak{p})$, giving (1). If $x$ is not in any minimal prime, then we can use the argument in the proof of the theorem to show that $\mathrm{ht}(\mathfrak{p}/xR) < \mathrm{ht}(\mathfrak{p}) = k$ by extending a chain for $\mathfrak{p}/xR$. Hence $\mathrm{ht}(\mathfrak{p}/xR) = k-1$.

**129 Proposition**

*Let $(R, \mathfrak{m}, k)$ be a local noetherian ring. Let $x \in \mathfrak{m} - \mathfrak{m}^2$. Write $\overline{R} := R/xR$, $\overline{m} := \mathfrak{m}/xR$. Then*

$$\dim_k(\overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2) = \dim_k(\mathfrak{m}/\mathfrak{m}^2) - 1.$$

Proof Let $x_1, \ldots, x_n \in \mathfrak{m}$ be such that their images $\overline{x}_1, \ldots, \overline{x}_n$ in $\overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2$ are a basis. Now $\mathfrak{m} = \mathfrak{m}^2 + Rx_1 + \cdots + Rx_n + Rx$, so by Nakayama's lemma, $x, x_1, \ldots, x_n$ generate $\mathfrak{m}$ and their images span $\mathfrak{m}/\mathfrak{m}^2$. To show these images are linearly independent, we must show that if $ax + a_1x_1 + \cdots a_nx_n \in \mathfrak{m}^2$ then $a, a_1, \ldots, a_n \in \mathfrak{m}$. Sending this linear combination to $\overline{R}$ gives $\overline{a}_1\overline{x}_1 + \cdots + \overline{a}_n\overline{x}_n \in \overline{\mathfrak{m}}$, so $\overline{a}_i = 0$ for each $i$, i.e. $a_i \in \mathfrak{m}$. It follows that $a_0x \in \mathfrak{m}^2$, but $x \notin \mathfrak{m}^2$ by assumption, so $a_0 \in \mathfrak{m}$, completing the proof.

**130 Theorem**

*Let $(R, \mathfrak{m}, k)$ be a local noetherian ring with Krull dimension $n$. Then $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = n$ if and only if $\mathfrak{m}$ is generated by $n$ elements.*

Proof ($\Rightarrow$) is Nakayama's lemma. For ($\Leftarrow$), suppose false. Then $\dim_k(\mathfrak{m}/\mathfrak{m}^2) < n$. Hence $\mathfrak{m}$ is generated by $< n$ elements by Nakayama's lemma, so $n > \mathrm{ht}(\mathfrak{m}) = \mathrm{Kdim}(R)$ by the GPIT, a contradiction.

**131 Definition**

A local noetherian ring $(R, \mathfrak{m}, k)$ is regular if $\mathfrak{m}$ can be generated by $\mathrm{Kdim}(R)$ elements, i.e. if $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = \mathrm{Kdim}(R)$. A (not necessarily local) noetherian ring $R$ is regular if $R_{\mathfrak{m}}$ is regular for all maximal ideals $\mathfrak{m}$.

**132 Proposition**

*Let $(R, \mathfrak{m}, k)$ be a regular local noetherian ring and $x \in \mathfrak{m} - \mathfrak{m}^2$. Then $R/xR$ is regular and $\mathrm{Kdim}(R/xR) = \mathrm{Kdim}(R) - 1$.*

Proof Let $n := \mathrm{Kdim}(R)$. Then $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = n$ and (with $\overline{R} := R/I$ and $\overline{\mathfrak{m}} := (\mathfrak{m} + xR)/xR$) $\dim_k(\overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2) = n-1$. Hence $\overline{\mathfrak{m}}$ is generated by $n-1$ elements, so $\mathrm{ht}(\overline{\mathfrak{m}}) \leq n-1$, i.e. $\mathrm{Kdim}(\overline{R}) \leq n-1$. However, we also showed $\mathrm{Kdim}(\overline{R})$ is either $n$ or $n-1$, so $\mathrm{Kdim}(\overline{R}) = n-1 = \mathrm{Kdim}(\overline{m}/\overline{m}^2)$, so $\overline{R}$ is regular with the suggested dimension.

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## November 12th, 2014: Krull's Intersection Theorem

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**133 Theorem (Krull's Intersection Theorem)**

*Let $R$ be noetherian, $I$ an ideal in $R$, $M$ a finitely generated $R$-module, and define $N := \cap_{n=1}^{\infty} I^n M \subset M$. Then $N = IN$.*

Proof Define $\mathcal{L}$ as the set of all submodules $L \subset M$ such that $L \cap N = IN$. $\mathcal{L}$ is non-empty since it contains $IN$. Let $L$ be a maximal member of $\mathcal{L}$, which exists since $R$ is noetherian. We will show that $I^n M \subset L$ for $n \gg 0$. Let $x \in I$ and define $M_n := \{m \in M : x^n m \in L\}$. Note that $M_n \subset M_{n+1}$. Since $M$ is finitely generated over a noetherian ring, this ascending chain eventually stabilizes, say $M_n = M_{n+1} = \cdots$. We will now show $IN = (x^n M + L) \cap N$, which implies by maximality that $x^n M \subset L$. For the claim, $IN \subset L \cap N \subset (x^n M + L) \cap N$. On the other hand, given $m \in (x^n M + L) \cap N$, write $m = x^n m' + \ell$ where $m' \in M, \ell \in L$. Then

$xm \in xN \subset IN \subset L$. This gives $x^{n+1}m' \in L$, i.e. $m' \in M_{n+1} = M_n$, so $x^n m' \in L$. Therefore $m \in L$, so $m \in L \cap N = IN$, giving the reverse inclusion.

We have shown $x^n M \subset L$, for any arbitrary $x \in I$, with $n$ depending on $x$. In particular, this is true for the finitely many generators $x_i$ of $I$, and taking $n$ sufficiently large, each $x_i^n M \subset M$. Indeed, increasing $n$ even further and applying the multinomial theorem, $I^n M \subset L$. Now $N \subset I^n M \subset L$, so $IN = L \cap N = N$.

**134 Theorem**

*Let $R$ be noetherian, $I$ a proper ideal in $R$, and $M$ a finitely generated $R$-module. If $R$ is a domain and $M$ is a torsion-free $R$-module, then $\cap_{n=1}^{\infty} I^n M = 0$.*

Proof Let $N := \cap_{n=1}^{\infty} I^n M$. Since $R$ is noetherian and $M$ is finitely generated, $N$ is finitely generated, say by $m_1, \ldots, m_k$. By Krull's Intersection Theorem, $IN = N$, so there are elements $a_{ij} \in I$ such that $m_i = \sum_{j=1}^{k} a_{ij} m_j$ for all $1 \leq i \leq k$. If $A = (a_{ij})$, we have

$$(\mathrm{id} - A) \begin{pmatrix} m_1 \\ \vdots \\ m_k \end{pmatrix} = 0.$$

Multiplying by the adjugate, $\det(\mathrm{id} - A)$ annihilates all the $m_i$'s, hence it annihilates $N$. But $\det(\mathrm{id} - A) = 1 + a$ for some $a \in I$ by Laplace expansion. Since $I$ is proper, $a \neq -1$, but $M$ is torsion-free, so $N = 0$.

**135 Corollary**

*If $R$ is a noetherian domain and $I$ is an ideal in $R$, then $\cap_{n=0}^{\infty} I^n = 0$.*

Proof Apply the preceding theorem to $M = R$.

**136 Remark**

We used a linear-algebraic fact in the proof of the previous theorem. We discuss it further here. Let $A = (a_{ij})$ be an $n \times n$ matrix with entries in a commutative ring $R$. Define determinants and minors in the usual way, eg. $A^{ij}$ is the matrix obtained by deleting row $i$ and column $j$. We define the adjoint matrix or adjugate, $\mathrm{adj}(A)$, to be the $n \times n$ matrix with $(i,j)$th entry $(-1)^{i+j} \det(A^{ji})$. For each $i$, we have $\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A^{ij})$. This type of reasoning gives $\det(A) I = A(\mathrm{adj}\, A) = (\mathrm{adj}\, A)A$.

We also mention the "principal of permanence of identities" or perhaps the "Lefshetz principle". Let's say we have two "polynomials functions", $f(a_1, \ldots, a_n)$ and $g(a_1, \ldots, a_n)$. Suppose that for every field $K$ and every choice of $a_1, \ldots, a_n \in K$ we have $f(a_1, \ldots, a_n) = g(a_1, \ldots, a_n)$. We want to say that for every commutative ring $R$ and every $a_1, \ldots, a_n \in R$, these polynomials are equal. Here our polynomials have integer coefficients, so they make sense in any ring.

Let $R = \mathbb{Z}[x_1, \ldots, x_n]$ and let $K = \mathrm{Frac}(R)$. By assumption, plugging in coefficients in $K$ yields an identity. In particular, $f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n)$. Now if $a_1, \ldots, a_n$ are elements in any ring, there is a homomorphism $\phi\colon \mathbb{Z}[x_1, \ldots, x_n] \to R$ given by $\phi(x_i) = a_i$ for arbitrary $a_i$. But then

$$f(a_1, \ldots, a_n) = \phi(f(x_1, \ldots, x_n)) = \phi(g(x_1, \ldots, x_n)) = g(a_1, \ldots, a_n).$$

Indeed, it is enough to have the identity $f = g$ for the complex numbers because $\mathbb{C}$ contains a subring isomorphic to $\mathbb{Z}[x_1, \ldots, x_n]$ for all $n$.

