26 §23.3: Harish-Chandra's Theorem

This section seemed especially dense to the seminar. The following is simply an expanded version of Humphreys' account. (Author: Josh Swanson.)

Notation *In this section, L is a semisimple Lie algebra over an algebraically closed field of characteristic 0, H is a maximal toral subalgebra, Φ is a root system, $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ is a base, Λ is the set of integral weights, $\lambda_1, \ldots, \lambda_\ell$ is the set of fundamental dominant weights in Λ, $\mathfrak{U}(L)$ and $\mathfrak{U}(H)$ are universal enveloping algebras, $\mathfrak{Z}$ is the center of $\mathfrak{U}(L)$, $Z(\lambda)$ is the standard cyclic module of highest weight $\lambda \in H^*$, $\mathfrak{S}(V)$ is the symmetric algebra on V, $\mathfrak{P}(V) := \mathfrak{S}(V^*)$ is the algebra of polynomial functions on V, $\mathfrak{T}(V)$ is the tensor algebra of V, $G = \operatorname{Int} L$ is the set of inner automorphisms of L generated by exponentials of adjoints of ad-nilpotent elements of L.*

Section 23.2 introduced characters of the center $\mathfrak{Z}$ of $\mathfrak{U}(L)$ determined by $\lambda \in H^*$. In particular, for a maximal vector $v^+ \in Z(\lambda)$, for each $z \in \mathfrak{Z}$ there is a unique constant $\chi_\lambda(z)$ such that $z \cdot v^+ = \chi_\lambda(z) v^+$. Now $\chi_\lambda \colon \mathfrak{Z} \to \mathbb{F}$ is an $\mathbb{F}$-algebra homomorphism. For instance, it sends $1 \in \mathfrak{U}(L)$ to $1 \in \mathbb{F}$. That section ended by giving a sufficient condition for two weights $\lambda, \mu \in \Lambda$ to give equal characters, namely if $\lambda + \delta$ is $\mathcal{W}$-conjugate to $\mu + \delta$, then $\chi_\mu = \chi_\lambda$. (While λ, μ were integral weights in this part of §23.2, below we will amplify this result to all $\lambda, \mu \in H^*$.) The point of section 23.3 is to give the converse:

Theorem 121 (Harish-Chandra) *If $\lambda, \mu \in H^*$ and $\chi_\lambda = \chi_\mu$, then $\lambda + \delta$ and $\mu + \delta$ are $\mathcal{W}$-conjugates.*

We begin by giving an alternate definition of the characters. To describe this, first note that every $\mu \colon H \to \mathbb{F}$ may be viewed as an associative algebra homomorphism $\mu \colon \mathfrak{U}(H) \to \mathbb{F}$, which sends a monomial $h_1 \cdots h_k$ to $\mu(h_1) \cdots \mu(h_k)$. We think of μ as evaluating a polynomial in the h_i's at $h_i = \mu(h_i)$. Applying this observation to $\lambda + \delta$, we will show:

Proposition 122 *There exists an explicit algebra homomorphism $\psi \colon \mathfrak{Z} \to \mathfrak{U}(H)$ such that*

$$\chi_\lambda(z) = (\lambda + \delta)(\psi(z))$$

for all $\lambda \in H^*$ and $z \in \mathfrak{Z}$.

Proof Pick the usual basis for L, written as $\{h_i, 1 \le i \le \ell; x_\alpha, y_\alpha, \alpha \succ 0\}$. Corollary 17.3C of the PBW theorem allows us to order these basis elements, say with y_α's first, h_i's next, and x_α's last, to give a basis for $\mathfrak{U}(L)$ given by the weakly increasing monomials from our basis, namely monomials of the form $\prod_{\alpha \succ 0} y_\alpha^{i_\alpha} \prod_i h_i^{k_i} \prod_{\alpha \succ 0} x_\alpha^{j_\alpha}$. Write some $z \in \mathfrak{Z}$ in this basis and consider how a particular monomial acts on a highest weight vector $v^+ \in Z(\lambda)$. If any $j_\alpha > 0$, that x_α will annihilate v^+, so such monomials may be ignored. If each $j_\alpha = 0$, the h_i's will simply scale v^+, while if some $i_\alpha > 0$, the resulting monomial will land v^+ in a lower weight space and will not contribute to the eigenvalue (indeed, the sum of all such monomials will kill v^+). In this way, only monomials coming entirely from h_i's contribute to $\chi_\lambda(z)$, and each h_i multiplies v^+ by $\lambda(h_i)$. Precisely, if we let $\xi\colon \mathfrak{U}(L) \to \mathfrak{U}(H)$ be the linear map which sends a basis monomial in $h_1, \ldots, h_\ell$ to itself and all other basis elements to 0, we've just observed that