**137 Lemma**

*Let $A = (a_{ij})$ be an $n \times n$ matrix with entries in a commutative ring $R$. Let $C$ be an $n \times 1$ matrix over $R$. If $AC = 0$, then $\det(A)C = 0$.*

Proof $\det(A)C = \mathrm{adj}(A)AC = 0$.

**138 Lemma**

*If $A = (a_{ij})$ is an $n \times n$ matrix over $R$ and $M$ is an $R$-module generated by $k$ elements, then there is an injective homomorphism $R/\operatorname{Ann}(M) \hookrightarrow M^{\oplus k}$.*

PROOF Write $M = Rm_1 + \cdots + Rm_k$. Define $\phi\colon R \to M^{\oplus k}$ by $\phi(x) = (xm_1, \ldots, xm_k)$.

$$\ker \phi = \{x : xm_i = 0 \text{ for all } i\} = \{x : xM = 0\} = \operatorname{Ann}(M).$$

**139 Proposition**

*Let $A = (a_{ij})$ be an $n \times n$ matrix with entries in an $R$-module $M$. Let $C$ be an $n \times 1$ matrix over $M$. If $M$ is generated by the entries in $C$ and $AC = 0$, then $\det(A)M = 0$.*

PROOF Say $M = Rm_1 + \cdots + Rm_n$ where $C = (m_1; \ldots; m_n)$. By hypothesis, $\det(A)C = \operatorname{adj}(A)AC = 0$, so $\det(A)m_i = 0$ for all $i$, so $\det(A)M = 0$.

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## November 14th, 2014: Consequences of Krull's Intersection Theorem

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**140 Remark**

We attempted to finish the proof of the fact above that in a noetherian ring each prime is minimal over an ideal generated by as many elements as its height above. We did not quite finish, though we added an example afterwards.

**141 Theorem**

*Let $R$ be noetherian, $\mathfrak{p} \in \operatorname{spec} R$. Suppose $\operatorname{ht}(\mathfrak{p}) = n$ and $\mathfrak{p}$ is minimal over $(x_1, \ldots, x_n)$.*

*(1) If $a_1, \ldots, a_k \in \mathfrak{p}$, then $\operatorname{ht}(\mathfrak{p}/(a_1, \ldots, a_k)) \geq n - k$.*

*(2) For every $k$, $\operatorname{ht}(\mathfrak{p}/(x_1, \ldots, x_k)) = n - k$.*

PROOF Let $I := (a_1, \ldots, a_k)$ and write $m := \operatorname{ht}_{R/I}(\mathfrak{p}/I)$. By a previous corollary, $\mathfrak{p}/I$ is minimal over some $(\overline{b_1}, \ldots, \overline{b_m})$ for some $b_i \in R$. Therefore $\mathfrak{p}$ is minimal over $(a_1, \ldots, a_k, b_m, \ldots, b_m)$. By the generalized Krull's theorem, $\operatorname{ht}(\mathfrak{p}) \leq m + k$, i.e. $m \geq n - k$, giving (1). For (2), since $\mathfrak{p}/(x_1, \ldots, x_k)$ is minimal over $(\overline{x_{k+1}}, \ldots, \overline{x_n})$, $\operatorname{ht}(\mathfrak{p}/(x_1, \ldots, x_k)) \leq n - k$. Combined with (1), $\operatorname{ht}(\mathfrak{p}/(x_1, \ldots, x_k)) = n - k$.

**142 Corollary**

*(... to Krull's Intersection Theorem.) Let $(R, \mathfrak{m}, k)$ be a local noetherian ring, $I$ an ideal in $R$, $M$ a finitely generated $R$-module. If $I$ is proper, then $\cap_{n=1}^{\infty} I^n M = 0$.*

PROOF It suffices to prove this when $I = \mathfrak{m}$. Write $N := \cap_{n=1}^{\infty} \mathfrak{m}^n M$. By Krull's Intersection Theorem, $\mathfrak{m}N = N$. By Nakayama, $N = 0$.

**143 Corollary**

*Let $(R, \mathfrak{m}, k)$ be a local noetherian ring, $x \in R$. Suppose $R$ is not a domain. If $xR$ is a prime ideal, then it is minimal.*

PROOF Suppose not. Let $\mathfrak{p} \subsetneq xR$ be prime. If $a \in \mathfrak{p}$, then $a = xy$ for some $y \in R$. Since $x \notin \mathfrak{p}$, $y \in \mathfrak{p}$. Therefore $a \in x\mathfrak{p}$, so $\mathfrak{p} \subset x\mathfrak{p}$, giving $\mathfrak{p} = x\mathfrak{p} = x^2\mathfrak{p} = \cdots$. Hence $\mathfrak{p} \subset \cap_{n=1}^{\infty} x^n R$. By the previous corollary, this intersection is 0, so $\mathfrak{p} = 0$. This contradicts the fact that $R$ is not a domain.

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## November 17th, 2014: Hilbert Series

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**144 Remark**

We again tried to prove the converse to the GPIT above and got stuck. Next time.

**145 Proposition**

*Regular local rings are Cohen-Macaulay. Specifically, if $(R, \mathfrak{m}, k)$ is a regular local ring of Krull dimension $n$, then $R$ is Cohen-Macaulay of depth $n$.*

PROOF Let $\mathfrak{m} = x_1R + \cdots + x_nR$. It suffices to show $x_1, \ldots, x_n$ form a maximal regular sequence. $x_1 \in \mathfrak{m} - \mathfrak{m}^2$ since $\mathfrak{m}$ is generated by no fewer than $n$ elements. $R/(x_1)$ is regular local of dimension $n-1$, so by induction $\overline{x_2}, \cdots, \overline{x_n}$ form a regular sequence on $R/(x_1)$. But $R$ is a domain, so $x_1$ is regular in $R$. Hence $x_1, x_2, \ldots, x_n$ is a regular sequence in $R$. This is maximal because $(x_1, \ldots, x_n) = \mathfrak{m}$ is the maximal ideal.

**146 Theorem**

*Let $(R, \mathfrak{m}, k)$ be a local noetherian ring of Krull dimension $n$. Then $R$ is regular if and only if $R/\mathfrak{m} \oplus \mathfrak{m}/\mathfrak{m}^2 \oplus \mathfrak{m}^2/\mathfrak{m}^3 \oplus \cdots$ is isomorphic to $k[x_1, \ldots, x_n]$. Explaining and proving this result is our next goal.*

**147 Lemma**

*Let $A := A_0 \oplus A_1 \oplus \cdots$ be a graded commutative ring such that $A_0$ has finite length as a module over itself. Then $A$ is noetherian if and only if it is finitely generated as an algebra over $A_0$.*

PROOF ($\Leftarrow$): $A_0[x_1, \ldots, x_n]$ is noetherian since $A_0$ is, so the quotient $A$ is. ($\Rightarrow$): Since $A$ is noetherian, $A_{\geq 1} := \oplus_{i=1}^{\infty} A_i$ is an ideal, so is finitely generated, say by $x_1, \ldots, x_n$. We may take each $x_i$ homogeneous of positive degree. We will show by induction on $k$ that $A_k \subset A_0[x_1, \ldots, x_n]$. This is trivial for $k = 0$. For $k \geq 1$, $A_k \subset A_{\geq 1} = x_1A + \cdots + x_nA$, and we can write $A_k \subset x_1A_{k-\deg x_1} + \cdots + x_nA_{k-\deg x_n}$. Inductively, this is $\subset x_1A_0[x_1, \ldots, x_n] + \cdots + x_nA_0[x_1, \ldots, x_n] \subset A_0[x_1, \ldots, x_n]$.

**148 Definition (Hilbert Functions for Graded Noetherian Rings)**

Let $A := A_0 \oplus A_1 \oplus \cdots$ be a graded commutative ring such that $A_0$ has finite length as a module over itself. Suppose $M$ is a finitely generated graded $A$-module. Because $M$ is finitely generated, $M_{-n} = 0$ for $n \gg 0$. Also, $M_{\geq i} := \oplus_{j=i}^{\infty} M_j$ is a submodule of $M$. Now $M_{\geq i}/M_{\geq i+1} \cong M_i$ as $A_0$-modules. Since $M_{\geq i}$ is finitely generated, $M_i$ is a finitely generated $A_0$-module, hence is of finite length $\ell(M_i)$

Hence it makes sense to define the Hilbert series of $M$ as the formal Laurent series

$$\boxed{H(M; t)} := \sum_{n \in \mathbb{Z}} \ell(M_n) t^n \in \mathbb{Z}[[t]][t^{-1}].$$

Let $\boxed{\mathrm{Gr}(A)}$ denote the category of graded $A$-modules with degree-preserving morphisms. Let $\boxed{\mathrm{gr}(A)}$ denote the full subcategory of noetherian modules.

Given $M \in \mathrm{Gr}(A)$, write $\boxed{M(p)} := M$ as an $A$-module but with a different grading, namely $M(p)_i := M_{p+i}$.

**149 Lemma**

*If $A := A_0 \oplus A_1 \oplus \cdots$ is noetherian with $\ell(A_0) < \infty$, $L, M, N \in \mathrm{gr}(A)$ are finitely generated graded $A$-modules, and*

$$0 \to L \to M \to N \to 0$$

*is an exact sequence in $\mathrm{gr}(A)$, then*

$$H(M; t) = H(L; t) + H(N; t).$$

PROOF For all $n$, $0 \to L_n \to M_n \to N_n \to 0$ is an exact sequence of finite length $A_0$-modules, so $\ell(M_n) = \ell(L_n) + \ell(N_n)$.