$$\chi_\lambda(z) = \lambda(\xi(z)), \qquad \forall z \in \mathfrak{Z}.$$

In fact, $\xi|_{\mathfrak{Z}}$ is an algebra homomorphism, as follows. We observed that χ_λ is an algebra homomorphism, so that $\lambda(\xi(z_1 z_2)) = \lambda(\xi(z_1))\lambda(\xi(z_2))$, for all $\lambda \in H^*$. Since H is abelian, by the above usage of the PBW theorem, $\mathfrak{U}(H)$ is just the commutative polynomial algebra $\mathbb{F}[h_1, \ldots, h_\ell] = \mathfrak{S}(H)$. Hence we may consider $\lambda(\xi(z_1 z_2))$ as evaluating a polynomial in the h_i's at $(\lambda(h_1), \ldots, \lambda(h_\ell))$, so that $\xi(z_1 z_2) \equiv \xi(z_1)\xi(z_2)$ as polynomials under every evaluation. Since $\mathbb{F}$ is infinite, we have equality as formal polynomials, i.e. $\xi(z_1 z_2) = \xi(z_1)\xi(z_2)$.

Next consider the algebra automorphism $\eta\colon \mathfrak{U}(H) \to \mathfrak{U}(H)$ given by $h_i \mapsto h_i - 1$. (It has inverse induced by $h_i \mapsto h_i + 1$.) We claim that $\psi := \eta \circ \xi|_{\mathfrak{Z}}$ has the desired property.

To see this, first recall from §13.3 that $\delta := \frac{1}{2}\sum_{\alpha \succ 0} \alpha$ satisfies $\delta = \sum_{i=1}^{\ell} \lambda_i$. Further recall that $\lambda(h_j) = \langle \lambda, \alpha_j \rangle$ for all $\lambda \in H^*$. (I haven't found a proposition or lemma explicitly giving this. One argument is to see the final sentences of §9.4 and Proposition 8.4(e), which together show that $\langle \beta, \alpha \rangle = r - q = \beta(h_\alpha)$ for all non-proportional roots α, β. For proportional roots, say $\alpha = \beta = \alpha_i$, we have on the one hand $\langle \alpha_i, \alpha_i \rangle = 2(\alpha_i, \alpha_i)/(\alpha_i, \alpha_i) = 2$. On the other hand, Proposition 8.3(g) gives $h_i = 2t_i/(t_i, t_i)$, and the discussion after Corollary 8.2 gives by definition $\alpha_i(h_i) = (t_i, h_i)$, so that $\alpha_i(h_i) = 2$. Extend this to all $\beta \in H^*$ using linearity in the first argument.) By definition we have $\langle \lambda_i, \alpha_j \rangle = \delta_{ij}$, so that altogether we have $\delta(h_j) = \sum_{i=1}^{\ell} \lambda_i(h_j) = \sum_i \langle \lambda_i, \alpha_j \rangle = \sum_i \delta_{ij} = 1$. We may now compute

$$\begin{aligned}(\lambda + \delta)(h_i - 1) &= (\lambda + \delta)(h_i) - (\lambda + \delta)(1) \\ &= \lambda(h_i) + \delta(h_i) - 1 = \lambda(h_i) + 1 - 1 \\ &= \lambda(h_i).\end{aligned}$$

That is, $(\lambda + \delta)(h_i - 1) = \lambda(h_i)$, which extends to the full algebra $\mathfrak{U}(H)$ since the h_i are an algebraic generating set, i.e. $(\lambda+\delta)(\eta(h)) = \lambda(h)$ for all $h \in \mathfrak{U}(H)$. Since ξ maps $\mathfrak{Z}$ into $\mathfrak{U}(H)$, we further have $(\lambda+\delta)(\psi(z)) = \lambda(\xi(z))$ for all $z \in \mathfrak{Z}$. Combined with the displayed equation above, this gives the result.