**150 Theorem (Hilbert, Serre)**

*Let $A := A_0[x_1, \ldots, x_n]$ be a graded ring with $\ell(A_0) < \infty$ and $d_i := \deg(x_i)$ (so $A$ is noetherian). If $M \in \mathrm{gr}(A)$, then*

$$H(M; t) = \frac{f(t)}{\prod_{i=1}^{n}(1 - t^{d_i})}$$

*for some $f(t) \in \mathbb{Z}[t, t^{-1}]$.*

PROOF Induct on $n$. If $n = 0$, $\ell(M) < \infty$, so $H(M; t)$ will only have a finite number of terms, so is in $\mathbb{Z}[t, t^{-1}]$. Take $n \geq 1$. Consider the exact sequence

$$0 \to K_s \to M_s \xrightarrow{\cdot x_n} M(d_n)_s \to C(d_n)_s \to 0$$

where $\cdot x_n$ is multiplication by $x_n$ and $C_{s+d_n}$ denotes its cokernel. Then

$$H(M; t) = H(K; t) + t^{d_n} H(M; t) - t^{d_n} H(C; t).$$

Therefore $(1 - t^{d_n})H(M; t) = H(K; t) - t^{d_n} H(C; t)$. Since $x_n K = x_n C = 0$, the right-hand side is $g(t)/\prod_{i=1}^{n-1}(1 - t^{d_i})$ with $g(t) \in \mathbb{Z}[t, t^{-1}]$.

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# November 19th, 2014: Hilbert Series and GK Dimension

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**151 Remark**

We first prove the converse to the GPIT above and then continue our discussion of Hilbert series.

**152 Lemma**

*Let $\mathfrak{p} \in \operatorname{spec} R$ and $x \in \mathfrak{p}$. Write $\overline{R} := R/(x)$, $\overline{\mathfrak{p}} := \mathfrak{p}/(x)$. If $x$ is not in any minimal prime, then* $\mathrm{ht}(\mathfrak{p}) \geq \mathrm{ht}(\overline{\mathfrak{p}}) + 1$.

PROOF If $k := \mathrm{ht}(\overline{p}) = \infty$, the result is true. So, suppose $k < \infty$. Let $\overline{\mathfrak{p}} = \overline{\mathfrak{p}_0} \supsetneq \overline{\mathfrak{p}_1} \supsetneq \cdots \supsetneq \overline{\mathfrak{p}_k}$ be a chain of primes in $\overline{R}$ of length $k$. The preimages in $R$ of the $\overline{\mathfrak{p}_i}$'s give a chain of primes $\mathfrak{p} = \mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \cdots \supsetneq \mathfrak{p}_k$. Since $x \in \mathfrak{p}_k$, $\mathfrak{p}_k$ is not a minimal prime, so $\mathfrak{p}_k$ contains a minimal prime, so $\mathrm{ht}(\mathfrak{p}) \geq k + 1$.

**153 Proposition**

*If $\mathfrak{p}$ is a height $n$ prime in a noetherian ring $R$, then $\mathfrak{p}$ is minimal over an ideal generated by $n$ elements.*

PROOF If $n = 0$, $\mathfrak{p}$ is a minimal prime, so is minimal over $\{0\} = (\varnothing)$. Suppose $n \geq 1$. Since $R$ is noetherian, it has only a finite number of minimal primes, $\mathfrak{q}_1, \ldots, \mathfrak{q}_k$, say. Since $n > 0$, $\mathfrak{p}$ is not one of these minimal primes, so by the prime avoidance lemma, $\mathfrak{p} \subsetneq \mathfrak{q}_1 \cup \cdots \cup \mathfrak{q}_k$, so we have $x \in \mathfrak{p} - (\mathfrak{q}_1 \cup \cdots \cup \mathfrak{q}_k)$. By the lemma, $n - 1 \geq \mathrm{ht}(\mathfrak{p}/(x))$. By the induction hypothesis, $\mathfrak{p}/(x)$ is then a minimal prime over an ideal generated by $n - 1$ elements, $(\overline{x_1}, \ldots, \overline{x_{n-1}})$. We claim $\mathfrak{p}$ is minimal over $(x, x_1, \ldots, x_{n-1})$. Indeed, if $\mathfrak{p} \supset \mathfrak{q} \supset (x, x_1, \ldots, x_{n-1})$, then $\mathfrak{p}/(x) \supset \mathfrak{q}/(x) \supset (\overline{x_1}, \ldots, \overline{x_{n-1}})$, so by minimality of $\mathfrak{p}/(x)$, $\mathfrak{p}/(x) = \mathfrak{q}/(x)$, so $\mathfrak{p} = \mathfrak{q}$.

**154 Remark**

From now on, $A := A_0 \oplus A_1 \oplus \cdots$ is a graded noetherian ring with $\ell(A_0) < \infty$. We showed last time that $A$ is generated algebraically over $A_0$ by finitely many homogeneous elements, say $A = A_0[x_1, \ldots, x_n]$ with $d_i := \deg x_i$. If $M \in \mathrm{gr}(A)$, then we had shown

$$H(M; t) := \sum_j \ell(M_j) t^j = \frac{f(t)}{\prod_{i=1}^{n}(1 - t^{d_i})}$$

for $f(t) \in \mathbb{Z}[t, t^{-1}]$.

**155 Definition**

We define $\boxed{d(M)}$ to be the order of the pole of $H(M;t)$ at $t=1$. For instance, if $\ell(M)<\infty$, $d(M)=0$.

(This is the $\boxed{\text{Gelfand-Kirillov Dimension}}$ in this context, though this doesn't seem to be common terminology among commutative algebraists.)

**156 Corollary**

*If $x \in A$ is homogeneous and $\operatorname{Ann}_M(x)=0$, then $d(M/xM)=d(M)-1$.*

Proof Let $n := \deg x$. There is a short exact sequence

$$0 \to M(-n) \overset{\cdot x}{\to} M \to M/xM \to 0.$$

Now $H(M;t)=t^n H(M;t)+H(M/xM;t)$, so $H(M/xM;t)=(1-t^n)H(M;t)$. Hence the order of the pole at $t=1$ on the right is one less than the order on the left.

**157 Proposition**

*If $0 \to L \to M \to N \to 0$ is an exact sequence in $\operatorname{gr}(A)$, then $d(M)=\max\{d(L),d(N)\}$.*

Proof Write $m := d(M)$, $n := d(N)$, $\ell := d(L)$. Then $H(M;t)=H(L;t)+H(N;t)$, so we may write

$$\frac{f(t)}{(1-t)^m f_2(t)} = \frac{g(t)}{(1-t)^\ell g_2(t)} + \frac{h(t)}{(1-t)^n h_2(t)}$$

where $f(1)g(1)h(1)\neq 0$, $f_2(t)g_2(t)h_2(t)\in\mathbb{Z}[t]$, and $f_2(1)g_2(1)h_2(1)\neq 0$. Suppose $\ell<n$. Now

$$f(t)(1-t)^{n-m} = \frac{f_2(t)g(t)(1-t)^{n-\ell}}{g_2(t)} + \frac{f_2(t)h(t)}{h_2(t)}.$$

At $t=1$, this gives $n-m=0$. If $n<\ell$, then $m=\ell$. If $n=\ell$, the result also works.

**158 Lemma**

*Let $V$ and $W$ be $\mathbb{Z}$-graded vector spaces such that $\dim(V_i)<\infty$ and $\dim(W_i)<\infty$ for all $i$, and for $i \ll 0$, $V_i=W_i=0$. Define the Hilbert series of $V$ and $W$ exactly as before. Then*

$$H(V\otimes W;t)=H(V;t)H(W;t).$$

Proof We have $(V\otimes W)_n := \oplus_i (V_i \otimes W_{n-i})$. This corresponds precisely to polynomial multiplication of the underlying dimensions.

**159 Proposition**

*If $S := k[x_1,\ldots,x_n]$ is the polynomial ring with its standard grading $\deg x_i := 1$, then $H(S;t)=\frac{1}{(1-t)^n}$.*

Proof We induct on $n$. Trivial for $n=0$. For $n=1$, this is just the geometric series. In general, we may apply the lemma to get $H(S;t)=H(k[x]^{\otimes n};t)=\frac{1}{(1-t)^n}$.

**160 Remark**

Note that

$$\frac{1}{(1-t)^d} = \sum_{n=0}^{\infty} \binom{d+n-1}{d-1} t^n.$$

Indeed, letting $S$ be as in the preceding proposition, this says $\dim S_n = \binom{d+n-1}{d-1}$. $S_n$ has basis $\{x_1^{i_1}\cdots x_d^{i_d}\}$ for $i_1+\cdots+i_d=n$. We prove by example. If $d=4, n=8$, then we associate $x_1^2x_2^2x_3^2x_4^2$ to $11\star 22\star 33\star 44$. Evidently, the number of monomials is the number of ways to insert $3=4-1$ $\star$'s in a string of length $8+3$, i.e. $\binom{8+3}{3}$. This is the "stars and bars" bijection counting compositions of a certain length.

# November 21st, 2014: Associated Graded Rings of ($\mathfrak{m}$-adic) Filtrations

**161 Aside**

Another construction of the $p$-adics: $\mathbb{Z}[[x]]/(x-p)$.

**162 Theorem**

*Let $(R, \mathfrak{m}, k)$ be a local noetherian ring. Then* $\mathrm{Kdim}(R)$ *is the smallest integer $s$ such that $\mathfrak{m}$ is a minimal prime over an ideal generated by $s$ elements.*

Proof Given $s$ minimal with corresponding ideal $(x_1, \ldots, x_s)$, the GPIT says that $\mathrm{ht}\,\mathfrak{m} \leq s$. If $\mathrm{ht}(\mathfrak{m}) < s$, then $\mathfrak{m}$ is minimal over an ideal generated by $< s$ elements, contradicting the minimality of $s$. Hence $s = \mathrm{ht}(\mathfrak{m}) = \mathrm{Kdim}(R)$.

**163 Remark**

Recall our earlier notation, $A := A_0 \oplus A_1 \oplus \cdots$ is a graded commutative ring, $\ell(A_0) < \infty$—so $A$ is noetherian—, $A = A_0[x_1, \ldots, x_n]$, $M \in \mathrm{gr}(A)$, $d(M)$ is the order of the pole at $t = 1$ of $H(M; t) = \sum_{n=0}^{\infty} \ell(M_n) t^n$. We had shown that given a short exact sequence of $M$'s, $d$ of the middle is the maximum of the $d$'s of the left and right terms.

**164 Theorem**

*With the notation and hypotheses above, suppose further that $A$ is generated over $A_0$ by homogeneous elements of degree 1, i.e. $\deg x_i = 1$. (This is not an essential assumption, but makes the arguments less tedious.) If $M$ is finitely generated and graded with $d := d(M)$, then $H(M; t) = f(t)/(1-t)^d$ for some $f \in \mathbb{Z}[t, t^{-1}]$ with $f(1) \neq 0$. If $f(t) = a_0 + a_1 t + \cdots + a_r t^r$, then there is a polynomial $q_M(t)$ of degree $d - 1$ such that for $n \geq r + 1 - d$, $\ell(M_n) = q_M(n)$.*

Proof $\ell(M_n)$ is the coefficient of $t^n$ in $f(t)/(1-t)^d$, which is given by

$$a_0 \binom{d+n-1}{d-1} + a_1 \binom{d+n-2}{d-1} + \cdots + a_r \binom{d+n-r-1}{d-1}.$$

Since $\binom{m}{d-1} = \frac{1}{(d-1)!} m(m-1) \cdots (m-d+2)$ is a polynomial in $m$ of degree $d-1$, $\ell(M_n)$ is a polynomial of degree $d - 1$ in $n$, say $q_M(n)$. Furthermore, $q_M(t) = \frac{f(1)}{(d-1)!} t^{d-1} + \cdots$.

**165 Example**

Let $g \in k[x_1, \ldots, x_n]$ be homogeneous of degree $r$. Let $R := k[x_1, \ldots, x_n]/(g)$. Then $q_R(t) = \frac{r}{(n-1)!} t^{n-1} + \cdots$, so this $r$ is telling us the degree of $\{g = 0\} \subset \mathbb{P}^{n-1}$.

**166 Definition**

Let $(R, \mathfrak{m}, k)$ be local noetherian. $R[\mathfrak{m}t] \subset R[t]$ as follows, where in $R[t]$, $R$ has degree 0 and $t$ has degree 1. Set

$$\boxed{R[\mathfrak{m}t]} := R \oplus \mathfrak{m}t \oplus \mathfrak{m}^2 t^2 \oplus \cdots \subset R[t].$$

Define the $\boxed{\text{associated graded ring}}$ as

$$\boxed{\mathrm{Gr}_{\mathfrak{m}}(R)} := \frac{R[\mathfrak{m}t]}{(m)}.$$

It is easy to see that

$$\mathrm{Gr}_{\mathfrak{m}}(R) \cong R/\mathfrak{m} \oplus \mathfrak{m}/\mathfrak{m}^2 \oplus \mathfrak{m}^2/\mathfrak{m}^3 \oplus \cdots,$$

with multiplication defined as follows. If $x \in \mathfrak{m}^p - \mathfrak{m}^{p+1}$ and $y \in \mathfrak{m}^q - \mathfrak{m}^{q+1}$, then

$$(x + \mathfrak{m}^{p+1})(y + \mathfrak{m}^{q+1}) := xy + \mathfrak{m}^{p+q+1}.$$

**167 Definition**

If $R \supset I_1 \supset I_2 \supset \cdots$ is a chain of ideals in a ring $R$ such that $I_p I_q \subset I_{p+q}$, we call this a filtration on $R$ and the associated graded ring is $R/I_1 \oplus I_1/I_2 \oplus I_2/I_3 \oplus \cdots$. If $(R, \mathfrak{m})$ is local, we have a filtration $R \supset \mathfrak{m} \supset \mathfrak{m}^2 \supset \cdots$, called the $\mathfrak{m}$-adic filtration. Hence the associated graded ring above is just the graded ring associated to the $\mathfrak{m}$-adic filtration.

**168 Proposition**

*Let $(R, \mathfrak{m}, k)$ be local noetherian and suppose $\mathfrak{a}$ is an ideal generated by $s$ elements such that $\mathfrak{m}^n \subset \mathfrak{a}$ for $n \gg 0$. Then there is a surjective homomorphism of graded $R/\mathfrak{a}$-algebras*

$$\phi\colon (R/\mathfrak{a})[x_1, \ldots, x_s] \twoheadrightarrow \mathrm{Gr}_{\mathfrak{a}}(R) = R/\mathfrak{a} \oplus \mathfrak{a}/\mathfrak{a}^2 \oplus \cdots,$$

*where* $\deg x_i := 1$.

Proof Since $R$ is noetherian and $\mathfrak{m}^n \subset \mathfrak{a}$ for some $n$, $R/\mathfrak{a}$ has finite length. By hypothesis, $\mathfrak{a} = Rx_1 + \cdots + Rx_s$, so $\mathfrak{a}/\mathfrak{a}^2 = (R/\mathfrak{a})x_1 + \cdots + (R/\mathfrak{a})x_s$. Define $\phi$ by declaring $\phi|_{R/\mathfrak{a}} = \mathrm{id}$ and $\phi(x_i) := x_i + \mathfrak{a}^2 \in \mathfrak{a}/\mathfrak{a}^2 \subset \mathrm{Gr}_{\mathfrak{a}}(R)$, extending $\phi$ to be $R/\mathfrak{a}$-linear. $\phi$ is a homomorphism of graded $R/\mathfrak{a}$-algebras and is surjective because $\mathrm{Gr}_{\mathfrak{a}}(R)$ is generated as an $R/\mathfrak{a}$-algebra by $\mathfrak{a}/\mathfrak{a}^2 = (R\phi(x_1) + \cdots + R\phi(x_s))/\mathfrak{a}^2$.

**169 Corollary**

*If $(R, \mathfrak{m}, k)$ is local noetherian with ideal $\mathfrak{a}$ generated by $s$ elements such that $\mathfrak{m}^n \subset \mathfrak{a}$ for $n \gg 0$, then*

$$H(\mathrm{Gr}_{\mathfrak{a}}(R); t) = \frac{f(t)}{(1-t)^d}$$

*for some $d \leq s$ and $f \in \mathbb{Z}[t^{\pm 1}]$, $f(1) \neq 0$.*

**170 Lemma**

*Let $(R, \mathfrak{m}, k)$ be a noetherian local ring. If $\mathrm{Gr}_{\mathfrak{m}}(R)$ is a domain, then so is $R$.*

Proof Let $0 \neq x, y \in R$. Since $\cap_{n=0}^{\infty} \mathfrak{m}^n = 0$, there exist integers $p, q$ such that $x \in \mathfrak{m}^p - \mathfrak{m}^{p+1}$ and $y \in \mathfrak{m}^q - \mathfrak{m}^{q+1}$. By hypothesis, $0 \neq (x + \mathfrak{m}^{p+1})(y + \mathfrak{m}^{q+1}) = xy + \mathfrak{m}^{p+q+1}$, so $xy \notin \mathfrak{m}^{p+q+1}$, and in particular $xy \neq 0$.

Note: the converse fails.

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# November 24th, 2014: (Class Canceled.)

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**171 Remark**

Class canceled.

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# November 26th, 2014: Regular Local Rings are Domains with $\mathrm{Gr}_{\mathfrak{m}}(R) \cong k[X_1, \ldots, X_n]$

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**172 Remark**

We first wrap up a loose end, namely that regular local rings are domains. We then continue where we left off last lecture and show that the associated graded ring of the $\mathfrak{m}$-adic filtration of a noetherian local ring $R$ is a polynomial ring if and only if $R$ is regular.

Note: today's entry was pieced together from notes Paul sent out after the fact. I was gone during this lecture. It's likely this was not all covered in full detail in class.