Note: a rough description of $(\lambda + \delta)(\psi(z))$ is that it evaluates z viewed as a "polynomial" in x_α, h_i, y_α (ordered!) at $x_\alpha = 0$, $y_\alpha = 0$, and $h_i = (\lambda + \delta)(h_i - 1) = \lambda(h_i)$. ∎

Our next goal is to show that $\psi\colon \mathfrak{Z} \to \mathfrak{U}(H)$ maps surjectively onto the $\mathcal{W}$-invariants of $\mathfrak{U}(H)$. While $\mathcal{W}$ acts on H^*, it is not immediately clear how

$\mathcal{W}$ actually acts on H. It is natural to define the Killing form identification $F\colon H^* \to H$ given by $F(\phi) := t_\phi$ (in the notation of §8.2; for instance, $F(\alpha_k) = t_{\alpha_k}$) to be an isomorphism of $\mathcal{W}$-modules. That is, $\sigma \cdot h := F(\sigma \cdot F^{-1}(h))$. This action respects evaluation of linear functionals in the following sense, which is implicit in Humphreys but does not seem to appear formally anywhere:

Lemma 123 *For all $\lambda \in H^*$, $\sigma \in \mathcal{W}$, and $h \in H$, we have*

$$\lambda(\sigma \cdot h) = (\sigma^{-1} \cdot \lambda)(h).$$

PROOF Write $t_i := t_{\alpha_i}$. The t_i are a basis, being dual to the $\alpha_i^\vee := 2\alpha_i/(\alpha_i, \alpha_i)$. By linearity, we may then verify the identity for all λ and h by considering the case when $\lambda = \alpha_i$ and $h = t_k$. Moreover, we may then check it on generators $\sigma = \sigma_{\alpha_j} =: \sigma_j$ after noting that

$$\begin{aligned}\lambda((\sigma_1\sigma_2) \cdot h) &= \lambda(\sigma_1 \cdot (\sigma_2 \cdot h)) \\ &= (\sigma_1^{-1} \cdot \lambda)(\sigma_2 \cdot h) \\ &= (\sigma_2^{-1} \cdot (\sigma_1^{-1} \cdot \lambda))(h) \\ &= ((\sigma_1\sigma_2)^{-1} \cdot \lambda)(h).\end{aligned}$$

That is, we have reduced to the case $\alpha_i(\sigma_j \cdot t_k) = (\sigma_j \cdot \alpha_i)(t_k)$. First compute $\sigma_j \cdot t_k$ using F above:

$$\sigma_j \cdot t_k \mapsto \sigma_j \cdot \alpha_k = \alpha_k - \langle \alpha_k, \alpha_j \rangle \alpha_j \mapsto t_k - \langle \alpha_k, \alpha_j \rangle t_j = \sigma_j \cdot t_k.$$

Now compute

$$\begin{aligned}\alpha_i(\sigma_j \cdot t_k) &= \alpha_i(t_k - \langle \alpha_k, \alpha_j \rangle t_j) \\ &= \alpha_i(t_k) - \langle \alpha_k, \alpha_j \rangle \alpha_i(t_j)\end{aligned}$$

and

$$\begin{aligned}(\sigma_j \cdot \alpha_i)(t_k) &= (\alpha_i - \langle \alpha_i, \alpha_j \rangle \alpha_j)(t_k) \\ &= \alpha_i(t_k) - \langle \alpha_i, \alpha_j \rangle \alpha_j(t_k).\end{aligned}$$

Hence we must show $(\alpha_k, \alpha_j)\alpha_i(t_j) = (\alpha_i, \alpha_j)\alpha_j(t_k)$. First recall that by definition $\phi(h) = (t_\phi, h)$ for all $\phi \in H^*$ and $h \in H$. Further recall that by definition $(t_\alpha, t_\beta) = (\alpha, \beta)$. Hence $\alpha_i(t_j) = (t_i, t_j) = (\alpha_i, \alpha_j)$ and $\alpha_j(t_k) = (t_j, t_k) = (\alpha_j, \alpha_k) = (\alpha_k, \alpha_j)$, which gives the desired equality. ■

As in the introduction to §23.2, we may extend this $\mathcal{W}$-action from H to $\mathfrak{U}(H)$. A formal verification shows that the lemma remains true in this setting, i.e. when $h \in \mathfrak{U}(H)$. We are now ready to show:

Proposition 124 *$\psi(z)$ is $\mathcal{W}$-invariant, so $\psi\colon \mathfrak{Z} \to \mathfrak{S}(H)^{\mathcal{W}}$.*

Proof If $\lambda, \mu \in \Lambda$, Corollary' of §23.2 says that if $\lambda+\delta$ is $\mathcal{W}$-conjugate to $\mu+\delta$, then $\chi_\lambda = \chi_\mu$. Indeed, every $\mathcal{W}$-conjugate of $\lambda+\delta$ is of the form $\mu+\delta$ with $\mu \in \Lambda$—use $\mu = \sigma(\lambda+\delta)-\delta$, which is in Λ since $\mathcal{W}$ preserves inner products and $\delta \in \Lambda$. Hence in this case

$$(\lambda+\delta)(\psi(z)) = \chi_\lambda(z) = \chi_\mu(z) = (\mu+\delta)(\psi(z)), \qquad z \in \mathfrak{Z}$$

so $\psi(z)$ is the same on all $\mathcal{W}$-conjugates of $\lambda+\delta$ for all $\lambda \in \Lambda$. Since $\Lambda = \Lambda - \delta$, $\psi(z)$ is the same on all $\mathcal{W}$-conjugates of all $\lambda \in \Lambda$, i.e. $(\sigma \cdot \lambda)(\psi(z)) = \lambda(\psi(z))$ for all $\lambda \in \Lambda$ and $\sigma \in \mathcal{W}$. From the lemma, we then have $\lambda(\sigma \cdot \psi(z)) = (\sigma^{-1} \cdot \lambda)(\psi(z)) = \lambda(\psi(z))$, which is to say that all $\mathcal{W}$-conjugates of $\psi(z)$, viewed as polynomials in the h_i, have the same evaluations at all Λ.

Consider now the $\mathcal{W}$-invariant polynomial $z' := \frac{1}{|\mathcal{W}|}\sum_{\sigma\in\mathcal{W}} \sigma \cdot \psi(z) \in \mathfrak{S}(H)$. Note that z' and $\psi(z)$ yield the same evaluations at each Λ. Now z' and $\psi(z)$ agree at infinitely many inputs, so we have equality of formal polynomials, i.e. $z' = \psi(z)$, so $\psi(z)$ is $\mathcal{W}$-invariant, completing the result.

We may incidentally now amplify the result of Corollary' of §23.2, namely we may replace $\lambda, \mu \in \Lambda$ with $\lambda, \mu \in H^*$. To see this, note that we have $\lambda(\psi(z)) = \lambda(\sigma \cdot \psi(z)) = (\sigma^{-1} \cdot \lambda)(\psi(z))$ for all $\lambda \in H^*$ (not just $\lambda \in \Lambda$!) and $\sigma \in \mathcal{W}$. Hence

$$\chi_\lambda(z) = (\lambda+\delta)(\psi(z)) = (\mu+\delta)(\psi(z)) = \chi_\mu(z)$$

for all $\lambda, \mu \in H^*$ where $\lambda+\delta$ and $\mu+\delta$ are $\mathcal{W}$-conjugate. ■

To prove Harish-Chandra's theorem, we first consider what goes wrong when $\lambda+\delta$ and $\mu+\delta$ are not conjugate:

Lemma 125 *Let $\lambda_1, \lambda_2 \in H^*$ lie in distinct $\mathcal{W}$-orbits. Then λ_1, λ_2 take distinct values at some element of $\mathfrak{S}(H)^{\mathcal{W}}$.*

Proof We must find some $\mathcal{W}$-invariant polynomial in the h_i's for which evaluating at $h_i \mapsto \lambda_1(h_i)$ and $h_i \mapsto \lambda_2(h_i)$ yield distinct values. By (multivariable) Lagrange interpolation, there is a polynomial which is 1 at λ_1 and vanishes at every other $\mathcal{W}$-conjugate of λ_1 as well as every $\mathcal{W}$-conjugate of λ_2. The sum of the $\mathcal{W}$-conjugates of this polynomial is $\mathcal{W}$-invariant and evaluates to $|\mathcal{W}| \neq 0$ at λ_1 and to 0 at λ_2. ■

Now suppose $\lambda, \mu \in H^*$ and that $\chi_\lambda = \chi_\mu$. From the Proposition, we then have

$$(\lambda+\delta)(\psi(z)) = \chi_\lambda(z) = \chi_\mu(z) = (\mu+\delta)(\psi(z)).$$

That is, $\lambda+\delta$ and $\mu+\delta$ agree on $\psi(\mathfrak{Z}) \subset \mathfrak{S}(H)^{\mathcal{W}}$. If $\psi(\mathfrak{Z}) = \mathfrak{S}(H)^{\mathcal{W}}$, then $\lambda+\delta$ and $\mu+\delta$ must be $\mathcal{W}$-conjugate to avoid creating a contradiction through the lemma. Thus the following will prove the theorem:

Proposition 126 *$\psi\colon \mathfrak{Z} \to \mathfrak{S}(H)^{\mathcal{W}}$ is surjective.*

Proof The main idea is to express ψ in terms of the surjection (isomorphism) θ from §23.1. There are several natural isomorphisms which allow us to turn θ into a map from $\mathfrak{Z}$ to $\mathfrak{S}(H)^{\mathcal{W}}$, though this composite doesn't quite yield ψ. However, on certain homogeneous elements they will only differ by lower order terms, which will allow an inductive argument to finish off the result.

Our first goal is to explain the following (non-commutative) diagram:

$$\begin{array}{ccccccc} \mathfrak{S}(L)^G & \longleftrightarrow & \widetilde{\mathfrak{S}}(L)^G & \overset{\pi}{\longleftrightarrow} & \mathfrak{Z} & \overset{\psi}{\longrightarrow} & \mathfrak{S}(H)^{\mathcal{W}} \\ \updownarrow & & & & & & \updownarrow \\ \mathfrak{P}(L)^G & & & \overset{\theta}{\longrightarrow} & & & \mathfrak{P}(H)^{\mathcal{W}} \end{array}$$

Recall that $\mathfrak{P}(V) := \mathfrak{S}(V^*)$, so if V^* has a G-action, a linear isomorphism $V \to V^*$ can be declared to be a G-module isomorphism, which induces a G-module isomorphism $\mathfrak{S}(V) \cong \mathfrak{S}(V^*) = \mathfrak{B}(V)$. In our case, H^* is a $\mathcal{W}$-module and L is a $G = \operatorname{Int} L$-module. The Killing form identifications $L \cong L^*$ and $H \cong H^*$ then yield a G-module isomorphism $\mathfrak{P}(L) \cong \mathfrak{S}(L)$ and a $\mathcal{W}$-module isomorphism $\mathfrak{P}(H) \cong \mathfrak{S}(H)$. In particular, $\mathfrak{S}(L)^G \cong \mathfrak{P}(L)^G$ and $\mathfrak{S}(H)^{\mathcal{W}} \cong \mathfrak{P}(H)^{\mathcal{W}}$. By composing these and $\theta\colon \mathfrak{P}(L)^G \to \mathfrak{P}(H)^{\mathcal{W}}$, we now have a map $\mathfrak{S}(L)^G \to \mathfrak{S}(H)^{\mathcal{W}}$.

As for $\mathfrak{S}(L)^G \leftrightarrow \widetilde{\mathfrak{S}}(L)^G$, recall the discussion at the beginning of §17.1. In the characteristic 0 case we identified a certain subset $\widetilde{\mathfrak{S}}(V) \subset \mathfrak{T}(V)$ of "symmetric tensors" with $\mathfrak{S}(V)$. Precisely, $\widetilde{\mathfrak{S}}(V)$ is obtained by considering the S_n-invariant tensors in the degree n piece of $\mathfrak{T}(L)$ under the natural S_n-action. Averaging over the S_n-action allows us to symmetrize the nth homogeneous component of a given element of $\mathfrak{T}(L)$. The canonical surjection $\sigma\colon \mathfrak{T}(V) \to \mathfrak{S}(V) = \mathfrak{T}(V)/I$ (where I is generated by all $x \otimes y - y \otimes x$) now restricts to a linear isomorphism $\sigma\colon \widetilde{\mathfrak{S}}(V) \to \mathfrak{S}(V)$ whose inverse is given by this symmetrizing process. In the case when $V = L$ carries the above G-module structure, we see that I is G-invariant, so $\sigma\colon \widetilde{\mathfrak{S}}(L) \to \mathfrak{S}(L)$ is a G-module isomorphism. This yields $\mathfrak{S}(L)^G \cong \widetilde{\mathfrak{S}}(L)^G$.