**173 Theorem**

*A regular local ring is a domain.*

PROOF Let $(R, \mathfrak{m}, k)$ be a regular local ring of dimension $n$. We argue by induction on $n$. If $n = 0$, then $R$ is a field. If $n \geq 1$, then $\mathfrak{m} \neq 0$ and $\mathfrak{m}^2 \neq \mathfrak{m}$, for instance since $\mathfrak{m}/\mathfrak{m}^2$ has $k$-dimension $n \geq 1$. Suppose to the contrary $R$ is not a domain. Pick $x \in \mathfrak{m} - \mathfrak{m}^2$. Now $R/(x)$ is a regular local ring of dimension $n-1$, hence a domain, so $xR$ is prime. By one of our corollaries of Krull's Intersection Theorem, $xR$ is a minimal prime. Letting $x \in \mathfrak{m} - \mathfrak{m}^2$ vary, we may cover $\mathfrak{m} - \mathfrak{m}^2$ by minimal primes, of which there are finitely many. Hence take

$$\mathfrak{m} - \mathfrak{m}^2 \subset \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_k$$

with each $\mathfrak{p}_i$ a minimal prime and $k$ minimal. That is, $\mathfrak{m} \subset \mathfrak{m}^2 \cup \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_k$.

If $\mathfrak{m}^2 \subset \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_k$, then $\mathfrak{m}$ would be contained in $\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_k$, hence by the prime avoidance lemma, $\mathfrak{m} \subset \mathfrak{p}_i$ for some $i$. But then $\mathfrak{m} = \mathfrak{p}_i$ is a minimal prime, so $n = \mathrm{Kdim}(R) = 0$. Hence there is some $x_0 \in \mathfrak{m}^2 - \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_k$.

If $\mathfrak{p}_i \subset \mathfrak{m}^2 \cup \mathfrak{p}_1 \cup \cdots \cup \widehat{\mathfrak{p}_i} \cup \cdots \cup \mathfrak{p}_k$ (where $\widehat{-}$ indicates the ommission of $-$), then $\mathfrak{m} - \mathfrak{m}^2 \subset \mathfrak{p}_1 \cup \cdots \cup \widehat{\mathfrak{p}_i} \cup \cdots \cup \mathfrak{p}_n$, so $k$ is not minimal, a contradiction. Hence we may pick

$$x_i \in \mathfrak{p}_i - \mathfrak{m}^2 \cup \mathfrak{p}_1 \cup \cdots \cup \widehat{\mathfrak{p}_i} \cup \cdots \cup \mathfrak{p}_k.$$

Let $y := x_1 + x_0 x_2 \cdots x_k$. Since $x_0 x_2 \cdots x_k \in \mathfrak{m}^2 \subset \mathfrak{m}^2 \cup \mathfrak{p}_2 \cup \cdots \cup \mathfrak{p}_k$ and $x_1 \notin \mathfrak{m}^2 \cup \mathfrak{p}_2 \cup \cdots \cup \mathfrak{p}_k$, we have $y \notin \mathfrak{m}^2 \cup \mathfrak{p}_2 \cup \cdots \cup \mathfrak{p}_k$. Now $x_0 \in \mathfrak{m}$ and $x_i \in \mathfrak{p}_i \subset \mathfrak{m}$ for each $i$, so $y \in \mathfrak{m} \subset \mathfrak{m}^2 \cup \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_k$, forcing $y \in \mathfrak{p}_1$. However, $x_1 \in \mathfrak{p}_1$, so $x_0 x_2 \cdots x_k \in \mathfrak{p}_1$, but each of $x_0, x_2, \ldots, x_k$ is not in $\mathfrak{p}_1$, a contradiction.

**174 Corollary**

*Let $(R, \mathfrak{m}, k)$ be a regular local ring of dimension $n$ and suppose $\mathfrak{m} = (x_1, \ldots, x_n)$. Then $x_1, \ldots, x_n$ is a regular sequence on $R$ and a system of parameters.*

PROOF Since $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = n$, we have $x_i \in \mathfrak{m} - \mathfrak{m}^2$ for all $i$. By induction the successive quotients $R/(x_1, \ldots, x_i)$ are regular local rings of dimension $n - i$. Since $R/(x_1, \ldots, x_i)$ is a domain, $x_{i+1}$ is regular on $R/(x_1, \ldots, x_i)$.

**175 Theorem (Rees)**

*Let $x_1, \ldots, x_n$ be a regular sequence on a ring $R$ and define $\mathfrak{a} := (x_1, \ldots, x_n)$. Let $F$ be a degree $d$ element of $R[X_1, \ldots, X_n]$ (where $\deg X_i := 1$). If $F(x_1, \ldots, x_n) \in \mathfrak{a}^{d+1}$, then all the coefficients of $F$ belong to $\mathfrak{a}$.*

PROOF Induct on $n$. If $n = 1$, then $\mathfrak{a} = Rx_1$ and $F(X_1) \in RX_1^d$, so $F(X_1) = rX_1^d$. If $F(x_1) = rx_1^d \in \mathfrak{a}^{d+1} = Rx_1^{d+1}$, then $rx_1^d \in Rx_1^{d+1}$. Since $x_1$ is regular it follows that $r$ is a multiple of $x_1$, so $r \in \mathfrak{a}$. So, suppose $n \geq 2$.

Claim: let $\mathfrak{b} := (x_1, \ldots, x_{n-1})$. Then $x_n$ is regular on $R/\mathfrak{b}^j$ for all $j \geq 1$. Proof: induct on $j$. The case $n = 1$ is true since $x_1, \ldots, x_n$ is a regular sequence. Suppose $j \geq 2$ and suppose for some $y$ that $x_n y \in \mathfrak{b}^j$. Since $x_n y \in \mathfrak{b}^{j-1}$ and $x_n$ is regular on $\mathfrak{b}^{j-1}$, inductively $y \in \mathfrak{b}^{j-1}$. Elements of $\mathfrak{b}^{j-1}$ are $R$-linear combinations of words in $x_1, \ldots, x_{n-1}$ of length $j-1$, so $y = G(x_1, \ldots, x_{n-1})$ for some $G \in R[X_1, \ldots, X_{n-1}]_{j-1}$. We may apply the theorem to $x_n G(X_1, \ldots, X_{n-1})$ using $\mathfrak{b}$, so the coefficients of $x_n G(X_1, \ldots, X_{n-1})$ belong to $\mathfrak{b}$. Thus $y \in \mathfrak{b}^j$, so $x_n$ is regular on $R/\mathfrak{b}^j$.

With $n \geq 2$ fixed as above, we now induct on $d$. The case $d = 0$ is trivial. Suppose $d \geq 1$. We will show it suffices to prove the lemma under the assumption $F(x_1, \ldots, x_n) = 0$. Indeed, if $F(x_1, \ldots, x_n) \in \mathfrak{a}^{d+1}$, we may write this element as a sum of words of length $d+1$ in $x_1, \ldots, x_n$, so that $F(x_1, \ldots, x_n) = G(x_1, \ldots, x_n)$ for some $G(X_1, \ldots, X_n) \in R[X_1, \ldots, X_n]_{d+1}$. Collecting terms, we may write $G(X_1, \ldots, X_n) = X_1 G_1 + \cdots + X_n G_n$ where $G_i \in R[X_1, \ldots, X_n]_d$. Now consider the polynomial

$$F'(X_1, \ldots, X_n) := F(X_1, \ldots, X_n) - \sum_{i=1}^{n} x_i G(X_1, \ldots, X_n),$$

which is homogeneous of degree $d$ and $F'(x_1, \ldots, x_n) = 0$. The coefficients of the sum are each multiples of some $x_i$, so are in $\mathfrak{a}$. Hence the coefficients of $F'$ belong to $\mathfrak{a}$ if and only if the coefficients of $F$ do, so indeed we may assume $F(x_1, \ldots, x_n) = 0$.

Collect terms to write $F(X_1, \ldots, X_n) = G + X_n H$ for $G \in R[X_1, \ldots, X_{n-1}]_d$ and $H \in R[X_1, \ldots, X_n]_{d-1}$. Since

$$0 = F(x_1, \ldots, x_n) = G(x_1, \ldots, x_{n-1}) + x_n H(x_1, \ldots, x_n)$$

and since $G$ is degree $d$, $x_n H(x_1, \ldots, x_n) \in \mathfrak{b}^d$. By the above claim, $H(x_1, \ldots, x_n) \in \mathfrak{b}^d \subset \mathfrak{a}^d$. Using our induction hypothesis on $d$, the coefficients of $H$ belong to $\mathfrak{a}$. On the other hand, $H(x_1, \ldots, x_n) \in \mathfrak{b}^d$, so there is some $H'(X_1, \ldots, X_{n-1}) \in R[X_1, \ldots, X_{n-1}]_d$ such that $H(x_1, \ldots, x_n) = H(x_1, \ldots, x_n)$. Evidently, $(G + x_n H')(x_1, \ldots, x_{n-1}) = F(x_1, \ldots, x_n) = 0$, so applying the induction hypothesis (on $n$) to $G + x_n H'$, each coefficient belongs to $\mathfrak{b} \subset \mathfrak{a}$. The coefficients of $x_n H'$ belong to $\mathfrak{a}$, so the coefficients of $G$ belong to $\mathfrak{a}$. As above, the coefficients of $H$ belong to $\mathfrak{a}$, so the same is true of the coefficients of $G + X_n H = F$, which completes the proof.