For the final map, there is a linear isomorphism $\pi\colon \widetilde{\mathfrak{S}}(L)^G \to \mathfrak{Z}$ given as follows. Recall that $\mathfrak{U}(L)$ was constructed as the quotient of $\mathfrak{T}(L)$ by a certain two-sided ideal J, so let $\pi\colon \mathfrak{T}(L) \to \mathfrak{U}(L)$ be the canonical projection. G acts on $\mathfrak{T}(L)$ and it leaves the ideal J invariant since J is generated by all $x \otimes y - y \otimes x - [x, y]$ and $g \in G \subset \operatorname{Aut}(L)$ satisfies

$$g \cdot (x \otimes y - y \otimes x - [x, y]) = (g \cdot x) \otimes (g \cdot y) - (g \cdot y) \otimes (g \cdot x) - [g \cdot x, g \cdot y].$$

Hence G acts on the quotient $\mathfrak{U}(L)$, so $\pi\colon \mathfrak{T}(L) \to \mathfrak{U}(L)$ is a surjection of G-modules. It is clear that $\widetilde{\mathfrak{S}}(L)$ is G-invariant, so $\pi\colon \widetilde{\mathfrak{S}}(L) \to \mathfrak{U}(L)$ is a G-module morphism (though not an algebra morphism). Indeed, this last map is a linear isomorphism, as follows. Write U_n for the part of $\mathfrak{U}(L)$ coming from at most n-fold tensors in $\mathfrak{T}(L)$ and V^n for the part of $\widetilde{\mathfrak{S}}(L)$ coming from precisely n-fold tensors in $\mathfrak{T}(L)$. Note that $\widetilde{\mathfrak{S}}(L) = \oplus_{i\geq 0} V^i$ and $\mathfrak{U}(L) = \cup_{i\geq 0} U_i$. Corollary E of Theorem 17.3 (PBW) says that the restriction of π to V^n is a linear isomorphism onto a vector space complement of U_{n-1} in U_n. Supposing inductively that π is an isomorphism from $\oplus_{i=0}^{n-1} V^i$ onto U_{n-1}, we then have that π is an isomorphism from $\oplus_{i=0}^{n} V^i$ onto U_n, which gives the result as $n \to \infty$. We now take G-invariants of $\pi\colon \widetilde{\mathfrak{S}}(L) \to \mathfrak{U}(L)$. From Lemma 23.2, $\mathfrak{U}(L)^G = \mathfrak{Z}$, giving the advertised linear isomorphism $\pi\colon \widetilde{\mathfrak{S}}(L)^G \to \mathfrak{Z}$.

To get a feel for the preceding maps and to see how to proceed, we pause to work through an example.

Example *Let $L = \mathfrak{sl}(2, \mathbb{F})$, with the standard basis $(x, y, h) \subset L$. One may compute the dual of this basis in L relative to the Killing form, giving $(\frac{1}{4}y, \frac{1}{4}x, \frac{1}{8}h) \subset L$. Recall the Killing form identification of L with L^*, given by $\phi \mapsto t_\phi$ where $\phi(v) = (t_\phi, v)$ for all $v \in L$. Another way to obtain dual vectors given a basis $v_1, \dots, v_n$ is by sending $v_i \in L$ to $v_i^* \in L^*$ where $v_i^*(v_j) = \delta_{ij}$. This procedure yields a basis dual to that obtained by the Killing form identification since*

$$(t_{v_i^*}, v_j) = v_i^*(v_j) = \delta_{ij}.$$

We may therefore compute the Killing form identification of (x^*, y^*, h^*) in L^* by computing the dual basis of (x, y, h) in L as above. That is, $\frac{1}{4}y \leftrightarrow x^*$, $\frac{1}{4}x \leftrightarrow y^*$, $\frac{1}{8}h \leftrightarrow h^*$ under the Killing form identification.

The fundamental dominant weight λ is dual to $\alpha^\vee = \alpha$ (here $(\alpha, \alpha) = 2$), from which one finds $\lambda = \frac{1}{2}\alpha$. A trace polynomial calculation (see §23.1) allows us to compute θ on particular elements, which yields $\theta(h^{*^2} + x^* y^*) = \lambda^2$. Under $\mathfrak{P}(L)^G \to \mathfrak{S}(L)^G$, we then find $h^{*2} + x^* y^* \mapsto \frac{1}{8^2}h^2 + \frac{1}{4^2}yx$. The map $\mathfrak{S}(L)^G \to \widetilde{\mathfrak{S}}(L)^G$ is given by symmetrizing with respect to the S_2 action, which yields

$$\begin{aligned}
\frac{1}{2}\left(\frac{1}{64}h\otimes h + \frac{1}{16}y\otimes x + \frac{1}{64}h\otimes h + \frac{1}{16}x\otimes y\right)& \\
&= \frac{1}{64}h\otimes h + \frac{1}{32}(x\otimes y + y\otimes x) \in \widetilde{\mathfrak{S}}(L)^G \\
&\mapsto \frac{1}{64}h^2 + \frac{1}{32}(xy + yx) \in \mathfrak{Z}
\end{aligned}$$

where we have applied π in the final step. To compute the image of this latter element under ψ, we must write it as a polynomial in the y, h, x (in this order), so we rewrite xy as $[xy] + yx = h + yx$ to get $\frac{1}{64}h^2 + \frac{2}{32}yx + \frac{1}{32}h \in \mathfrak{Z}$. Now ξ applies $x = 0, y = 0$ to obtain $\frac{1}{64}(h^2 + 2h)$, and η replaces h with $h - 1$ to obtain $\frac{1}{64}(h^2 - 1) \in \mathfrak{S}(H)^{\mathcal{W}}$, which is the image under ψ. Finally, consider sending this to $\mathfrak{P}(H)^{\mathcal{W}}$. One may check that $\lambda = h^*$, so that applying the Killing form identification sends λ to $\frac{1}{8}h$, giving $\lambda^2 - \frac{1}{64}$. Since this is not λ^2 and we went all the way around the diagram, it indeed does not commute. However, the "discrepancy" is measured by an invariant (here, $\frac{1}{64}$) of lower "degree" than the element we started with.

Incidentally, here the Weyl group is generated by the element $\alpha \mapsto -\alpha$, and it follows that $\mathfrak{P}(H)^{\mathcal{W}}$ consists precisely of even-degree pieces, so α^2 or equivalently λ^2 generates $\mathfrak{P}(H)^{\mathcal{W}}$.

Now we complete the proof. Since $\psi(1) = 1$, ψ is surjective in degree 0. Let $\theta' \colon \mathfrak{S}(L)^G \to \mathfrak{S}(H)^{\mathcal{W}}$ and $\psi' \colon \mathfrak{S}(L)^G \to \mathfrak{S}(H)^{\mathcal{W}}$ denote the composite of θ or ψ with the various isomorphisms above. We will shortly show that $\theta' - \psi'$ sends homogeneous elements of degree n to elements of degree at most $n-1$. We may then induct: if ψ' surjects onto elements of degree $\leq n-1$, then any degree n element is of the form $\theta'(v)$, which differs from $\psi'(v)$ by an element in the image of ψ', so the original element is in the image of ψ'.

For the last claim, recall how $\theta \colon \mathfrak{P}(L)^G \to \mathfrak{P}(H)^{\mathcal{W}}$ was defined. Namely, write $f \in \mathfrak{P}(L)$ as a polynomial in the duals of a basis $\{v_i\}$ for L containing a basis $\{h_i\}$ for H and set the variables not in H to zero. Essentially the same procedure is used for $\theta' \colon \mathfrak{S}(L)^G \to \mathfrak{S}(H)^{\mathcal{W}}$, except without the duals.

The map $\psi' \colon \mathfrak{S}(L)^G \to \mathfrak{S}(H)^{\mathcal{W}}$ is much the same, except we must reorder the monomials so their variables appear in weakly increasing order before setting the variables not in H to zero, and then we apply the η map. If we begin with a homogeneous element of degree n, we may reorder the variables in a monomial of degree n at the cost of introducing lower degree terms, since $v_i v_j = [v_i, v_j] - v_j v_i$ where $[v_i, v_j] \in L$ may be rewritten in the v_i basis. Replacing each h_i with $h_i - 1$ similarly may be done at the cost of introducing lower-degree terms while preserving the degree n terms. Hence ψ' and θ' differ by terms of degree at most $n-1$, which proves the claim, proposition, and theorem. ∎