**176 Theorem**

*Let $x_1, \ldots, x_n$ be a regular sequence on a ring $R$ and define $\mathfrak{a} := (x_1, \ldots, x_n)$. Write $\overline{x_i} := x_i + \mathfrak{a}^2 \in \mathfrak{a}/\mathfrak{a}^2$. Then there is an isomorphism of graded $R/\mathfrak{a}$-algebras*

$$\phi\colon (R/\mathfrak{a})[X_1, \ldots, X_n] \overset{\cong}{\to} \mathrm{Gr}_{\mathfrak{a}}(R) \qquad (= R/\mathfrak{a} \oplus \mathfrak{a}/\mathfrak{a}^2 \oplus \cdots)$$

*given by sending $R/\mathfrak{a}$ to $R/\mathfrak{a}$ and $x_i$ to $\overline{x_i}$.*

Proof Write $\mathfrak{a}[X_1, \ldots, X_n]$ for the ideal in $R[X_1, \ldots, X_n]$ consisting of polynomials with coefficients in $\mathfrak{a}$. Let $\psi$ be the composite

$$\psi\colon R[X_1, \ldots, X_n] \to (R/\mathfrak{a})[X_1, \ldots, X_n] \overset{\phi}{\to} \mathrm{Gr}_{\mathfrak{a}}(R).$$

It suffices to show $\ker \psi = \mathfrak{a}[X_1, \ldots, X_n]$, and $\supseteq$ is obvious. Since $\psi$ is a homomorphism of graded rings, its kernel is a graded ideal. Hence it suffices to show that for all $d \geq 0$, if $F \in R[X_1, \ldots, X_n]_d \cap \ker \psi$, then $F \in \mathfrak{a}[X_1, \ldots, X_n]$. Since $\deg F = d$, $\psi(F) \in \mathrm{Gr}_{\mathfrak{a}}(R)_d = \mathfrak{a}^d/\mathfrak{a}^{d+1}$. Since by assumption $\psi(F) = 0$, evidently $F(x_1, \ldots, x_n) \in \mathfrak{a}^{d+1}$. The preceding theorem assures us that $F(X_1, \ldots, X_n) \in \mathfrak{a}[X_1, \ldots, X_n]$, completing the proof.

**177 Theorem**

*Let $(R, \mathfrak{m}, k)$ be a local noetherian ring of dimension $n$. Then $R$ is a regular local ring if and only if there is an isomorphism of graded rings*

$$\mathrm{Gr}_{\mathfrak{m}}(R) \cong k[X_1, \ldots, X_n]$$

*(where $\deg X_i := 1$).*

Proof ($\Rightarrow$) Suppose $R$ is regular of dimension $n$. Then $\mathfrak{m} = (x_1, \ldots, x_n)$ for some elements $x_1, \ldots, x_n$. By the proposition from last lecture, there is a surjective homomorphism of graded $(R/\mathfrak{m})$-algebras $\phi\colon (R/\mathfrak{m})[X_1, \ldots, X_n] \twoheadrightarrow \mathrm{Gr}_{\mathfrak{m}}(R)$. By the corollary above, $x_1, \ldots, x_n$ is a regular sequence on $R$, so by the preceding theorem, $\phi$ is an isomorphism.

($\Leftarrow$) The existence of such an isomorphism implies that $\mathfrak{m}$ is generated by $n$ elements, so $R$ is regular.

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# December 1st, 2014: Minimal Projective Resolutions of Noetherian Local Rings

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**178 Definition**

Let $(R, \mathfrak{m}, k)$ be a noetherian local ring, $M$ a finitely generated $R$-module. A projective resolution of $M$
$$\cdots \xrightarrow{d_1} P_1 \xrightarrow{d_0} P_0 \xrightarrow{\epsilon = d_{-1}} M \to 0$$
(i.e. an exact sequence of $R$-modules with each $P_i$ projective) is called a minimal projective resolution if additionally the induced maps $P_0/\mathfrak{m}P_0 \to M/\mathfrak{m}M$ and $P_i/\mathfrak{m}P_i \to \ker(d_{i-1})/\operatorname{im}\ker(d_{i-1})$ are isomorphisms. That is, $\ker(P_{i+1} \xrightarrow{d_i} P_i) \subset \mathfrak{m}P_{i+1}$ for all $i \geq 0$ (where $P_{-1} := M$).

**179 Proposition**

*Let $(R, \mathfrak{m}, k)$ be a noetherian local ring. Every finitely generated $R$-module $M$ has a minimal projective resolution.*

Proof Let $P_0$ be the free $R$-module with basis $x_1, \ldots, x_n$ and suppose the images of $m_1, \ldots, m_n \in M$ in $M/\mathfrak{m}M$ form a basis and $\epsilon\colon P_0 \to M$ is given by $\epsilon(x_i) := m_i$. By Nakayama's lemma, $M = R\mathfrak{m}_1 + \cdots + R\mathfrak{m}_n$, so $\epsilon$ is surjective and $P_0/\mathfrak{m}P_0 = kx_1 \oplus \cdots \oplus kx_n$, $M/\mathfrak{m}M = km_1 \oplus \cdots \oplus km_n$, as required. Next define $K_0 := \ker(\epsilon\colon P_0 \to M)$. Define $P_1 \to K_0$ in a similar way, i.e. $P_1$ is free and $P_1/\mathfrak{m}P_1 \xrightarrow{\sim} K_0/\mathfrak{m}K_0$, etc.

**180 Proposition**

*Let $(R, \mathfrak{m}, k)$ be a noetherian local ring, $M$ a finitely generated $R$-module. Let $\cdots \to P_1 \to P_0 \to M \to 0$ be a minimal projective resolution. Then*
$$\operatorname{Tor}^n_R(k, M) \cong P_n/\mathfrak{m}P_n$$

Proof Apply $k \otimes_R -$ to the projective resolution to get terms of the form $k \otimes_R P_n = R/\mathfrak{m} \otimes_R P_n \cong P_n/\mathfrak{m}P_n \to P_{n-1}/\mathfrak{m}P_{n-1}$. Claim: $\ker(P_i \to P_{i-1}) \subset \mathfrak{m}P_i$. From our definition, $\overline{d_{i-1}}\colon P_i/\mathfrak{m}P_i \xrightarrow{\sim} K_{i-1}/\mathfrak{m}K_{i-1}$ where $K_{i-1} = \ker(P_{i-1} \to P_{i-2})$. If $x \in P_i$ and $d_{i-1}(x) = 0$, then under $\overline{d_{i-1}}$, $x$ must be sent to zero, so it must be in $\mathfrak{m}P_i$. Hence $K_i \subset \mathfrak{m}P_i$. Therefore the map $P_n \to P_{n-1}/\mathfrak{m}P_{n-1}$ is zero, so $P_n/\mathfrak{m}P_n \to P_{n-1}/\mathfrak{m}P_{n-1}$ is zero.

Now the complex $k \otimes P_* = \cdots \xrightarrow{0} P_1/\mathfrak{m}P_1 \xrightarrow{0} P_0/\mathfrak{m}P_0 \to 0$ consists entirely of 0 maps. The result follows.

**181 Proposition**

*Let $(R, \mathfrak{m}, k)$ be a local noetherian ring. The minimal projective resolution of a finitely generated $R$-module $M$ is*
$$\cdots \to R \otimes_k \operatorname{Tor}^R_1(k, M) \to R \otimes_k M/\mathfrak{m}M \to M \to 0.$$

**182 Proposition**

*Let $(R, \mathfrak{m}, k)$ be local noetherian, $M$ a finitely generated $R$-module, and $P_* \to M$ its minimal projective resolution. Then*
$$\operatorname{Tor}^R_n(k, M)^* \cong \operatorname{Ext}^n_R(M, k) \cong \operatorname{Hom}_k(P_n/\mathfrak{m}P_n, k).$$

Proof Apply $\operatorname{Hom}_R(-, k)$ to the projective resolution to get
$$0 \to \operatorname{Hom}_R(P_0, k) \to \operatorname{Hom}_R(P_1, k) \to \cdots,$$
i.e.
$$0 \to \operatorname{Hom}_k(M/\mathfrak{m}M, k) \to \operatorname{Hom}_k(P_1/\mathfrak{m}P_1, k) \xrightarrow{d_1^*} \cdots.$$
Suppose $\overline{f}\colon P_1/\mathfrak{m}P_1 \to k$ is an $R$-module homomorphism induced by some $f\colon P_1 \to k$. What is $d_1^*(\overline{f})$? Since $f \circ d_1\colon P_2 \to P_1 \to k$ has $d_1(P_2) \subset \mathfrak{m}P_1$ and $\mathfrak{m}$ annihilates $k$, $f \circ d_1 = 0$, so $d_1^*(\overline{f}) = 0$. So,
$$\begin{aligned}\operatorname{Ext}^n_R(M, k) &= H^n(\cdots \xrightarrow{0} \operatorname{Hom}_k(P_n/\mathfrak{m}P_n, k) \xrightarrow{0} \cdots) \\ &= \operatorname{Hom}_k(P_n/\mathfrak{m}P_n, k).\end{aligned}$$

**183 Theorem**

*If $(R, \mathfrak{m}, k)$ is a noetherian local ring such that* $\operatorname{pdim}(k) = n$, *then* $\operatorname{pdim}(M) \leq n$ *for all $R$-modules $M$.*

Proof From the hypotheses, $\operatorname{Tor}_i^R(k, M) = 0$ for all $i > n$ and all $M$. Therefore $\operatorname{Ext}_R^i(M, k) = 0$ for all $i > n$ and all finitely generated $M$. But the $i$th term in the minimal resolution of $M$ is the free $R$-module generated by $\operatorname{Ext}_R^i(M, k)^*$, so if $M$ is finitely generated, $\operatorname{pdim}(M) \leq n$. To extend the result to non-finitely generated modules, we appeal to a theorem of Kaplansky, namely if $\operatorname{pdim}(I) \leq n$ for all ideals $I$, then $\operatorname{pdim}(M) \leq n$ for all $M$.

**184 Remark**

Our next goal is to prove that $\operatorname{gldim}(R) = n$ if $(R, \mathfrak{m}, k)$ is a regular local ring of dimension $n$. By the preceding theorem, it suffices to show that $\operatorname{pdim}(k) = n$, since the global dimension of $R$ is

$$\boxed{\operatorname{gldim}}(R) := \sup_M \operatorname{pdim}(M)$$

where $M$ runs over all $R$-modules. (It can be shown that one may restrict $M$ to ideals of $R$, so that if $R$ is noetherian, one can restrict $M$ to finitely generated $R$-modules.)

We were originally going to use the Koszul complex (some preliminary definitions follow), though Paul decided later to use another approach.

**185 Definition**

Let $F := Re_1 \oplus \cdots \oplus Re_n$ be a free $R$-module of rank $n$. The exterior algebra of $F$ over $R$ is

$$\boxed{\Lambda(e_1, \ldots, e_n)} := T_R(F)/(e_i \otimes e_i, e_i \otimes e_j + e_j \otimes e_i).$$

Here $T_R(F) := R \oplus F \oplus (F \otimes_R F) \oplus F^{\otimes 3} \oplus \cdots = R\langle e_1, \ldots, e_n\rangle$ denotes the free algebra over $R$ on $e_1, \ldots, e_n$. That is, $T_R(F)$ is a free $R$-module with basis given by the set of words in the letters $e_1, \ldots, e_n$ and multiplication given by concatenation of words extended $R$-bilinearly. Note that $e_i$ and $e_j$ in $T_R(F)$ do not commute when $i \neq j$.

**186 Lemma**

*$\Lambda(e_1, \ldots, e_n)$ is a graded $R$-algebra with $\{e_{i_1} \cdots e_{i_p} : 1 \leq i_1 < \cdots < i_p \leq n\}$ a free $R$-basis for $\Lambda(e_1, \ldots, e_n)_p$. Hence $\operatorname{rk}(\Lambda_p) = \binom{n}{p}$, so $\operatorname{rk}(\Lambda_*) = 2^n$.*

Proof Exercise.

**187 Remark**

We were going to define

$$\cdots \to \Lambda_2 \xrightarrow{d} \Lambda_1 \xrightarrow{d} R \to R/(x_1, \ldots, x_n) \to 0$$

with $d(e_{i_1} \cdots e_{i_p}) := \sum_{r=1}^p (-1)^r x_{i_r} e_{i_1} \cdots \widehat{e_{i_r}} \cdots e_{i_p}$. One may check $d^2 = 0$ directly. One would have to check exactness as well. This may be used to give a projective resolution of $R/(x_1, \ldots, x_n)$ of length exactly $n$.

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# December 3rd, 2014: Towards the Global Dimension of Regular Local Rings

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**188 Remark**

If $(R, \mathfrak{m}, k)$ is local noetherian, a projective resolution

$$\cdots \to P_n \to \cdots \xrightarrow{d_1} P_1 \xrightarrow{d_0} P_0 \xrightarrow{d_{-1}} M \to 0$$

is minimal if $\ker d_i \subset \mathfrak{m} P_{i+1}$ for all $i$. This is a slightly more succinct way of stating our previous definition. The main properties of minimal projective resolutions are that $k \otimes_R P_*$ and $\operatorname{Hom}_R(P_*, k)$ have trivial induced differentials. These properties appeared in proofs last lecture.

**189 Remark**

Last time we said we would use the Koszul complex to prove that the global dimension of a regular local ring is its Krull dimension. We'll instead use the following lemmas.

**190 Lemma**

*Every finitely generated projective module over a local noetherian ring $(R, \mathfrak{m}, k)$ is free.*

*Note: Paul says it's true without the finitely generated hypothesis, but the proof is more involved.*

Proof Let $M$ be a finitely generated projective module and $m_1, \ldots, m_n$ a generating set for $M$ with $n$ minimal. Let $F := \oplus_{i=1}^n Re_i$ be free of rank $n$. Let $f\colon F \to M$ be the homomorphism defined by $f(e_i) := m_i$. Hence we have $K$ such that $0 \to K \to F \xrightarrow{f} M \to 0$. By Nakayama, $m_1, \ldots, m_n$ is a basis for $M/\mathfrak{m}M$. This sequence splits since $M$ is projective, so $F \cong K \oplus M$, and $F/\mathfrak{m}F \cong K/\mathfrak{m}K \oplus M/\mathfrak{m}M$. Now $\dim_k(M/\mathfrak{m}M) = n = \dim_K(F/\mathfrak{m}F)$, so $K/\mathfrak{m}K = 0$. By Nakayama, $K = 0$.

**191 Lemma**

*Let $R$ be any ring (commutative or not). If $F$ is a free $R$-module, $M$ is a non-projective $R$-module, and $0 \to K \to F \to M \to 0$ is exact, then* $\operatorname{pdim}(M) = 1 + \operatorname{pdim}(K)$.

Proof If $n > 0$, then $0 = \operatorname{Ext}^n(F, N) \to \operatorname{Ext}^n(K, N) \to \operatorname{Ext}^{n+1}(M, N) \to \operatorname{Ext}^{n+1}(F, N) = 0$, so the middle arrow is an isomorphism. Since $\operatorname{pdim}(M)$ is the largest $n$ such that $\operatorname{Ext}^n(M, N) \neq 0$ for some $N$, the result follows.

An alternate half-proof: let $\cdots \to P_0 \to K \to 0$ be a projective resolution of $K$. We can replace the end of the resolution with $P_0 \to F \to M \to 0$ and get a projective resolution of $M$. This immediately gives $\operatorname{pdim}(M) \leq \operatorname{pdim}(K) + 1$. There doesn't seem to be a quick analogous way to get the reverse inequality.

**192 Lemma**

*Let $(R, \mathfrak{m}, k)$ be local noetherian, $x \in \mathfrak{m}$, and $M$ a finitely generated $R$-module such that $x$ acts regularly on $M$. If $M/xM$ is a free $R/xR$ module, then $M$ is a free $R$-module.*

Proof Let $m_1, \ldots, m_n \in \mathfrak{m}$ be a set of elements whose images in $M/xM$ form a basis. Since $M/xM \twoheadrightarrow M/\mathfrak{m}M$, the images of $m_1, \ldots, m_n$ span $M/\mathfrak{m}M$. By Nakayama, $m_1, \ldots, m_n$ generate $\mathfrak{m}$. Claim: $\{m_1, \ldots, m_n\}$ is a basis for $M$. Suppose $\sum_{i=1}^n r_i m_i = 0$. Projecting to $M/xM$, each $r_i = 0 \in M/xM$, so each $r_i = xs_i$ for some $s_i \in R$. Therefore $0 = \sum r_i m_i = x \sum s_i m_i$. Since $x$ acts regularly on $M$, $\sum s_i m_i = 0$. Now let $F$ be the free $R$-module with basis $e_1, \ldots, e_n$ and let $0 \to K \to F \xrightarrow{f} M \to 0$ be the exact sequence where $f\colon e_i \mapsto m_i$. We have shown that if $\sum r_i e_i \in K$, then $\sum r_i e_i = x \sum s_i e_i$ and that $\sum s_i e_i \in K$. That is, $K = xK$, so since $x \in \mathfrak{m}$, $K = \mathfrak{m}K$. By Nakayama, $K = 0$.

**193 Proposition**

*Let $(R, \mathfrak{m}, k)$ be local, $x \in \mathfrak{m}$ a regular element, and $M$ an $R$-module on which $x$ acts regularly.*

*(1)* $\operatorname{pdim}_{R/xR}(M/xM) \leq \operatorname{pdim}_R(M)$.

*(2) If additionally $R$ is noetherian and $M$ is finitely generated, then* $\operatorname{pdim}_{R/xR}(M/xM) = \operatorname{pdim}_R(M)$.

Proof (1) If $\operatorname{pdim}(M) = \infty$, the result is true. Suppose $\operatorname{pdim}(M) = n < \infty$. We induct on $n$. If $n = 0$, since $M$ is projective, $M/xM$ is projective (since $F = M \oplus N$, we have $F/xF = M/xM \oplus N/xN$, so $M/xM$ is projective). Hence the proposition is true when $n = 0$. Now take $n \geq 1$. Let $0 \to K \to F \xrightarrow{f} M \to 0$ be exact with $F$ free. Since $M$ is not projective, by the preceding lemma, $\operatorname{pdim}_R(K) = \operatorname{pdim}_R(M) - 1$. By induction, $\operatorname{pdim}_{R/xR}(K/xK) \leq \operatorname{pdim}_R(K)$. Now apply $R/xR \otimes_R -$ to the right-most three terms of the preceding exact sequence and identify the resulting kernel to get

$$0 \to (K + xF)/xF \to F/xF \xrightarrow{f} M/xM \to 0.$$

The left-hand term is $K/(K \cap xF)$. If $xf \in K$, then $0 = \alpha(xf) = x\alpha(f)$, so $\alpha(f) = 0$ since $x$ acts regularly. Hence $K \cap xF = xK$. Therefore our sequence is isomorphic to

$$0 \to K/xK \to F/xF \to M/xM \to 0.$$

Since $\mathrm{pdim}(M) \geq 1$, the above lemma implies $M/xM$ is not projective. It now follows that $\mathrm{pdim}_{R/xR}(K/xK) = \mathrm{pdim}_{R/xR}(M/xM) - 1$, which combined with the preceding considerations gives the result.

(2) Essentially, our argument from (1) gives equality in general if it gives equality in the base case, which follows from one of our lemmas. More precisely, if $\mathrm{pdim}_{R/xR}(M/xM) = \infty$, the result follows from (1). Suppose $\mathrm{pdim}_{R/xR}(M/xM) = n < \infty$. If $n = 0$, the lemma shows $M$ is projective also, giving the result. So, suppose $n \geq 1$ and induct. Using the same exact sequence $0 \to K \to F \xrightarrow{f} M \to 0$ as before, we have $0 \to K/xK \to F/xF \to M/xM \to 0$ exact, and $M/xM$ is not projective, so again $\mathrm{pdim}_{R/xR}(K/xK) = \mathrm{pdim}_{R/xR}(M/xM) - 1$. Inductively, $\mathrm{pdim}_{R/xR}(K/xK) = \mathrm{pdim}_R(K)$ and the result follows.

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# December 5th, 2014: Regular Local Rings are Cohen-Macaulay

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**194 Remark**

Recall the proposition from the end of last lecture. We will use it again today. It said the projective dimension of a quotient of a module over a local noetherian ring by a cyclic submodule generated by a regular element in the maximal ideal is weakly smaller than the projective dimension of the module. Also, if the ring is noetherian and the module is finitely generated, equality holds.

**195 Theorem**

*If $(R, \mathfrak{m}, k)$ is a noetherian local ring, then*

$$\mathrm{gldim}(R) = \mathrm{pdim}_R(k).$$

Proof By definition, $\mathrm{gldim}(R) \geq \mathrm{pdim}(k)$. On the other hand, we showed earlier that $\mathrm{pdim}(k) \geq \mathrm{pdim}(M)$ for all $R$-modules $M$, so $\mathrm{pdim}(k) \geq \mathrm{gldim}(R)$, forcing equality.

**196 Theorem**

*If $(R, \mathfrak{m}, k)$ is a regular local ring, then* $\mathrm{gldim}(R) = \mathrm{Kdim}(R)$.

Proof From the preceding result, it suffices to show $\mathrm{pdim}_R(k) = n$ where $n := \mathrm{Kdim}(R)$. Induct on $n$. If $n = 0$, then $R = k$ and $\mathrm{pdim}_k(k) = 0 = n$. Take $n \geq 1$. Suppose $x \in \mathfrak{m} - \mathfrak{m}^2$, so $R/xR$ is regular of dimension $n - 1$, hence $\mathrm{pdim}_{R/xR}(k) = \mathrm{gldim}(R/xR) = n - 1$. If $n = 1$, this is 0, so $R/xR = k$ and evidently $0 \to xR \to R \to k \to 0$ is a projective resolution of minimal length ($x$ is a non-zero divisor since $R$ is a domain), so $\mathrm{pdim}_R(k) = 1 = n$. So, suppose $n \geq 2$.

Since $k \cong (R/xR)/(\mathfrak{m}/xR)$, we see $\mathrm{pdim}_{R/xR}(\mathfrak{m}/xR) = n - 2$. Since $(xR)/(x\mathfrak{m}) \cong k$, $\mathrm{pdim}_{R/xR}(xR/x\mathfrak{m}) = n - 1$. Since there is an exact sequence

$$0 \to xR/x\mathfrak{m} \to \mathfrak{m}/x\mathfrak{m} \to \mathfrak{m}/xR \to 0,$$

it follows from the long exact sequence for Ext that $\mathrm{pdim}_{R/xR}(\mathfrak{m}/x\mathfrak{m}) \leq n - 1$.

Since $\mathrm{Ann}_{\mathfrak{m}}(x) = 0$, $\mathrm{pdim}_R(\mathfrak{m}) = \mathrm{pdim}_{R/xR}(\mathfrak{m}/x\mathfrak{m}) \leq n - 1$. It follows that $\mathrm{pdim}_R(k) = \mathrm{pdim}_R(\mathfrak{m}) + 1 = n$, completing the proof.

**197 Corollary**

*A regular local ring of dimension $n$ is Cohen-Macaulay of depth $n$.*

Proof Let $(R, \mathfrak{m}, k)$ be a regular local ring of dimension $n$. The depth of $R$ is again the maximal length of a regular sequence on $R$. Since $R$ has dimension $n$, $\operatorname{depth}(R) = n$. Thus $H^n_{\mathfrak{m}}(R) \neq 0$. However, $H^p_{\mathfrak{m}}(R) = \varinjlim \operatorname{Ext}^p_R(R/\mathfrak{m}^t), R)$ which is zero for $p > n$ because $R$ has global dimension $n$. Therefore $H^p_{\mathfrak{m}}(R) \neq 0$ if and only if $p = n$, so $R$ is Cohen-Macaulay of depth $n$.

**198 Theorem**

*Let $(R, \mathfrak{m}, k)$ be local noetherian of dimension $n$. The following are equivalent:*

*(1) $R$ is regular.*

*(2)* $\operatorname{gldim}(R) = n$.

*(3)* $\operatorname{pdim}_R(k) < \infty$.

Proof (1) $\Rightarrow$ (2) was the preceding theorem. (2) $\Rightarrow$ (3) is clear since $\operatorname{gldim}(R) = \operatorname{pdim}_R(k)$. For (3) $\Rightarrow$ (1), let $t$ be the minimal number of generators for $\mathfrak{m}$. If $t = 0$, we're done, so suppose $t \geq 1$ and $\operatorname{pdim}_R(k) < \infty$. By the Auslander-Buchsbaum formula, $\operatorname{pdim}_R(k) + \operatorname{depth}(k) = \operatorname{depth}(R)$. Here $\operatorname{depth}(k) = 0$. If $\operatorname{pdim}_R(k) = 0$, then we showed before that $\operatorname{pdim}_R(M) = 0$ for all $R$-modules, which says $R$ is semisimple and local, so $R = k$. Since $t \geq 1$ by assumption, $\operatorname{pdim}_R(k) \geq 1$. Hence $\operatorname{depth}(R) \geq 1$, so there exists a regular element, so $R$ does not consist only of units and zero-divisors, and in particular $\mathfrak{m}$ does not consist of zero-divisors. If $x_1, \ldots, x_t$ generate $\mathfrak{m}$, one of the $x_i$'s must then be regular. So, pick $x \in \mathfrak{m} - \mathfrak{m}^2$ regular. To show $R$ is regular, it is enough to show $R/xR$ is regular. Since $x$ is regular, $\operatorname{Ann}_{\mathfrak{m}}(x) = 0$. Hence $\operatorname{pdim}_R(\mathfrak{m}) = \operatorname{pdim}_{R/xR}(\mathfrak{m}/x\mathfrak{m})$. Since $0 \to \mathfrak{m} \to R \to k \to 0$ and $\operatorname{pdim}_R(k) < \infty$, we get $\operatorname{pdim}_R(\mathfrak{m}) < \infty$, so the same is true of $\operatorname{pdim}_{R/xR}(\mathfrak{m}/x\mathfrak{m})$. To show $R/xR$ is regular, it is enough to show that $\operatorname{pdim}_{R/xR}(k) < \infty$ and induct. If $k$ is a direct summand of $\mathfrak{m}/x\mathfrak{m}$, then $\operatorname{pdim}_{R/xR}(k) < \infty$. Notice that $x \notin x\mathfrak{m}$ since $x \notin \mathfrak{m}^2$. Hence $\overline{x} \in \mathfrak{m}/x\mathfrak{m}$ is a non-zero element annihilated by $\mathfrak{m}$. Hence there is a homomorphism $\alpha$ such that

$$0 \to k = R/\mathfrak{m} \xrightarrow{\alpha} \mathfrak{m}/x\mathfrak{m} = N \to C \to 0,$$

where $\alpha(\overline{1}) = \overline{x}$. We show $\alpha$ splits. Since $\mathfrak{m}N = \mathfrak{m}(\mathfrak{m}/x\mathfrak{m}) = (\mathfrak{m}^2 + x\mathfrak{m})/x\mathfrak{m}$, $N/\mathfrak{m}N = \mathfrak{m}/(\mathfrak{m}^2 + x\mathfrak{m}) = \mathfrak{m}/\mathfrak{m}^2$. Now $\overline{x} \notin \mathfrak{m}N$, i.e. $\overline{\overline{x}}$ is a non-zero element of the $k$-vector space $N/\mathfrak{m}N$. Hence there exists $\overline{\beta}\colon N/\mathfrak{m}N \to k = R/\mathfrak{m}$ with $\overline{\beta}(\overline{\overline{x}}) = \overline{1}$. Let $\beta\colon N \to k$ be the composition $N \to N/\mathfrak{m}N \xrightarrow{\overline{\beta}} k$. Under this map, $x \mapsto \overline{1}$. Therefore $\beta\alpha(\overline{1}) = \beta(\overline{x}) = \overline{1}$, so we've constructed a splitting map, whence $k$ is indeed a direct summand of $\mathfrak{m}/x\mathfrak{m}$, which indeed shows that $\operatorname{pdim}_{R/xR}(k) < \infty$.

# List of Symbols

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